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ON OSCILLATION CRITERIA FOR FORCED NONLINEAR HIGHER ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, sufficient conditions are obtained for oscillation of all solutions of neutral differential equations of the form

$$\big[y(t)-p(t)y(t-\tau)\big]^{(n)}+\sum_{i=1}^{m}Q_i(t)G\big(y(t-\sigma_i)\big)=f(t) \tag{$*$}$$

and

$$\big[y(t)-p(t)y(t-\tau)\big]^{(n)}+\sum_{i=1}^{m}Q_i(t)G\big(y(t-\sigma_i)\big)=0 \tag{$**$}$$

for different ranges of $p(t)$, where $n \ge 2$. For $(*)$, one of the conditions states that $F(t)$ changes sign finitely, where $F \in C^{(n)}\big([0,\infty),\mathbb{R}\big)$ with $F^{(n)}(t)=f(t)$. In results concerning $(**)$, the nonlinearity of G, the nature of n and the range of $p(t)$ are closely related.

1. Introduction

In a recent paper [10], necessary and sufficient conditions are obtained for every bounded solution of

$$\big[y(t)-p(t)y(t-\tau)\big]^{(n)}+\sum_{i=1}^{m}Q_i(t)G\big(y(t-\sigma_i)\big)=f(t)\,,\qquad t\ge 0\,, \tag{1}$$

to oscillate or tend to zero as $t\to\infty$ for different ranges of $p(t)$. It is shown there, under some stronger conditions, that every solution of (1) oscillates or tends to zero as $t\to\infty$. In [10], a particular class of superlinear G is considered. However, similar results are obtained in [11] for superlinear/sublinear G under

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stronger conditions. In [10], [11], one of the conditions states the existence of a function $F \in C^{(n)}([0,\infty),\mathbb{R})$ such that $F^{(n)}(t) = f(t)$ and $\lim_{t\to\infty} F(t) = 0$. In this paper, we will not assume that $\lim_{t\to\infty} F(t) = 0$. However, $F(t)$ is allowed to change sign finitely. (This condition is made precise in the following.) As this condition is not applicable to the associated homogeneous equation

$$[y(t) - p(t)y(t-\tau)]^{(n)} + \sum_{i=1}^{m} Q_i(t)G(y(t-\sigma_i)) = 0\,, \qquad t \ge 0\,, \tag{2}$$

it is studied separately. In this paper, we are able to show that every solution of (1)/(2) oscillates under reasonably suitable conditions. While considering (2), different types of superlinear/sublinear G are taken. In (1)/(2), $n \ge 2$, $p, f \in C([0,\infty),\mathbb{R})$, $Q_i \in C([0,\infty),[0,\infty))$, $1 \le i \le m$, $G \in C(\mathbb{R},\mathbb{R})$ is nondecreasing and $uG(u) > 0$ for $u \ne 0$, $\tau > 0$ and $\sigma_i > 0$, $1 \le i \le m$.

The oscillatory and asymptotic behaviour of solutions of (1) with $G(u) = u$ are investigated in [8] under the assumption that f is a very rapidly oscillating function. In [6], [7], equation (1) is studied under the assumption that f is small in some sense. Equation (2) is considered in [2], [13] under strong assumptions on Q_i. Moreover, in most of these works, $p(t)$ lies in the range $-1 < p(t) \le 0$ or $0 \le p(t) < 1$.

By a *solution* of (1) we mean a real-valued continuous function y on $[T_y-\rho,\infty)$, for some $T_y \ge 0$, such that $y(t)-p(t)y(t-\tau)$ is n-times continuously differentiable and (1) is satisfied for $t \ge T_y$, where $\rho = \max\{\tau, \sigma_i :\ 1 \le i \le m\}$. A solution of (1) is said to be *oscillatory* if it has arbitrarily large zeros. It is called *nonoscillatory* otherwise.

The nonhomogeneous equation is considered in the second section and the homogeneous equation is studied in Section 3. We need the following assumptions in the sequel.

(H_1) There exists $F \in C^{(n)}([0,\infty),\mathbb{R})$ such that $F^{(n)}(t) = f(t)$ and $F(t)$ changes sign with $-\infty < \lambda = \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) = \mu < \infty$.

(H_1') There exists $F \in C^{(n)}([0,\infty),\mathbb{R})$ such that $F^{(n)}(t) = f(t)$ and $F(t)$ changes sign.

(H_2) For $u > 0$ and $\nu > 0$, there exists a $\delta > 0$ such that
$$G(u) + G(\nu) \ge \delta G(u+\nu)\,.$$

(H_2') For $u < 0$ and $\nu < 0$, there exists a $\delta > 0$ such that
$$G(u) + G(\nu) \le \delta G(u+\nu)\,.$$

(H_3) For $u > 0$ and $\nu > 0$, $G(u\nu) \le G(u)G(\nu)$.

(H_4) $G(-u) = -G(u)$, $u \in \mathbb{R}$.

