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# A SIMPLIFIED FORMULA FOR CALCULATION OF METRIC DIMENSION OF CONVERGING SEQUENCES

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ABSTRACT. In [MIŠÍK, L.—ŽÁČIK, T.: *A formula for calculation of metric dimension of converging sequences*, Comment. Math. Univ. Carolin. **40** (1999), 393 401] a formula for calculation of metric dimension of convex sequences converging to 0 was derived. In this paper, the formula is studied in more details. A much simpler formula is derived and mutual relations between two quantities determining metric dimension are described.

## Introduction

A frequent problem in mathematics is evaluation of the size of sets. An important indicator is the dimension of a set. There are more different concepts of dimension. All kinds of dimensions are equal on "nice" sets, for example on open sets in $\mathbb{R}^n$. In this contribution we will consider the metric dimension. It is the only kind of dimension which can be positive on countable sets. In the paper, we deal with a calculation of the (upper) metric dimension of sequences of real numbers.

According to the knowledge of the authors, the only method for calculation of the upper metric dimension was published by Hawkes ([H], this method was based on result presented in [BT]), and there was no method for calculation of the lower metric dimension. In [MZ2], formulae for calculation both, the upper and lower metric dimensions, were derived.

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Keywords: metric dimension, box-counting dimension, limit capacity, Kolmogorov dimension, Minkowski dimension, Bouligand dimension, entropy dimension, Hausdorff dimension, converging sequences, convex sequence, differentiable function.

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The main result of the paper is to derive a simpler (in comparison with [MZ2]) formula of calculation the metric dimension of a class of real sequences defined by convex differentiable functions. The use of the formula covers all calculations made in [KA].

## Preliminaries

Let $X$ be a totally bounded metric space. *Lower metric dimension* $\underline{\dim}$ of totally bounded metric space $X$ was introduced by Pontryagin and Snirelman in [PS] by

$$\underline{\dim}\, X = \liminf_{r\to 0+} \frac{\log N(r,X)}{-\log r}$$

and *upper metric dimension* $\overline{\dim}$ by Kolmogorov and Tihomirov in [KT] by

$$\overline{\dim}\, X = \limsup_{r\to 0+} \frac{\log N(r,X)}{-\log r}\,.$$

Here $N(r,X)$ denotes the least number of open balls of diameter $r$ covering the space $X$. In the case $\underline{\dim}\, X = \overline{\dim}\, X$, we define the *metric dimension* of $X$ $\dim X = \underline{\dim}\, X$.

Metric dimension is sometimes called limit capacity, entropy dimension, box-counting dimension, Kolmogorov dimension, Minkowski dimension, and Bouligand dimension.

We remark that

$$\operatorname{td} X \le \operatorname{hd} X \le \underline{\dim}\, X \le \overline{\dim}\, X\,,$$

where td denotes the topological dimension and hd the Hausdorff dimension.

It is well known that there exist countable sets of positive metric dimension. There are even examples of sets whose metric dimension equals to that of the original space. One can think that countable sets of positive metric dimension are very rare and of complicated structure. In fact, as shown in [MZ1], the opposite is true.

**THEOREM.** *Every infinite totally bounded space $X$ contains a converging sequence $\{a_n\}_{n=1}^{\infty}$ such that*

$$\overline{\dim}\{a_n\}_{n=1}^{\infty} = \overline{\dim}\, X$$

We remark that an analogue of this theorem does not hold for lower metric dimension.

In [MZ2] the concepts of convex sequence and its associate function are introduced.

By a *convex sequence* (*c-sequence*, in short) we mean a decreasing sequence of positive real numbers converging to zero which is convex as a function defined on the set of all positive integers. Any $C^1$ convex decreasing function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $a_n = f(n)$ is called an *associated function* to the given convex sequence $\{a_n\}_{n}^{\infty}{}_1$.

