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ON OSCILLATORY FOURTH ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS I

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ABSTRACT. In this paper, oscillatory and asymptotic property of solutions of a class of fourth order neutral differential equations

$$\big(r(t)\big(y(t)+p(t)y(t-\tau)\big)''\big)''+q(t)G\big(y(t-\sigma)\big)=f(t) \tag{$*$}$$

and

$$\big(r(t)\big(y(t)+p(t)y(t-\tau)\big)''\big)''+q(t)G\big(y(t-\sigma)\big)=0$$

are studied under the assumption $\int\limits_0^\infty \frac{t}{r(t)}\,\mathrm{d}t<\infty$ for various ranges of $p(t)$. Sufficient conditions are obtained for the existence of bounded positive solutions of $(*)$.

1. Introduction

In [2], Kusano and Naito have studied oscillatory behaviour of solutions of a class of fourth order nonlinear differential equations of the form

$$\big(r(t)y''(t)\big)''+y(t)F\big(y^2(t),t\big)=0\,,$$

where r and F are continuous and positive on $[0,\infty)$ and $(0,\infty)\times[0,\infty)$ respectively, under the assumption that

(H_1) $\int\limits_0^\infty \frac{t}{r(t)}\,\mathrm{d}t<\infty.$

The object of this paper is to study, under the assumption (H_1), oscillatory behaviour of solutions of a class of fourth order nonlinear neutral differential

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equations of the form

$$\big[r(t)\big(y(t)+p(t)y(t-\tau)\big)''\big]''+q(t)G\big(y(t-\sigma)\big)=0\,, \tag{1}$$

where $r\in C\big([0,\infty),(0,\infty)\big)$, $p\in C\big([0,\infty),\mathbb{R}\big)$, $q\in C\big([0,\infty),[0,\infty)\big)$, $G\in C(\mathbb{R},\mathbb{R})$ is nondecreasing and $uG(u)>0$ for $u\neq 0$, $\tau>0$ and $\sigma\geq 0$. The associated forced equation

$$\big[r(t)\big(y(t)+p(t)y(t-\tau)\big)''\big]''+q(t)G\big(y(t-\sigma)\big)=f(t)\,, \tag{2}$$

where $f\in C\big([0,\infty),\mathbb{R}\big)$, is also studied under the assumption (H_1). Different ranges of $p(t)$ and different type of forcing functions are considered. In recent papers [3], [4], Parhi and Rath have discussed oscillation and asymptotic behaviour of solutions of nth order neutral differential equations of the form

$$\big[y(t)+p(t)y(t-\tau)\big]^{(n)}+q(t)G\big(y(t-\sigma)\big)=f(t)$$

and

$$\big[y(t)+p(t)y(t-\tau)\big]^{(n)}+q(t)G\big(y(t-\sigma)\big)=0\,.$$

Equations (1) and (2) cannot be termed particular case of the above equations in view of (H_1). Indeed, the study of (1) and (2) is very interesting. Necessary and sufficient conditions for oscillation of (1)/(2) are obtained in this paper.

By a *solution* of (1) we understand a function $y\in C\big([-\rho,\infty),\mathbb{R}\big)$ such that $y(t)+p(t)y(t-\tau)$ is twice continuously differentiable, $r(t)\big(y(t)+p(t)y(t-\tau)\big)''$ is twice continuously differentiable and equation (1) is satisfied for $t\geq 0$, where $\rho=\max\{\tau,\sigma\}$ and $\sup\big\{|y(t)|:\ t\geq t_0\big\}>0$ for every $t_0\geq 0$. A solution of (1) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*.

2. Some lemmas

In this section we prove some lemmas which play an important role in the next section.

Remark. From (H_1) it follows that

$$\int_0^\infty \frac{\mathrm{d}t}{r(t)}<\infty\,.$$

Lemma 2.1. *If $u(t)$ is an eventually positive twice continuously differentiable function such that $r(t)u''(t)$ is twice continuously differentiable and $\big(r(t)u''(t)\big)'' \le 0$ but $\not\equiv 0$ for large t, where $r \in C\big([0,\infty),(0,\infty)\big)$, then one of the following cases holds for large t:*

(a) $u'(t) > 0$, $u''(t) > 0$ *and* $\big(r(t)u''(t)\big)' > 0$,
(b) $u'(t) > 0$, $u''(t) < 0$ *and* $\big(r(t)u''(t)\big)' > 0$,
(c) $u'(t) > 0$, $u''(t) < 0$ *and* $\big(r(t)u''(t)\big)' < 0$,
(d) $u'(t) < 0$, $u''(t) > 0$ *and* $\big(r(t)u''(t)\big)' > 0$.

The proof is immediate and hence is omitted.

Lemma 2.2. *Let* (H_1) *hold. Suppose that the conditions of Lemma* 2.1 *hold. Then*

(i) *the following inequalities hold for large t in the case* (c) *of Lemma* 2.1*:*

$$u'(t) \ge -\big(r(t)u''(t)\big)'R(t), \qquad u'(t) \ge -r(t)u''(t)\int_t^\infty \frac{\mathrm{d}s}{r(s)},$$

$$u(t) \ge ktu'(t) \qquad \text{and} \qquad u(t) \ge -k\big(r(t)u''(t)\big)'tR(t),$$

where $k > 0$ and $R(t) = \int_t^\infty \frac{s-t}{r(s)}\,\mathrm{d}s$

and

(ii) $u(t) \ge r(t)u''(t)R(t)$ *for large t in case* (d) *of Lemma* 2.1.

Proof. We may note that $R(t) < \infty$ due to (H_1).

(i) For $s \ge t$, $\big(r(s)u''(s)\big)' \le \big(r(t)u''(t)\big)'$ and hence $r(s)u''(s) \le r(t)u''(t) + \big(r(t)u''(t)\big)'(s-t)$. Thus

$$0 < u'(s) \le u'(t) + \big(r(t)u''(t)\big)' \int_t^s \frac{(\theta - t)}{r(\theta)}\,\mathrm{d}\theta .$$

Taking limit as $s \to \infty$, the first inequality is obtained. For $s \ge t$, $r(s)u''(s) \le r(t)u''(t)$ and hence

$$0 < u'(s) \le u'(t) + r(t)u''(t)\int_t^s \frac{1}{r(\theta)}\,\mathrm{d}\theta .$$

Taking limit as $s \to \infty$, we obtain the second inequality. For $t > t_0 > 0$,

$$u(t) > u(t) - u(t_0) = \int_{t_0}^t u'(s)\,\mathrm{d}s > u'(t)(t - t_0) > ktu'(t),$$

where $0 < k < 1$. Hence the third inequality is obtained. One may have the fourth inequality from the first and the third ones.

(ii) For $s > \theta > t$, $r(s)u''(s) > r(\theta)u''(\theta)$ and hence

$$-u'(\theta) > r(\theta)u''(\theta) \int_{\theta}^{s} \frac{\mathrm{d}x}{r(x)}\,.$$

Taking limit as $s \to \infty$, we obtain

$$-u'(\theta) \geq -r(\theta)u''(\theta) \int_{\theta}^{\infty} \frac{\mathrm{d}x}{r(x)}\,.$$

Further integrating from t to s yields

$$\begin{aligned} u(t) &\geq \int_{t}^{s} r(\theta)u''(\theta)\Bigg(\int_{\theta}^{\infty} \frac{\mathrm{d}x}{r(x)}\Bigg)\,\mathrm{d}\theta \\ &\geq r(t)u''(t)\int_{t}^{s}\Bigg(\int_{\theta}^{\infty} \frac{\mathrm{d}x}{r(x)}\Bigg)\,\mathrm{d}\theta \\ &> r(t)u''(t)\left[\int_{t}^{s} \frac{\theta}{r(\theta)}\,\mathrm{d}\theta - t\int_{t}^{\infty} \frac{\mathrm{d}\theta}{r(\theta)}\right]. \end{aligned}$$

Taking limit as $s \to \infty$, we get

$$u(t) \geq r(t)u''(t)\int_{t}^{\infty} \frac{\theta - t}{r(\theta)}\,\mathrm{d}\theta = r(t)u''(t)R(t)\,.$$

This is the required inequality and hence the lemma is proved. □

Remark. Since $R(t) < \int_{t}^{\infty} \frac{s}{r(s)}\,\mathrm{d}s$, then $R(t) \to 0$ as $t \to \infty$ in view of (H_1).