In the above mentioned paper the following formulae for calculation both *upper* and *lower metric dimension* of any convex sequence $A = \{a_n\}_{n-1}^{\infty}$ by means of its associated function $f$ are derived

$$\underline{\dim} A = \liminf_{x\to\infty} \frac{\log\left(x - \frac{f(x)}{f'(x)}\right)}{-\log(-f'(x))},$$

and

$$\overline{\dim} A = \limsup_{x\to\infty} \frac{\log\left(x - \frac{f(x)}{f'(x)}\right)}{-\log(-f'(x))}. \tag{1}$$

Two corollaries of this formulae enable us to calculate the metric dimension under the existence of all limits occurring:

$$\overline{\dim} A = \underline{\dim} A = \dim A = \lim_{x\to\infty} \frac{\log\left(x - \frac{f(x)}{f'(x)}\right)}{-\log(-f'(x))} \tag{2}$$

$$\dim A - \underline{\dim} A = \dim A = \max\left\{ \lim_{x\to\infty} \frac{\log x}{-\log(-f'(x))}\,,\ 1 - \lim_{x\to\infty} \frac{\log(f(x))}{\log(-f'(x))} \right\}$$

(see [MZ2; Theorem, Corollary 1, Corollary 2]).

In this paper these formulae are analysed in more details, and a much simpler formula is derived.

## Main results

We will start with a simple generalization of [MZ2; Corollary 2].

**Proposition 1.** *Let $A = \{a_n\}_{n-1}^{\infty}$ be a c-sequence with an associate function $f$. Then*

$$\max\left\{ \liminf_{x\to\infty} \frac{\log x}{\log(-f'(x))}\,,\ \liminf_{x\to\infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}} \right\} \le \underline{\dim} A$$

$$\le \max\left\{ \limsup_{x\to\infty} \frac{\log x}{\log(-f'(x))}\,,\ \limsup_{x\to\infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{f'(x)}} \right\} = \overline{\dim} A\,.$$

A proof of this proposition is omitted since it is similar to that of [MZ2; Corollary 2].

**Lemma 1.** *Let $f$ be a decreasing convex function with $\lim_{x\to\infty} f(x) = 0$ and let the inequalities*

$$0 < a < \frac{\log x}{\log \frac{1}{-f'(x)}} < b \tag{3}$$

*hold for all $x > x_0$, for some $x_0 > 0$. Then*

$$b + 1 - \frac{b}{a} \le \liminf_{x\to\infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}} \le \limsup_{x\to\infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}} \le \min\{b, 1\}\,.$$

Proof. Since $f(x) < 1$ for sufficiently large $x$, $\limsup_{x\to\infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}}$ is bounded from above by $1$. The assumption $a \ge 1$ in (3), after integration similar to that used later to derive (5), leads to a contradiction. This allows us to consider only the case $0 < a < b \le 1$.

Now, we prove the lower estimation. Firstly, let $0 < a < b < 1$.

Denote by $\alpha = \frac{1-a}{a}$ and $\beta = \frac{1-b}{b}$. From (3) we obtain

$$-t^{-\beta-1} < f'(t) < -t^{-\alpha-1} \qquad (t > x_0)\,. \tag{4}$$

Integrating (4) in the interval $(x, \infty)$ gives

$$\int_x^\infty -t^{-\beta-1}\,\mathrm{d}t < \int_x^\infty f'(t)\,\mathrm{d}t < \int_x^\infty -t^{-\alpha-1}\,\mathrm{d}t\,,$$

or

$$-\frac{x^{-\beta}}{\beta} < -f(x) < -\frac{x^{-\alpha}}{\alpha}\,,$$

since $\lim_{t\to\infty} f(t) = 0$ and $\alpha > 0$, $\beta > 0$. Hence,

$$\frac{x^{-\alpha}}{\alpha} < f(x) < \frac{x^{-\beta}}{\beta} \qquad (x > x_0)\,. \tag{5}$$

Rewriting (4) and (5) gives the forms

$$x^{\beta+1} < \frac{1}{-f'(x)} < x^{\alpha+1}\,, \tag{4'}$$

$$\beta x^{\beta} < \frac{1}{f(x)} < \alpha x^{\alpha}\,. \tag{5'}$$

Now, we estimate $\frac{\log\frac{f(x)}{-f'(x)}}{\log\frac{1}{-f'(x)}} = 1 - \frac{\log\frac{1}{f(x)}}{\log\frac{1}{-f'(x)}}$.