LEMMA 2.3. *Let* (H_1) *hold. If the conditions of Lemma* 2.1 *hold, then there exist constants* $k_1 > 0$ *and* $k_2 > 0$ *such that* $k_1 R(t) \leq u(t) \leq k_2 t$ *for large* t.

Proof. Suppose that the four cases of Lemma 2.1 hold for $t \geq T_1 > 1$. If $g(t) = \int_{T_1}^{t} \frac{s(t-s)}{r(s)}\,\mathrm{d}s$, then $g'(t) = \int_{T_1}^{t} \frac{s}{r(s)}\,\mathrm{d}s$ and hence $g(t) < Lt$ for $t \geq T > T_1$

in view of (H_1). Integrating the inequality $\big(r(t)u''(t)\big)'' \le 0$, $t \ge T$, we obtain

$$u(t) \le u(T) + u'(T)(t-T) + (ru'')'(T) \int_T^t \left(\int_T^\theta \frac{s-T}{r(s)}\, \mathrm{d}s \right) \mathrm{d}\theta$$

$$+ r(T)u''(T) \int_T^t \left(\int_T^\theta \frac{\mathrm{d}s}{r(s)} \right) \mathrm{d}\theta$$

$$\le u(T) + u'(T)(t-T) + \big((ru'')'(T) + r(T)u''(T)\big) \int_T^t \frac{s(t-s)}{r(s)}\, \mathrm{d}s$$

because $T_1 > 1$. In the cases (a) and (d) of Lemma 2.1, we have

$$u(t) \le u(T) + u'(T)(t-T) + L\big((ru'')'(T) + r(T)u''(T)\big)t\,.$$

Thus $u(t) \le K_2 t$ for large t, where $K_2 > 0$ is a constant. For the case (b),

$$u(t) \le u(T) + u'(T)(t-T) + L(ru'')'(T)t\,.$$

Hence $u(t) \le K_2 t$ for large t. Similarly, we may show that $u(t) \le K_2 t$ for large t in the case (c). On the other hand, $u(t) \ge K_1 R(t)$ for large t in the case (a) because $R(t) < \int_0^\infty \frac{t}{r(t)}\, \mathrm{d}t < \infty$, where $K_1 > 0$ is a constant. Since $R(t) < \int_t^\infty \frac{s}{r(s)}\, \mathrm{d}s$, then $R(t) \to 0$ as $t \to \infty$ in view of (H_1). Hence, in the case (b), $u(t) > u(t_1) > K_1 R(t)$ for $t \ge t_1 > 0$. Consider the case (c). From Lemma 2.2, we have

$$u(t) \ge -k\big(r(t)u''(t)\big)' t R(t) \ge -k\big(r(t_1)u''(t_1)\big)' t R(t) > K_1 R(t)$$

for $t \ge t_1$. In the case (d), we obtain from Lemma 2.2 that $u(t) \ge r(t)u''(t)R(t) \ge r(t_1)u''(t_1)R(t) \ge k_1 R(t)$ for $t \ge t_1$. Thus the lemma is proved. □

LEMMA 2.4. *Let z be a real-valued twice continuously differentiable function on $[0,\infty)$ such that $r(t)z''(t)$ is twice continuously differentiable with $(r(t)z''(t))'' \le 0$ for large t. If $z(t) > 0$ eventually, then one of the following cases holds for large t:*

(a) $z'(t) > 0$, $z''(t) > 0$ *and* $\big(r(t)z''(t)\big)' > 0$,

(b) $z'(t) > 0$, $z''(t) < 0$ *and* $\big(r(t)z''(t)\big)' > 0$,

(c) $z'(t) > 0$, $z''(t) < 0$ *and* $\big(r(t)z''(t)\big)' < 0$,

(d) $z'(t) < 0$, $z''(t) > 0$ *and* $\big(r(t)z''(t)\big)' > 0$.

If $z(t) < 0$ for large t, then either one of the cases (b)–(d) *holds or one of the following cases hold for large t:*

(e) $z'(t) < 0$, $z''(t) < 0$ *and* $\big(r(t)z''(t)\big)' > 0$,

(f) $z'(t) < 0$, $z''(t) < 0$ *and* $\big(r(t)z''(t)\big)' < 0$.

Proof. Sine $\big(r(t)z''(t)\big)'' \le 0$ for large t, then $z(t) > 0$ or $z(t) < 0$ for large t. If $z(t) > 0$ for large t, then the first part of the lemma follows from Lemma 2.1. If $z(t) < 0$ for large t, then it is immediate to see that one of the cases (b)–(f) holds for large t. Thus the proof of the lemma is complete. □

Lemma 2.5. ([1; p. 19]) *Let $p, y, z \in C\big([0,\infty),\mathbb{R}\big)$ be such that $z(t) = y(t) + p(t)y(t-\tau)$, $t \ge \tau \ge 0$, $y(t) > 0$ for $t \ge t_1 > \tau$, $\liminf\limits_{t\to\infty} y(t) = 0$ and $\lim\limits_{t\to\infty} z(t) = L$ exists. Let $p(t)$ satisfy one of the following conditions:*

(i) $0 \le p(t) \le p_1 < 1$,

(ii) $1 < p_2 \le p(t) \le p_3$,

(iii) $p_4 \le p(t) \le 0$,

where p_i is a constant, $1 \le i \le 4$. Then $L = 0$.

3. Sufficient conditions for oscillation

Sufficient conditions are obtained for oscillation of solutions of equations (1) and (2). We need the following conditions:

(H_2) For $u > 0$ and $\nu > 0$, there exists $\lambda > 0$ such that $G(u) + G(\nu) \ge \lambda G(u+\nu)$.

(H_3) $G(u\nu) = G(u)G(\nu)$ for $u, \nu \in \mathbb{R}$.

(H_4) $Q(t) = \min\{q(t), q(t-\tau)\}$.

(H_5) For $u > 0$, $\nu > 0$, $G(u)G(\nu) \ge G(u\nu)$.

(H_6) $G(-u) = -G(u)$, $u \in \mathbb{R}$.

(H_7) There exists a real valued twice continuously differentiable function F on $[0,\infty)$ such that rF'' is twice continuously differentiable with $\big(r(t)F''(t)\big)'' = f(t)$ and $F(t)$ changes sign.

(H_7') Suppose that F is the same as in (H_7). In addition,

$$-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty.$$

(H_8) There exists a real valued twice continuously differentiable function F on $[0,\infty)$ such that rF'' is twice continuously differentiable with $\big(r(t)F''(t)\big)'' = f(t)$ and $\lim\limits_{t\to\infty} F(t) = 0$.

Remark. (H_3) implies that $G(-u) = -G(u)$. Indeed, $G(1)G(1) = G(1)$ and $G(1) > 0$ imply that $G(1) = 1$. Further, $G(-1)G(-1) = G(1) = 1$ implies that $\big(G(-1)\big)^2 = 1$. Since $G(-1) < 0$, then $G(-1) = -1$. Hence $G(-u) = G(-1)G(u) = -G(u)$. On the other hand, $G(u\nu) = G(u)G(\nu)$ for $u > 0$ and $\nu > 0$ and $G(-u) = -G(u)$ imply that $G(xy) = G(x)G(y)$ for every $x, y \in \mathbb{R}$.

Remark. The prototype of G satisfying (H_2), (H_5) and (H_6) is

$$G(u) = \big(a + b|u|^{\lambda}\big)|u|^{\mu} \operatorname{sgn} u,$$

where $a \geq 1$, $b \geq 1$, $\lambda \geq 0$ and $\mu \geq 0$. However, the prototype of G satisfying (H_2) and (H_3) is $G(u) = |u|^{\gamma} \operatorname{sgn} u$, where $\gamma > 0$. This G also satisfies the assumptions (H_2), (H_5) and (H_6).

THEOREM 3.1. *Let $0 \leq p(t) \leq p < \infty$. Suppose that (H_1)–(H_4) hold. If*

(H_9) $\int_0^\infty h(t)Q(t)G\big(R(t-\sigma)\big)\,\mathrm{d}t = \infty$, *where* $h(t) = \min\{R^{\alpha}(t), R^{\alpha}(t-\tau)\}$ *and* $\alpha > 1$,

then all solutions of (1) *oscillate.*

P r o o f . Since $R(t) \to 0$ as $t \to \infty$, then $h(t) \to 0$ as $t \to \infty$. Hence (H_9) implies that

$$\int_0^\infty Q(t)G\big(R(t-\sigma)\big)\,\mathrm{d}t = \infty.$$

If possible, let $y(t)$ be a nonoscillatory solution of (1). Then $y(t) > 0$ or < 0 for $t \geq t_0 > 0$. Let $y(t) > 0$ for $t \geq t_0$. Setting

$$z(t) = y(t) + p(t)y(t-\tau) \tag{3}$$

we obtain $0 < z(t) < y(t) + py(t-\tau)$ and

$$\big(r(t)z''(t)\big)'' = -q(t)G\big(y(t-\sigma)\big) \leq 0, \tag{4}$$

but $\not\equiv 0$ for $t \geq t_0 + \rho$. Hence Lemma 2.1 holds with $u(t)$ replaced by $z(t)$. Suppose that one of the cases (a), (b), (d) of Lemma 2.1 holds. Then, for $t \geq t_1 > t_0 + 2\rho$,

$$\begin{aligned} 0 &= \big(r(t)z''(t)\big)'' + G(p)\big(r(t-\tau)z''(t-\tau)\big)'' + q(t)G\big(y(t-\sigma)\big) \\ &\qquad\qquad + G(p)q(t-\tau)G\big(y(t-\tau-\sigma)\big) \\ &\geq \big(r(t)z''(t)\big)'' + G(p)\big(r(t-\tau)z''(t-\tau)\big)'' + \lambda Q(t)G\big(z(t-\sigma)\big) \\ &\geq \big(r(t)z''(t)\big)'' + G(p)\big(r(t-\tau)z''(t-\tau)\big)'' + \lambda G(k_1)Q(t)G\big(R(t-\sigma)\big) \end{aligned}$$

due to (H_2), (H_3), (H_4) and Lemma 2.3. Hence

$$\int_{t_1}^{\infty} Q(t)G\big(R(t-\sigma)\big)\, dt < \infty\,,$$

which is a contradiction. Suppose that the case (c) holds. The use of Lemmas 2.2 and 2.3 yields, for $t \ge t_2 > t_1$,

$$k\big(-r(t)z''(t)\big)' tR(t) \le z(t) \le k_2 t\,.$$

Hence

$$\begin{aligned} -\Big[\big((-r(t)z''(t))'\big)^{1-\alpha}\Big]' &= (\alpha-1)\big((-r(t)z''(t))'\big)^{-\alpha}\big(-r(t)z''(t)\big)'' \\ &\ge (\alpha-1)L^{\alpha}R^{\alpha}(t)q(t)G\big(y(t-\sigma)\big)\,, \end{aligned} \tag{5}$$

where $L = (k/k_2) > 0$. Thus

$$\begin{aligned} &-\Big[\big((-r(t)z''(t))'\big)^{1-\alpha}\Big]' - G(p)\Big[\big((-r(t-\tau)z''(t-\tau))'\big)^{1-\alpha}\Big]' \\ &\ge (\alpha-1)L^{\alpha}\big[R^{\alpha}(t)q(t)G\big(y(t-\sigma)\big) + G(p)R^{\alpha}(t-\tau)q(t-\tau)G\big(y(t-\tau-\sigma)\big)\big] \\ &\ge \lambda(\alpha-1)L^{\alpha}h(t)Q(t)G\big(z(t-\sigma)\big) \\ &\ge \lambda(\alpha-1)L^{\alpha}G(k_1)h(t)Q(t)G\big(R(t-\sigma)\big)\,. \end{aligned}$$

Consequently,

$$\int_{t_2}^{\infty} h(t)Q(t)G\big(R(t-\sigma)\big)\, dt < \infty\,,$$

which is a contradiction to (H_9). If $y(t) < 0$ for $t \ge t_0$, then we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \ge t_0$ and

$$\big[r(t)\big(x(t) + p(t)x(t-\tau)\big)''\big]'' + q(t)G\big(x(t-\sigma)\big) = 0\,.$$

Proceeding as above we obtain a similar contradiction. Thus the theorem is proved. □

THEOREM 3.2. *Suppose that* $0 \le p(t) \le p < 1$. *If* (H_1) *and* (H_3) *hold and if*

(H_{10}) $\int_0^{\infty} R^{\alpha}(t)G\big(R(t-\sigma)\big)q(t)\, dt = \infty$, $\alpha > 1$,

then every solution of (1) *oscillates or tends to zero as* $t \to \infty$.

Proof. Since $R(t) \to 0$ as $t \to \infty$, then (H_{10}) implies that

$$\int_0^{\infty} G\big(R(t-\sigma)\big)q(t)\, dt = \infty \tag{6}$$

and hence

$$\int_0^\infty q(t)\,\mathrm{d}t = \infty\,. \tag{7}$$

Let $y(t)$ be a nonoscillatory solution of (1). Let $y(t) > 0$ for $t \ge t_0 > 0$. The case $y(t) < 0$ for $t \ge t_0$ is similarly dealt with. We set $z(t)$ as in (3) to obtain $z(t) > 0$ and (4) for $t \ge t_0 + \rho$. Hence Lemma 2.1 holds. Consider the cases (a) and (b) of Lemma 2.1. In either case $z(t)$ is increasing. Hence for $t \ge t_0 + 2\rho$,

$$(1-p)z(t) < z(t) - p(t)z(t-\tau) = y(t) - p(t)p(t-\tau)y(t-2\tau) \le y(t)\,. \tag{8}$$

Thus $y(t) > (1-p)k_1R(t)$ for $t \ge t_1 > t_0 + 2\rho$ by Lemma 2.3. Consequently, from (4) we obtain

$$\int_{t_2}^\infty q(t)G\big(R(t-\sigma)\big)\,\mathrm{d}t < \infty\,,$$

where $t_2 > t_1 + \sigma$, a contradiction to (6). For the case (c) of Lemma 2.1 we proceed as in the proof of Theorem 3.1 to obtain (5). Since z is increasing, then we have (8). Hence $y(t) > (1-p)k_1R(t)$ for $t \ge t_1 > t_0 + 2\rho$ by Lemma 2.3. Consequently,

$$-\Big[\big((-r(t)z''(t))'\big)^{1-\alpha}\Big]' \ge (\alpha-1)L^\alpha G\big((1-p)k_1\big)R^\alpha(t)q(t)G\big(R(t-\sigma)\big)$$

for $t \ge t_2 > t_1 + \rho$. Integrating the above inequality, we get

$$\int_{t_2}^\infty q(t)R^\alpha(t)G\big(R(t-\sigma)\big)\,\mathrm{d}t < \infty\,,$$

a contradiction to (H_{10}). In the case (d) of Lemma 2.1, $\lim\limits_{t\to\infty} z(t)$ exists. If $\liminf\limits_{t\to\infty} y(t) > 0$, then from (4) it follows that

$$\int_0^\infty q(t)\,\mathrm{d}t < \infty\,,$$

which is a contradiction to (7). Hence $\liminf\limits_{t\to\infty} y(t) = 0$. Consequently, $\lim\limits_{t\to\infty} z(t) = 0$ by Lemma 2.5. Since $z(t) \ge y(t)$, then $\lim\limits_{t\to\infty} y(t) = 0$. Thus the theorem is proved. □

Remark. (H_9) implies (H_{10}).