$$1 - \frac{\log \alpha x^{\alpha}}{\log x^{\beta+1}} < \frac{\log\frac{f(x)}{-f'(x)}}{\log\frac{1}{-f'(x)}} < 1 - \frac{\log \beta x^{\beta}}{\log x^{\alpha+1}},$$

$$1 - \frac{\alpha\log x + \log\alpha}{(\beta+1)\log x} < \frac{\log\frac{f(x)}{-f'(x)}}{\log\frac{1}{-f'(x)}} < 1 - \frac{\beta\log x + \log\beta}{(\alpha+1)\log x}.$$

Letting $x$ tend to infinity we obtain

$$1 - \frac{\alpha}{\beta+1} \le \liminf_{x\to\infty} \frac{\log\frac{f(x)}{-f'(x)}}{\log\frac{1}{-f'(x)}} \le \limsup_{x\to\infty} \frac{\log\frac{f(x)}{-f'(x)}}{\log\frac{1}{-f'(x)}} \le 1 - \frac{\beta}{\alpha+1},$$

or, equivalently, returning to original values $a$ and $b$,

$$b + 1 - \frac{b}{a} \le \liminf_{x\to\infty} \frac{\log\frac{f(x)}{-f'(x)}}{\log\frac{1}{-f'(x)}} \le \limsup_{x\to\infty} \frac{\log\frac{f(x)}{-f'(x)}}{\log\frac{1}{-f'(x)}} \le a + 1 - \tfrac{a}{b}. \tag{6}$$

In the case $b = 1$ the inequalities $(4')$ and $(5')$, needed for the lower estimation of $\liminf_{x\to\infty}\frac{\log\frac{f(x)}{-f'(x)}}{\log\frac{1}{-f'(x)}}$, can be rewritten to the form:

$$x < \frac{1}{-f'(x)} < x^{\alpha+1}, \tag{$4_1'$}$$

$$\frac{1}{f(x)} < \alpha x^{\alpha}. \tag{$5_1'$}$$

Now,

$$\begin{aligned}\liminf_{x\to\infty} \frac{\log\frac{f(x)}{-f'(x)}}{\log\frac{1}{-f'(x)}} &\ge \liminf_{x\to\infty}\left(1 - \frac{\log(\alpha x^{\alpha})}{\log x}\right)\\ &= 1 - \lim_{x\to\infty}\frac{\log\alpha + \alpha\log x}{\log x}\\ &= 1 - \alpha = 1 - \frac{1-a}{a} = 2 - \frac{1}{a} = b + 1 - \frac{b}{a}.\end{aligned}$$

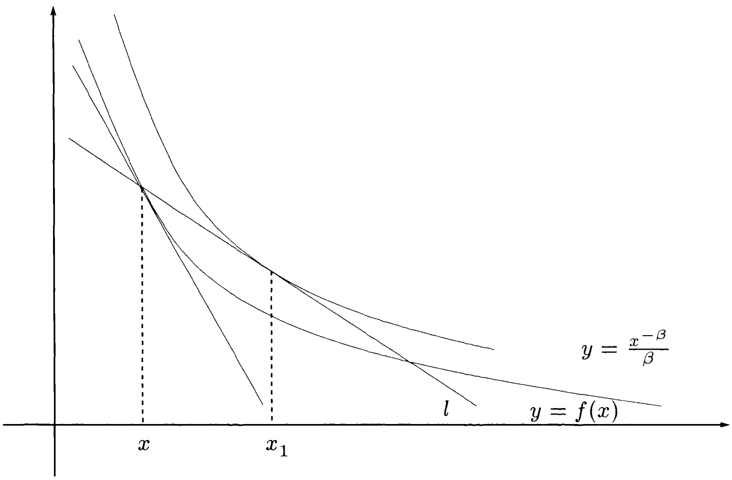
The lower estimation is now proved. The upper estimation is evidently true for $b = 1$. Now we prove the upper estimation in the case $b < 1$ using (5) and convexity of $f$.