THEOREM 3.3. *Let $-1 < p \le p(t) \le 0$. If* (H_1), (H_3) *and* (H_{10}) *hold, then every solution of* (1) *oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1). Let $y(t) > 0$ for $t \ge t_0 > 0$. Setting $z(t)$ as in (3) we obtain (4) for $t \ge t_0 + \rho$ and hence $z(t) > 0$ or < 0 for $t \ge t_1 > t_0 + \rho$. Let $z(t) > 0$ for $t \ge t_1$. Suppose that one of the cases (a), (b), (d) of Lemma 2.4 holds. From Lemma 2.3 we have $y(t) \ge z(t) \ge k_1 R(t)$ for $t \ge t_2 > t_1$ and hence (4) yields $\int_{t_3}^{\infty} q(t) G\big(R(t-\sigma)\big)\,\mathrm{d}t < \infty$, $t_3 > t_2 + \rho$, a contradiction to (6). We may note that (H_{10}) implies (6). Suppose that the case (c) holds. Proceeding as in the proof of Theorem 3.1 we obtain (5). Further, $y(t) \ge z(t) \ge k_1 R(t)$ for $t \ge t_2$ by Lemma 2.3. Hence, for $t \ge t_3 > t_2 + \rho$,

$$-\Big[\big((-r(t)z''(t))'\big)^{1-\alpha}\Big]' \ge (\alpha-1)L^{\alpha}G(k_1)R^{\alpha}(t)q(t)G\big(R(t-\sigma)\big).$$

Integrating the above inequality yields

$$\int_{t_3}^{\infty} q(t)R^{\alpha}(t)G\big(R(t-\sigma)\big)\,\mathrm{d}t < \infty,$$

a contradiction to (H_{10}).

If $z(t) < 0$ for $t \ge t_1$, then $y(t) < y(t-\tau)$ and hence $y(t)$ is bounded. Thus $z(t)$ is bounded. Consequently, none of the cases (e) and (f) of Lemma 2.4 arises. In the case (b) or (c), $-\infty < \lim_{t\to\infty} z(t) \le 0$. Then

$$\begin{aligned}
0 \ge \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}\big[y(t) + p(t)y(t-\tau)\big] \\
&\ge \limsup_{t\to\infty}\big[y(t) + py(t-\tau)\big] \\
&\ge \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\big(py(t-\tau)\big) \\
&= \limsup_{t\to\infty} y(t) + p\limsup_{t\to\infty} y(t-\tau) \\
&= (1+p)\limsup_{t\to\infty} y(t).
\end{aligned}$$

Hence $\lim_{t\to\infty} y(t) = 0$. In the case (d), $z(t) < \lambda < 0$ for $t \ge t_2 > t_1$. Hence $z(t) > py(t-\tau)$ implies that $y(t) > (\lambda/p)$ for $t \ge t_2$. Consequently, from (4) we obtain

$$G(\lambda/p)\int_{t_3}^{\infty} q(t)\,\mathrm{d}t < \infty, \qquad t_3 > t_2 + \sigma,$$

a contradiction to (7). If $y(t) < 0$ for $t \ge t_0$, then one may proceed as above to obtain $\lim_{t\to\infty} y(t) = 0$ or $\limsup_{t\to\infty} y(t) < 0$. Hence the proof of the theorem is complete. □

THEOREM 3.4. *Suppose that* $-\infty < p_1 \le p(t) \le p_2 < -1$. *If* (H_1), (H_3) *and* (H_{10}) *hold, then every bounded solution of* (1) *oscillates or tends to zero as* $t \to \infty$ *or* $\liminf_{t\to\infty} |y(t)| > 0$.

Proof. If $y(t)$ is a bounded solution of (1) such that $y(t) > 0$ for $t \ge t_0 > 0$, then from (4) it follows that $z(t) > 0$ or < 0 for $t \ge t_1 > t_0 + \rho$, where $z(t)$ is given by (3). If $z(t) > 0$ for $t \ge t_1$, then one of the cases (a)–(d) of Lemma 2.4 holds and we arrive at a contradiction in each case proceeding as in the proof of Theorem 3.3.

Suppose that $z(t) < 0$ for $t \ge t_1$. In the case (b) or (c) of Lemma 2.4, $-\infty < \lim_{t\to\infty} z(t) \le 0$. If $\lim_{t\to\infty} z(t) = 0$, then from the boundedness of $y(t)$ it follows that

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty} [y(t) + p(t)y(t-\tau)] \\ &\le \liminf_{t\to\infty} [y(t) + p_2 y(t-\tau)] \\ &\le \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} (p_2 y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p_2) \limsup_{t\to\infty} y(t)\,. \end{aligned}$$

Since $(1+p_2) < 0$, then $\lim_{t\to\infty} y(t) = 0$. Let $-\infty < \lim_{t\to\infty} z(t) < 0$. Then there exists $\beta < 0$ such that $\beta > z(t) > p_1 y(t-\tau)$. Hence, in the case (b) of Lemma 2.4, it follows from (4) that

$$G(\beta/p_1) \int_{t_3}^{\infty} q(t)\, \mathrm{d}t < \infty\,,$$

which is a contradiction to (7). We may note that (H_{10}) implies (7). However, such a contradiction cannot be obtained in the case (c) of Lemma 2.4. Since $\beta > z(t) > p_1 y(t-\tau)$, then $\liminf_{t\to\infty} y(t) > 0$. Further, in the case (d) one may proceed as in the proof of Theorem 3.3 to get a contradiction. However, either in the case (e) or in the case (f), $\lim_{t\to\infty} z(t) = -\infty$. Since $z(t) > p(t)y(t-\tau)$, then $\lim_{t\to\infty} y(t) = \infty$, a contradiction to the boundedness of $y(t)$.

The case $y(t) < 0$ for $t \ge t_0$ may similarly be dealt with. Thus the theorem is proved. □

THEOREM 3.5. *Let $0 \le p(t) \le p < \infty$. Suppose that* (H_1), (H_2), (H_4)-(H_7) *hold. If*

(H_{11}) $\int_{\sigma}^{\infty} h(t)Q(t)G\big(F^{+}(t-\sigma)\big)\,\mathrm{d}t = \infty = \int_{\sigma}^{\infty} h(t)Q(t)G\big(F^{-}(t-\sigma)\big)\,\mathrm{d}t$, *where* $h(t) = \min\{R^{\alpha}(t), R^{\alpha}(t-\tau)\}$, $\alpha > 1$,

then all solutions of (2) *oscillate.*

Proof. Since (H_1) implies that $R(t) \to 0$ as $t \to \infty$, then $h(t) \to 0$ as $t \to \infty$. Hence (H_{11}) implies that

$$\int_{\sigma}^{\infty} Q(t)G\big(F^{+}(t-\sigma)\big)\,\mathrm{d}t = \infty = \int_{\sigma}^{\infty} Q(t)G\big(F^{-}(t-\sigma)\big)\,\mathrm{d}t \tag{9}$$

Let $y(t)$ be a nonoscillatory solution of (2) such that $y(t) > 0$ for $t \ge t_0 > 0$. Set $w(t) = z(t) - F(t)$ for $t \ge t_0 + \rho$, where $z(t)$ is given by (3). Hence $0 < z(t) \le y(t) + py(t-\tau)$ for $t \ge t_0 + \rho$. Equation (2) may be written as

$$\big(r(t)w''(t)\big)'' = -q(t)G\big(y(t-\sigma)\big) \le 0 \tag{10}$$

for $t \ge t_0 + \rho$. Hence $w(t) > 0$ or < 0 for $t \ge t_1 > t_0 + \rho$. However, $w(t) < 0$ implies that $0 < z(t) < F(t)$, a contradiction to (H_7). Therefore $w(t) > 0$ for $t \ge t_1$. Consequently, Lemma 2.1 holds with $u(t)$ replaced by $w(t)$. Further, $z(t) \ge F^{+}(t)$, $t \ge t_1$. The use of (H_2), (H_4), (H_5) yields, for $t \ge t_2 > t_0 + 2\rho$,

$$\begin{aligned} 0 &\ge \big(r(t)w''(t)\big)'' + G(p)\big(r(t-\tau)w''(t-\tau)\big)'' + \lambda Q(t)G\big(z(t-\sigma)\big) \\ &\ge \big(r(t)w''(t)\big)'' + G(p)\big(r(t-\tau)w''(t-\tau)\big)'' + \lambda Q(t)G\big(F^{+}(t-\sigma)\big). \end{aligned}$$

If one of the cases (a), (b), (d) of Lemma 2.1 holds, then integrating the above inequality we get

$$\int_{t_2+\sigma}^{\infty} Q(t)G\big(F^{+}(t-\sigma)\big)\,\mathrm{d}t < \infty,$$

a contradiction to (9). If the case (c) of Lemma 2.1 holds, then the use of Lemmas 2.2 and 2.3 yields, for $t \ge t_3 > t_2$,

$$k\big(-r(t)w''(t)\big)'tR(t) \le w(t) \le k_2 t$$

and hence

$$\begin{aligned} -\Big[\big((-r(t)w''(t))'\big)^{1-\alpha}\Big]' &= (\alpha-1)\big((-r(t)w''(t))'\big)^{-\alpha}\big(-r(t)w''(t)\big)'' \\ &\ge \lambda(\alpha-1)L^{\alpha}R^{\alpha}(t)q(t)G\big(z(t-\sigma)\big), \end{aligned} \tag{11}$$

where $L = k/k_2$. Thus

$$-\Big[\big((-r(t)w''(t))'\big)^{1-\alpha}\Big]' - G(p)\Big[\big((-r(t-\tau)w''(t-\tau))'\big)^{1-\alpha}\Big]'$$
$$\geq \lambda(\alpha-1)L^{\alpha}h(t)Q(t)G\big(z(t-\sigma)\big)$$
$$\geq \lambda(\alpha-1)L^{\alpha}h(t)Q(t)G\big(F^{+}(t-\sigma)\big).$$

Integrating the above inequality we obtain

$$\int\limits_{t_3+\sigma}^{\infty} h(t)Q(t)G\big(F^{+}(t-\sigma)\big)\,\mathrm{d}t < \infty,$$

a contradiction to (H_{11}). If $y(t) < 0$ for $t \geq t_0$, then we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$ and

$$\big(r(t)\big(x(t)+p(t)x(t-\tau)\big)''\big)'' + q(t)G\big(x(t-\sigma)\big) = \tilde{f}(t),$$

where $\tilde{f}(t) = -f(t)$. If $\tilde{F}(t) = -F(t)$, then $\big(r(t)\tilde{F}''(t)\big)'' = -f(t) = \tilde{f}(t)$ and $\tilde{F}(t)$ changes sign. Further, $\tilde{F}^{+}(t) = F^{-}(t)$ and $\tilde{F}^{-}(t) = F^{+}(t)$. Proceeding as above we obtain a contradiction. Thus the proof of the theorem is complete. □

EXAMPLE. Consider

$$\big[\mathrm{e}^{t}\big(y(t) + (1+\mathrm{e}^{-t})y(t-\pi)\big)''\big]'' + (2+\mathrm{e}^{3t})y\big(t-\tfrac{3\pi}{2}\big) = -\,\mathrm{e}^{3t}\sin t, \qquad (12)$$

$t \geq 1$. Hence $1 < p(t) = 1+\mathrm{e}^{-t} < 2$, $Q(t) = 2+\mathrm{e}^{3(t-\pi)}$ and $R(t) = \mathrm{e}^{-t}$. Taking $\alpha = 2$, we get $h(t) = \mathrm{e}^{-2t}$. Further, $F(t) = (\mathrm{e}^{2t}\cos t)/50$. Since

$$F^{+}\big(t-\tfrac{3\pi}{2}\big) = \begin{cases} 0, & 2n\pi \leq t \leq (2n+1)\pi, \\ -\big(\mathrm{e}^{2t-3\pi}\sin t\big)/50, & (2n+1)\pi \leq t \leq 2(n+1)\pi \end{cases}$$

and

$$F^{-}\big(t-\tfrac{3\pi}{2}\big) = \begin{cases} \big(\mathrm{e}^{2t-3\pi}\sin t\big)/50, & 2n\pi \leq t \leq (2n+1)\pi, \\ 0, & (2n+1)\pi \leq t \leq 2(n+1)\pi \end{cases}$$

for $n = 0,1,2,\dots$, then

$$\int\limits_{3\pi/2}^{\infty} h(t)Q(t)G\big(F^{+}\big(t-\tfrac{3\pi}{2}\big)\big)\,\mathrm{d}t = \int\limits_{3\pi/2}^{\infty} \mathrm{e}^{-2t}\big(2+\mathrm{e}^{3(t-\pi)}\big)F^{+}\big(t-\tfrac{3\pi}{2}\big)\,\mathrm{d}t$$
$$> \frac{-\,\mathrm{e}^{-6\pi}}{50}\sum_{n=1}^{\infty}\int\limits_{(2n+1)\pi}^{2(n+1)\pi} \mathrm{e}^{3t}\sin t\,\mathrm{d}t$$
$$> \frac{\mathrm{e}^{-6\pi}}{500}\sum_{n=1}^{\infty}\mathrm{e}^{6(n+1)\pi} = \infty$$

and

$$\int_{3\pi/2}^{\infty} h(t)Q(t)G\big(F^{-}\big(t-\tfrac{3\pi}{2}\big)\big)\,\mathrm{d}t > \frac{\mathrm{e}^{-6\pi}}{50}\sum_{n=1}^{\infty}\int_{2n\pi}^{(2n+1)\pi} \mathrm{e}^{3t}\sin t\,\mathrm{d}t$$
$$> \frac{\mathrm{e}^{-3\pi}}{500}\sum_{n=1}^{\infty}\mathrm{e}^{6n\pi} = \infty\,.$$

Hence every solution of (12) oscillates by Theorem 3.5. In particular, $y(t) = \cos t$ is an oscillatory solution of the equation. Equation (12) may be put in the following form:

$$y^{(4)}(t) + \big(1+\mathrm{e}^{-t}\big)y^{(4)}(t-\pi) + 2y'''(t) + 2\big(1-\mathrm{e}^{-t}\big)y'''(t-\pi) + y''(t) + \big(1+\mathrm{e}^{-t}\big)y''(t-\pi) + \big(\mathrm{e}^{2t} + 2\,\mathrm{e}^{-t}\big)y\big(t-\tfrac{3\pi}{2}\big) = -\,\mathrm{e}^{2t}\sin t\,. \tag{13}$$

However, (13) cannot be put in the form

$$\big[y(t)+p(t)y(t-\tau)\big]^{(4)} + \sum_{i=1}^{m} Q_i(t)G\big(y(t-\sigma_i)\big) = f(t) \tag{14}$$

because of the presence of the terms $\big(1+\mathrm{e}^{-t}\big)y^{(4)}(t-\pi)$ and $y'''(t)$. Indeed, due to the presence of the term $\big(1+\mathrm{e}^{-t}\big)y^{(4)}(t-\pi)$, we have to take $p(t) = \big(1+\mathrm{e}^{-t}\big)$. Then we note that

$$\big[y(t) + \big(1+\mathrm{e}^{-t}\big)y(t-\pi)\big]^{(4)} = y^{(4)}(t) + \big(1+\mathrm{e}^{-t}\big)y^{(4)}(t-\pi) - 4\,\mathrm{e}^{-t}\,y'''(t-\pi) + 6\,\mathrm{e}^{-t}\,y''(t-\pi) - 4\,\mathrm{e}^{-t}\,y'(t-\pi) + \mathrm{e}^{-t}\,y(t-\pi)\,.$$

If we take for $p(t)$ a term other than $\big(1+\mathrm{e}^{-t}\big)$, then we cannot get $\big(1+\mathrm{e}^{-t}\big)y^{(4)}(t-\pi)$. Hence the results valid for (14) cannot be applied to (12). On the other hand, the results valid for (12) cannot be applied to (14) because we cannot take $r(t) \equiv 1$ in view of the assumption (H_1). Thus the present study is independent of the study in [3], [4].