Fix $x > x_0$, and let $x_1 > x$ be such that a line $l$ passing through points $(x, f(x))$ and $\left(x_1, \frac{x_1^{-\beta}}{\beta}\right)$ is the tangent line to the graph of $y = \frac{x^{-\beta}}{\beta}$ at point $\left(x_1, \frac{x_1{}^{\beta}}{\beta}\right)$, so

$$f(x) = \frac{x_1^{-\beta}}{\beta} - x_1^{-\beta-1}(x - x_1) < x_1^{-\beta}\left(1 + \tfrac{1}{\beta}\right),$$

i.e.,

$$\frac{1}{f(x)} > \frac{x_1^{\beta}}{1+\frac{1}{\beta}}\,.$$




By convexity of $f$ and (5), the gradient of line $l$ is greater than that of the tangent to the graph of $f$ at point $\bigl(x, f(x)\bigr)$, so $f'(x) < -x_1^{-\beta-1}$, i.e.,

$$\frac{1}{-f'(x)} < x_1^{\beta+1}\,.$$

Therefore, by decreasing nominator and increasing denominator in $\frac{\log \frac{1}{f(x)}}{\log \frac{1}{-f'(x}}$ we obtain estimation

$$1 - \frac{\log \frac{1}{f(x)}}{\log \frac{1}{-f'(x)}} < 1 - \frac{\log \frac{x_1^{\beta}}{1+\frac{1}{\beta}}}{\log x_1^{\beta+1}} = 1 - \frac{\beta \log x_1 - \log\bigl(1+\frac{1}{\beta}\bigr)}{(\beta+1)\log x_1}\,.$$

From this, as $x < x_1$,

$$\limsup_{x\to\infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}} \le 1 - \frac{\beta}{\beta+1} = \frac{1}{\beta+1} = b\,. \tag{7}$$

As for $a < b < 1$ it holds that $a + 1 - \frac{a}{b} > b$, $b$ provides a better upper bound of $\limsup\limits_{x\to\infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}}$ as the upper bound in (6).

This finishes the proof of lemma. $\square$

**Remark 1.** Note that the upper estimation and its proof depend only on $b$. So the upper estimation can be applied also in the case when $0 < \frac{\log x}{\log \frac{1}{-f'(x)}} < b$ for all $x > x_0$ instead of (3).

The following theorem provides a much simpler means of calculation of upper metric dimension as (1).

**THEOREM 1.** *Let $A = \{a_n\}_{n=1}^{\infty}$ be a c-sequence and let $f$ be its associated function. Then*

$$\overline{\dim} A = \limsup_{x \to \infty} \frac{\log x}{\log \frac{1}{-f'(x)}} .$$

Proof. Proof is a straightforward consequence of Proposition 1 and Lemma 1, where we take $b = \limsup_{x \to \infty} \frac{\log x}{\log \frac{1}{-f'(x)}} + \varepsilon$ for an arbitrary $\varepsilon > 0$. $\square$

Similarly, the following theorem enables a simpler calculation of metric dimension in comparison to (2).

**THEOREM 2.** *Let $A = \{a_n\}_{n=1}^{\infty}$ be a c-sequence and let $f$ be its associated function. Then*

$$\dim A = \lim_{x \to \infty} \frac{\log x}{\log \frac{1}{-f'(x)}}$$

*provided the limit exists.*

Proof. Let $\lim_{x \to \infty} \frac{\log x}{\log \frac{1}{-f'(x)}} = r$ and $r > 0$. Then for each $\varepsilon$, $0 < \varepsilon < r$, there is $x_0$ such that

$$r - \varepsilon < \frac{\log x}{\log \frac{1}{-f'(x)}} < r + \varepsilon$$

for each $x > x_0$. Lemma 1 then yields

$$r + \varepsilon + 1 - \frac{r + \varepsilon}{r - \varepsilon} \le \liminf_{x \to \infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}} \le \limsup_{x \to \infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}} \le r + \varepsilon .$$

If $\varepsilon$ tends to zero, both sides of previous inequalities tend to $r$. Proposition 1 then completes the proof in the case $r > 0$.

Let $r = 0$. Then for each $\varepsilon > 0$ there is $x_0$ such that

$$0 < \frac{\log x}{\log \frac{1}{-f'(x)}} < \varepsilon$$

for each $x > x_0$. By Lemma 1 and Remark 1,

$$-\infty \le \liminf_{x \to \infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}} \le \limsup_{x \to \infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}} \le \varepsilon$$

Therefore,

$$-\infty \le \liminf_{x\to\infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}} \le \limsup_{x\to\infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{-f'(x)}} \le 0\,.$$

From Proposition 1

$$\begin{aligned}\overline{\dim}\, A &= \max\left\{ \limsup_{x\to\infty} \frac{\log x}{-\log(-f'(x))}\,,\ \limsup_{x\to\infty} \frac{\log \frac{f(x)}{-f'(x)}}{\log \frac{1}{f'(x)}} \right\} \\ &= \max\left\{ 0\,,\ \limsup_{x\to\infty} \frac{\log \frac{f(x)}{f'(x)}}{\log \frac{1}{f'(x)}} \right\} = 0\,,\end{aligned}$$

hence $\dim A = 0$. The theorem is proved. □

## Examples

1) Let $\{a_n\}_{n-1}^{\infty} = \{1/n^{\alpha}\}_{n=1}^{\infty}$, $\alpha > 1$, with the associate function $f(x) = x^{\ \alpha}$. Then

$$\begin{aligned}\dim\{1/n^{\alpha}\} &= \lim_{x\to\infty} \frac{\log x}{-\log\bigl(-f'(x)\bigr)} = \lim_{x\to\infty} \frac{\log x}{-\log \alpha x^{\ \alpha-1}} \\ &= \lim_{x\to\infty} \frac{\log x}{-\log\alpha + (\alpha+1)\log x} = \frac{1}{\alpha+1}\,.\end{aligned}$$

2) Let $\{a_n\}_{n=1}^{\infty} = \{r^n\}_{n=1}^{\infty}$, $0 < r < 1$, with the associate function $f(x) = r^{x}$. Then

$$\begin{aligned}\dim\{r^n\} &= \lim_{x\to\infty} \frac{\log x}{-\log\bigl(-f'(x)\bigr)} = \lim_{x\to\infty} \frac{\log x}{-\log(-r^x \log r)} \\ &= \lim_{x\to\infty} \frac{\log x}{-x\log r - \log(-\log r)} = 0\,.\end{aligned}$$

3) Let $\{a_n\}_{n=1}^{\infty} = \{1/\log n\}_{n=1}^{\infty}$ with the associate function $f(x) = (\log x)^{\ 1}$. Then

$$\begin{aligned}\dim\{1/\log n\} &= \lim_{x\to\infty} \frac{\log x}{-\log\bigl(-f'(x)\bigr)} = \lim_{x\to\infty} \frac{\log x}{-\log \frac{1}{x(\log^2 x)}} \\ &= \lim_{x\to\infty} \frac{\log x}{\log x(\log^2 x)} = \lim_{x\to\infty} \frac{\log x}{\log x + 2\log(\log x)} = 1\,.\end{aligned}$$

4) Let $\{a_n\}_{n=1}^{\infty} = \left\{\frac{\pi}{2} - \arctan n\right\}_{n=1}^{\infty}$ with the associate function $f(x)$ $\frac{\pi}{2} - \arctan x$. Then

$$\begin{aligned}\lim_{x\to\infty} \frac{\log x}{-\log\bigl(-f'(x)\bigr)} &= \lim_{x\to\infty} \frac{\log x}{-\log \frac{1}{1+x^2}} \\ &= \lim_{x\to\infty} \frac{\log x}{\log(1+x^2)} = \frac{1}{2}\,.\end{aligned}$$

5) Let $\{a_n\}_{n=1}^{\infty} = \{1 - e^{-\frac{1}{n}}\}_{n=1}^{\infty}$ with the associate function $f(x) = 1 - e^{-\frac{1}{x}}$. Calculate

$$\lim_{x\to\infty} \frac{\log x}{-\log(-f'(x))} = \lim_{x\to\infty} \frac{\log x}{-\log\left(\frac{e^{-\frac{1}{x}}}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{\log x}{\log(x^2 e^{\frac{1}{x}})} = \lim_{x\to\infty} \frac{\log x}{\frac{1}{x} + \log x^2} = \frac{1}{2}.$$

## Open questions

We can hardly expect that the corresponding result for the lower dimension holds. The open problem remains to investigate stronger relations among $\liminf_{x\to\infty} \frac{\log x}{\log \frac{1}{f'(x)}}$, $\liminf_{x\to\infty} \frac{\frac{\log f(x)}{\log(-f'(x))}}{\log \frac{1}{-f'(x)}}$ and $\underline{\dim} A$ than are those formulated in Proposition 1. Another open problem is whether the reverse implication to that of Theorem 1 is true:

*If* $\dim A$ *exists, then also* $\lim_{x\to\infty} \frac{\log x}{\log \frac{1}{f'(x)}}$ *exists.*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