THEOREM 3.6. *Let $0 \le p(t) \le p < \infty$. Suppose that (H_1), (H_2), (H_4)-(H_6), (H_8) and (H_{11}) hold. Then every solution of (2) oscillates or tends to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 3.5, we obtain $w(t) > 0$ or < 0 for $t \ge t_1 > t_0 + \rho$. If $w(t) > 0$ for $t \ge t_1$, then we obtain a contradiction as in the proof of Theorem 3.5. If $w(t) < 0$ for $t \ge t_1$, then $y(t) \le z(t) < F(t)$ and hence $\limsup\limits_{t\to\infty} y(t) \le 0$ by (H_8). Consequently, $\lim\limits_{t\to\infty} y(t) = 0$. The proof of the theorem is complete. □

Remark. From the assumptions (H_{11}) and (H_8) it follows, respectively, that $F(t)$ changes sign and tends to zero as $t \to \infty$. Equation (2) does not admit a nonoscillatory solution due to Theorem 3.5. Hence Theorem 3.6 implies that only some oscillatory solutions could tend to zero as $t \to \infty$. In the following theorem $F(t) \to 0$ as $t \to \infty$ but need not change sign. Hence equation (2) may admit a nonoscillatory solution which tends to zero as $t \to \infty$.

THEOREM 3.7. *Let* $0 \le p(t) \le p < \infty$. *Let* (H_1), (H_2), (H_4)–(H_6) *and* (H_8) *hold. If*

(H_{12}) $\int\limits_{\sigma}^{\infty} h(t)Q(t)G(|F(t-\sigma)|)\,\mathrm{d}t = \infty$,

then every bounded solution of (2) *oscillates or tends to zero as* $t \to \infty$.

P r o o f . Proceeding as in the proof of Theorem 3.5, we obtain (10). Hence $w(t) > 0$ or < 0 for $t \ge t_1 > t_0 + \rho$. Let $w(t) > 0$. Thus $z(t) > F(t)$, $t \ge t_1$. Let $F(t) \ge 0$ for $t \ge t_2 > t_1$. The use of (H_2), (H_4), (H_5) yields

$$0 \ge \big(r(t)w''(t)\big)'' + G(p)\big(r(t-\tau)w''(t-\tau)\big)'' + \lambda Q(t)G\big(F(t-\sigma)\big)$$

for $t \ge t_3 > t_2 + \rho$. If one of the cases (a), (b), (d) of Lemma 2.1 holds, then

$$\int\limits_{t_3+\sigma}^{\infty} Q(t)G\big(F(t-\sigma)\big)\,\mathrm{d}t < \infty,$$

which is a contradiction to (H_{12}) because (H_{12}) implies that

$$\int\limits_{\sigma}^{\infty} Q(t)G\big(F(t-\sigma)\big)\,\mathrm{d}t = \infty.$$

If the case (c) of Lemma 2.1 holds, then we may proceed as in the proof of Theorem 3.5 to obtain

$$\int\limits_{t_3+\sigma}^{\infty} h(t)Q(t)G\big(F(t-\sigma)\big)\,\mathrm{d}t < \infty,$$

a contradiction to (H_{12}). Hence $w(t) < 0$ for $t \ge t_1$. Thus $y(t) < F(t)$. Consequently, $\liminf\limits_{t\to\infty} y(t) = 0$. In each of the cases (b) and (c) of Lemma 2.4, $\lim\limits_{t\to\infty} w(t)$ exists and hence $\lim\limits_{t\to\infty} z(t)$ exists. Since $y(t)$ is bounded, then $w(t)$ is bounded. In the case (d) of Lemma 2.4, $\lim\limits_{t\to\infty} w(t)$ exists and hence $\lim\limits_{t\to\infty} z(t)$ exists. The cases (e) and (f) of Lemma 2.4 do not hold since $w(t)$ is bounded. From Lemma 2.5 it follows that $\lim\limits_{t\to\infty} z(t) = 0$. Since $z(t) > y(t)$, then $\lim\limits_{t\to\infty} y(t) = 0$. Suppose that $F(t) \leq 0$ for $t \geq t_2$. In this case, $w(t) < 0$ implies that $0 < z(t) < F(t)$, a contradiction. Hence $w(t) > 0$ for $t \geq t_2$. Since $w(t)$ is bounded, the case (a) of Lemma 2.1 does not hold. Further $\lim_{t\to\infty} w(t)$ exists in each of the cases (b), (c) and (d). From (10) it follows that

$$\int_{t_2}^{\infty} q(t)G\big(y(t-\sigma)\big)\,\mathrm{d}t < \infty$$

in each of the cases (b) and (d). Hence $\liminf_{t\to\infty} y(t) = 0$ because (H_{12}) implies that $\int_{\sigma}^{\infty} q(t)\,\mathrm{d}t = \infty$. In the case (c) of Lemma 2.1, we obtain (11), which yields

$$\int_{t_2}^{\infty} h(t)q(t)G\big(y(t-\sigma)\big)\,\mathrm{d}t < \infty .$$

Hence $\liminf_{t\to\infty} y(t) = 0$; otherwise, $\int_{t_2}^{\infty} h(t)q(t)\,\mathrm{d}t < \infty$, a contradiction to (H_{12}). From Lemma 2.5 it follows that $\lim_{t\to\infty} z(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$. The case $y(t) < 0$ for $t \geq t_0$ is similarly dealt with. Thus the theorem is proved. □

THEOREM 3.8. *Let $-1 < p \leq p(t) \leq 0$. Suppose that (H_1) and (H_7') hold. If*

(H_{13}) $\int_{\sigma}^{\infty} R^{\alpha}(t)q(t)G\big(F^{+}(t-\sigma)\big)\,\mathrm{d}t = \infty = \int_{\sigma}^{\infty} q(t)G\big(F^{-}(t+\tau-\sigma)\big)\,\mathrm{d}t$

and

(H_{14}) $\int_{\sigma}^{\infty} R^{\alpha}(t)q(t)G\big(-F^{-}(t-\sigma)\big)\,\mathrm{d}t = -\infty = \int_{\sigma}^{\infty} q(t)G\big(-F^{+}(t+\tau-\sigma)\big)\,\mathrm{d}t,$

then a solution $y(t)$ of (2) *oscillates or $\liminf_{t\to\infty}\big(y(t) - y(t-\tau)\big) < 0$.*

Proof. Proceeding as in the proof of Theorem 3.5, we obtain $w(t) > 0$ or < 0 for $t \geq t_1 > t_0 + \rho$. If $w(t) > 0$, then $y(t) > F(t)$ and hence $y(t) \geq F^{+}(t)$, $t \geq t_1$. In each of the cases (a), (b) and (d) of Lemma 2.1, we obtain from (10) that

$$\int_{t_1+\sigma}^{\infty} q(t)G\big(F^{+}(t-\sigma)\big)\,\mathrm{d}t < \infty ,$$

a contradiction. In the case (c) of Lemma 2.1, we obtain from (11) that

$$\int_{t_2+\sigma}^{\infty} R^{\alpha}(t)q(t)G\big(F^{+}(t-\sigma)\big)\,\mathrm{d}t < \infty ,$$

a contradiction. Hence $w(t) < 0$ for $t \geq t_1$. We claim that $y(t)$ is bounded. If not, then there exists an increasing sequence $\{\sigma_n\}_{n=1}^{\infty}$ such that $\sigma_n \to \infty$ and $y(\sigma_n) \to \infty$ as $n \to \infty$ and $y(\sigma_n) = \max\{y(t) : t_1 \leq t \leq \sigma_n\}$. Hence

$$\begin{aligned} w(\sigma_n) &\geq y(\sigma_n) + py(\sigma_n - \tau) - F(\sigma_n) \\ &\geq (1+p)y(\sigma_n) - F(\sigma_n) \end{aligned}$$

implies that $w(\sigma_n) > 0$ for large n because $1 + p > 0$ and $F(t)$ is bounded. This is a contradiction. Hence $w(t)$ is bounded. Thus none of the cases (e) and (f) of Lemma 2.4 holds. Since $w(t) < 0$, then $y(t) > F^-(t+\tau)$. Hence, in each of the cases (b) and (d) of Lemma 2.4, we obtain from (10)

$$\int_{t_1+\sigma}^{\infty} q(t)G\big(F^-(t+\tau-\sigma)\big)\,\mathrm{d}t < \infty,$$

a contradiction. Suppose that the case (c) of Lemma 2.4 holds. None of the above considerations is possible in this case. However, $w(t) < 0$ implies that $y(t) - y(t-\tau) < F(t)$. Hence $\liminf_{t\to\infty}\big(y(t) - y(t-\tau)\big) \leq \liminf_{t\to\infty} F(t) < 0$. If $y(t) < 0$ for $t \geq t_0$, then one may proceed as above. Thus the proof of the theorem is complete. □

THEOREM 3.9. *Suppose that all the conditions of Theorem* 3.8 *are satisfied except* (H_7'), *which is replaced by* (H_8). *Then every solution of* (2) *oscillates or tends to zero as* $t \to \infty$.

P r o o f. If $w(t) > 0$, then a contradiction is obtained in each of the cases (a)–(d) of Lemma 2.1. Hence $w(t) < 0$ for $t \geq t_1 > t_0 + \rho$, that is, $z(t) < F(t)$. Since $z(t) \geq y(t) + py(t-\tau)$, $(1+p) > 0$ and $\limsup_{t\to\infty} z(t) \leq 0$, then $\lim_{t\to\infty} y(t) = 0$. Hence the proof is complete. □

THEOREM 3.10. *Let* $-\infty < p \leq p(t) \leq 0$. *If* ($H_1$), ($H_3$), ($H_7'$), ($H_{13}$) *and* ($H_{14}$) *hold, then a solution* $y(t)$ *of* (2) *oscillates or* $|y(t)| \to \infty$ *as* $t \to \infty$ *or* $\liminf_{t\to\infty}\big(y(t) + py(t-\tau)\big) < 0$.

The proof is similar to that of Theorem 3.8 and hence is omitted.

THEOREM 3.11. *Let* $-1 < p \leq p(t) \leq 0$. *Suppose that* (H_1), (H_3) *and* (H_8) *hold. If*

(H_{15}) $\int_{\sigma}^{\infty} q(t)R^{\alpha}(t)G\big(|F(t-\sigma)|\big)\,\mathrm{d}t = \infty, \ \alpha > 1,$

then every solution of (2) *oscillates or tends to zero or tends to* $\pm\infty$ *as* $t \to \infty$.

P r o o f. Proceeding as in the proof of Theorem 3.5, we get $w(t) > 0$ or < 0 for $t \geq t_1 > t_0 + \rho$. Let $w(t) > 0$ for $t \geq t_1$. Hence $y(t) \geq F(t)$. From (H_{15}) it follows that

$$\int_{\sigma}^{\infty} q(t)G\big(|F(t-\sigma)|\big)\ \mathrm{d}t = \infty\,, \qquad \int_{\sigma}^{\infty} q(t)R^{\alpha}(t)\ \mathrm{d}t = \infty \quad \text{and} \quad \int_{\sigma}^{\infty} q(t)\ \mathrm{d}t = \infty$$

because $F(t) \to 0$ and $R^{\alpha}(t) \to 0$ as $t \to \infty$. Let $F(t) \geq 0$ for $t \geq t_2 > t_1$. In each of the cases (a), (b) and (d) of Lemma 2.1, it follows from (10) that

$$\int_{t_2+\sigma}^{\infty} q(t)G\big(F(t-\sigma)\big)\ \mathrm{d}t < \infty\,,$$

which is a contradiction. In the case (c) of Lemma 2.1, we obtain from (11) that

$$\int_{t_2+\sigma}^{\infty} q(t)R^{\alpha}(t)G\big(F(t-\sigma)\big)\ \mathrm{d}t < \infty\,,$$

a contradiction to (H_{15}). Let $F(t) \leq 0$ for $t \geq t_2 > t_1$. If $w(t) > 0$ for $t \geq t_1$, then $\lim\limits_{t\to\infty} w(t) = \infty$ in the case (a) of Lemma 2.1. Hence $\lim\limits_{t\to\infty} z(t) = \infty$. Since $y(t) > z(t)$, then $\lim\limits_{t\to\infty} y(t) = \infty$. In each of the cases (b) and (c) of Lemma 2.1, $0 < \beta \leq \infty$, where $\beta = \lim\limits_{t\to\infty} w(t)$. If $\beta = \infty$, then $\lim\limits_{t\to\infty} y(t) = \infty$. If $0 < \beta < \infty$, then $\lim\limits_{t\to\infty} z(t) = \beta$. From (10) we get

$$\int_{t_2+\sigma}^{\infty} q(t)G\big(y(t-\sigma)\big)\ \mathrm{d}t < \infty\,. \tag{15}$$

in the case (b). Further, in the case (c), (11) yields

$$\int_{t_2+\sigma}^{\infty} q(t)R^{\alpha}(t)G\big(y(t-\sigma)\big)\ \mathrm{d}t < \infty\,.$$

Hence $\liminf\limits_{t\to\infty} y(t) = 0$. From Lemma 2.5 it follows that $\beta = 0$, a contradiction. In the case (d) of Lemma 2.1, $\lim\limits_{t\to\infty} w(t)$ exists and (15) holds. Then $\lim\limits_{t\to\infty} z(t) = 0$. Since $z(t) \geq y(t) + py(t-\tau)$ and $(1+p) > 0$, then $y(t)$ is bounded and hence $\limsup\limits_{t\to\infty} y(t) = 0$. Hence $\lim\limits_{t\to\infty} y(t) = 0$. Hence $w(t) < 0$ for $t \geq t_1$. The following analysis holds for $F(t) \geq 0$ or ≤ 0. As in the proof of Theorem 3.8, we may show that $y(t)$ is bounded and hence $w(t)$ is bounded. This implies that the cases (e) and (f) of Lemma 2.4 do not hold. In each of the cases (b), (c) and

(d) of Lemma 2.4, we proceed as follows: Since $w(t) < 0$, then $z(t) < F(t)$ and hence $\limsup_{t\to\infty} z(t) \le 0$. Thus

$$0 \ge \limsup_{t\to\infty} y(t) + py(t-\tau) \ge \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\big(py(t-\tau)\big)$$
$$= (1+p)\limsup_{t\to\infty} y(t)\,.$$

Since $(1+p) > 0$, then $\lim_{t\to\infty} y(t) = 0$. The proof for the case $y(t) < 0$ for $t \ge t_0$ is similar. Thus the theorem is proved. □

Example. Consider

$$\big[\mathrm{e}^t\big(y(t) + \mathrm{e}^{-1}(\mathrm{e}^{-t} - 1)y(t-1)\big)''\big]'' + 5\,\mathrm{e}^{9t}\,y^3(t-2) = \mathrm{e}^{-t}\,, \qquad t \ge 1\,. \tag{16}$$

If $F(t) = \frac{1}{4}\,\mathrm{e}^{-2t}$, then $\big(\mathrm{e}^t\,F''(t)\big)'' = \mathrm{e}^{-t}$. Further,

$$R(t) = \int_t^\infty \mathrm{e}^{-s}(s-t)\ \mathrm{d}s = \mathrm{e}^{-t}$$

implies, for $\alpha = 2$, that

$$\int_2^\infty q(t)R^\alpha(t)G\big(|F(t-\sigma)|\big)\ \mathrm{d}t = \frac{5\,\mathrm{e}^{12}}{64}\int_2^\infty \mathrm{e}^t\ \mathrm{d}t = \infty\,.$$

From Theorem 3.11 it follows that every solution of (16) oscillates or tends to zero as $t \to \infty$. Equation (16) may be written as

$$y^{(4)}(t) + \big(\mathrm{e}^{-(t+1)} - \mathrm{e}^{-1}\big)y^{(4)}(t) + 2y'''(t) - 2\big(\mathrm{e}^{-1} + \mathrm{e}^{-(t+1)}\big)y'''(t-1)$$
$$+ y''(t) + \big(\mathrm{e}^{-(t+1)} - \mathrm{e}^{-1}\big)y''(t-1) + 5\,\mathrm{e}^{8t}\,y^3(t-2) = \mathrm{e}^{-2t}\ .$$

The explanation given in the example following Theorem 3.5 also holds here.

Corollary 3.12. *Suppose that the conditions of Theorem* 3.10 *hold. Then every bounded solution of* (2) *oscillates or tends to zero as* $t \to \infty$.

Remark. Theorems 3.8, 3.11 and Corollary 3.12 do not hold for homogeneous equation (1).

4. Necessary conditions for oscillation

In this section we obtain conditions for the existence of bounded positive solutions of (2).

THEOREM 4.1. *Let $0 \le p(t) \le p < 1$. Suppose that G is Lipschitzian on intervals of the form $[a,b]$, $0 < a < b < \infty$ and $F(t)$ changes sign such that $-(1-p)/8 \le F(t) \le (1-p)/2$, where F is same as in* (H_7). *If* (H_1) *holds and*

(H_{16}) $\int\limits_0^\infty tq(t)\,\mathrm{d}t < \infty$,

then (2) *admits a positive bounded solution.*

Proof. Let t_0 be sufficiently large such that

$$L\int\limits_{t_0}^{\infty} tq(t)\,\mathrm{d}t < \frac{1}{2}(1-p) \qquad \text{and} \qquad \int\limits_{t_0}^{\infty}\frac{t}{r(t)}\,\mathrm{d}t < \frac{1}{2},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $\left[\frac{1-p}{8}, 1\right]$. Let $X = BC([t_0,\infty),\mathbb{R})$ be the Banach space of all real-valued bounded continuous functions on $[t_0,\infty)$ with sup norm. Let $S = \left\{x \in X : \frac{1-p}{8} \le x(t) \le 1,\ t \ge t_0\right\}$. Hence S is a complete metric space with the metric induced by the norm. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(t_0+\rho), & t \in [t_0, t_0+\rho],\\[2mm] -p(t)y(t-\tau) + \frac{1}{2}(1+p) + F(t) & \\ \quad -\int\limits_t^\infty \left(\frac{s-t}{r(s)}\int\limits_s^\infty (u-s)q(u)G\big(y(u-\sigma)\big)\,\mathrm{d}u\right)\mathrm{d}s, & t \ge t_0+\rho. \end{cases}$$

Hence, for $t \ge t_0$, $Ty(t) \le \frac{1}{2}(1+p) + \frac{1}{2}(1-p) = 1$ and

$$Ty(t) \ge -p + \frac{1}{2}(1+p) - \frac{1}{8}(1-p) - \frac{1}{4}(1-p) = \frac{1}{8}(1-p),$$

because, for $t \ge t_0$,

$$\begin{aligned} &\int\limits_t^\infty \left(\frac{s-t}{r(s)}\int\limits_s^\infty (u-s)q(u)G\big(y(u-\sigma)\big)\,\mathrm{d}u\right)\mathrm{d}s \\ &\qquad \le G(1)\int\limits_t^\infty \frac{s}{r(s)}\left(\int\limits_s^\infty uq(u)\,\mathrm{d}u\right)\mathrm{d}s \\ &\qquad \le G(1)\left(\int\limits_{t_0}^\infty tq(t)\,\mathrm{d}t\right)\left(\int\limits_{t_0}^\infty \frac{t}{r(t)}\,\mathrm{d}t\right) \\ &\qquad \le \frac{1}{4}(1-p). \end{aligned}$$

Thus $T\colon S \to S$. Further, for $x, y, \in S$,

$$\begin{aligned}\|Tx(t) - Ty(t)\| &\le p\|x-y\| + \frac{1}{4}(1-p)\|x-y\| \\ &\le \frac{(3p+1)}{4}\|x-y\|\end{aligned}$$

for $t \ge t_0$ implies that T is a contraction. Consequently, T has a unique fixed point y in S. It is the required solution of (2). Thus the theorem is proved. □

THEOREM 4.2. *Let* $-1 < p \le p(t) \le 0$. *If* (H_1) *holds,* G *is Lipschitzian on intervals of the form* $[a,b]$, $0 < a < b < \infty$, $F(t)$ *changes sign such that* $-\frac{1}{8}(1+p) \le F(t) \le \frac{1}{2}(1+p)$ *and* $\int\limits_0^\infty tq(t)\,\mathrm{d}t < \infty$, *then* (2) *admits a positive bounded solution.*

Proof. We choose t_0 sufficiently large so that

$$L\int\limits_{t_0}^{\infty} tq(t)\,\mathrm{d}t < \frac{1}{2}(1+p) \qquad \text{and} \qquad \int\limits_{t_0}^{\infty}\frac{t}{r(t)}\,\mathrm{d}t < \frac{1}{2},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $\left[\frac{1+p}{8}, 1\right]$. The rest of the proof is similar to that of Theorem 4.1. □

Two similar theorems may be obtained in other ranges of $p(t)$.

THEOREM 4.3. *Let* $0 \le p(t) \le p < 1$. *Suppose that* G *satisfies Lipschitz property on intervals of the form* $[a,b]$, $0 < a < b$. *If* (H_1), (H_8) *and* (H_{16}) *hold, then* (2) *admits a positive bounded solution.*

The proof is similar to that of Theorem 4.1. However, there are some changes in the setting. Let t_0 be sufficiently large so that

$$|F(t)| < \frac{1-p}{10} \quad \text{for } t \ge t_0, \qquad \int\limits_{t_0}^{\infty}\frac{t}{r(t)}\,\mathrm{d}t < \frac{1}{2} \quad \text{and} \quad L\int\limits_{t_0}^{\infty} tq(t)\,\mathrm{d}t < \frac{1-p}{10},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $\left[\frac{1-p}{20}, 1\right]$. For $y \in S = \left\{x \in BC([t_0,\infty),\mathbb{R}) : \frac{1-p}{20} \le x(t) \le 1\right\}$, we define Ty as in Theorem 4.1, where the term $\frac{1}{2}(1+p)$ is replaced by $\frac{1}{5}(1+4p)$.

5. Summary

In our results, no superlinearity or sublinearty conditions are imposed on G. However, if $p(t) \leq 0$, then the results are not satisfactory. Extra restriction on G could help in this case. Equations (1) and (2) are studied under the assumption $\int_0^\infty \frac{t}{r(t)}\,\mathrm{d}t = \infty$ in a separate paper. It would be interesting to study neutral differential equations with quasi-derivatives of the form

$$\big(r_3(t)\big(r_2(t)\big(r_1(t)\big(y(t)+p(t)y(t-\tau)\big)'\big)'\big)'\big)' + q(t)G\big(y(t-\sigma)\big) = f(t)\,.$$

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