(H_5) $\liminf\limits_{|u|\to\infty}\bigl(G(u)/u\bigr) > \alpha > 0$.

(H_6) $\liminf\limits_{|u|\to 0}\bigl(G(u)/u\bigr) > \beta > 0$.

(H_7) (i) $\int\limits_0^k \frac{du}{G(u)} < \infty$,

(ii) $\int\limits_0^{-k} \frac{du}{G(u)} < \infty$,

for every $k > 0$.

(H_8) $\int\limits_{\pm k}^{\pm\infty} \frac{du}{G(u)} < \infty$ for every $k > 0$.

(H_9) $\int\limits_0^{\infty}\Bigl(\sum\limits_{i=1}^{m} Q_i(t)\Bigr)\,dt = \infty$.

(H_{10}) $\int\limits_{\tau}^{\infty}\Bigl(\sum\limits_{i=1}^{m} Q_i^*(t)\Bigr)\,dt = \infty$,

where $Q_i^*(t) = \min\bigl\{Q_i(t), Q_i(t-\tau)\bigr\}$, $t \geq \tau$ and $1 \leq i \leq m$.

Remark.

(i) (H_1) implies that $F(t)$ is bounded.

(ii) The possibility that $\liminf\limits_{t\to\infty} F(t) = -\infty$ or $\limsup\limits_{t\to\infty} F(t) = \infty$ is included in (H_1').

(iii) (H_2) and (H_4) imply (H_2').

Remark. The prototype of G satisfying (H_2) and (H_4) is

$$G(u) = \bigl(a + b|u|^{\lambda}\bigr)|u|^{\mu}\operatorname{sgn} u\,,$$

where $a \geq 0$, $b \geq 0$, $\lambda \geq 0$ and $\mu \geq 0$ such that $a^2 + b^2 \neq 0$. It satisfies (H_3) if $a \geq 1$ and $b \geq 1$. Moreover, $G \in C(\mathbb{R}, \mathbb{R})$, $uG(u) > 0$ for $u \neq 0$ and $G(u)$ is nondecreasing. If $\lambda + \mu \geq 1$ and $b > 0$, then G satisfies (H_5). On the other hand, (H_6) holds if $\lambda + \mu \leq 1$ and $b > 0$. Further, $\lambda + \mu < 1$ and $b > 0$ imply that (H_7) holds, and $\lambda + \mu > 1$ and $b > 0$ imply that (H_8) holds because we may write $G(u) \geq b|u|^{\lambda+\mu}\operatorname{sgn} u$. If $G^1(u) = |u|^{\gamma}\operatorname{sgn} u$, where $\gamma > 0$, then $G^1 \in C(\mathbb{R}, \mathbb{R})$ with $uG^1(u) > 0$ for $u \neq 0$ and it is nondecreasing. Further, G^1 satisfies (H_2)–(H_4). It satisfies (H_5) if $\gamma \geq 1$ and (H_8) if $\gamma > 1$. Further, it satisfies (H_6) if $\gamma \leq 1$ and (H_7) if $\gamma < 1$.

2. Oscillation of nonhomogeneous equation

The oscillatory behaviour of solutions of equation (1) is studied in this section.

THEOREM 2.1. *Suppose that* $0 \le p(t) \le 1$ *and* (H_1) *holds. If*

(H_{11}) $\int\limits_{\rho}^{\infty} \Big[\sum\limits_{i=1}^{m} Q_i(t) G\big(F^+(t-\sigma_i)\big)\Big] \, \mathrm{d}t = \infty$,

(H_{12}) $\int\limits_{\rho}^{\infty} \Big[\sum\limits_{i=1}^{m} Q_i(t) G\big(F^-(t+\tau-\sigma_i)\big)\Big] \, \mathrm{d}t = \infty$,

(H_{13}) $\int\limits_{\rho}^{\infty} \Big[\sum\limits_{i=1}^{m} Q_i(t) G\big(-F^+(t+\tau-\sigma_i)\big)\Big] \, \mathrm{d}t = -\infty$
and
$\int\limits_{\rho}^{\infty} \Big[\sum\limits_{i=1}^{m} Q_i(t) G\big(-F^-(t-\sigma_i)\big)\Big] \, \mathrm{d}t = -\infty$,

where $F^+(t) = \max\{F(t), 0\}$ *and* $F^-(t) = \max\{-F(t), 0\}$, *then every solution of equation* (1) *oscillates.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1). Then there exists a $t_0 > T_y$ such that $y(t) > 0$ or $y(t) < 0$ for $t \ge t_0$. Let $y(t) > 0$ for $t \ge t_0$. Setting

$$w(t) = y(t) - p(t)y(t-\tau) - F(t) \tag{3}$$

for $t \ge t_0 + \tau$, we obtain

$$w^{(n)}(t) = -\sum_{i=1}^{m} Q_i(t) G\big(y(t-\sigma_i)\big) \le 0 \tag{4}$$

for $t \ge t_0 + \rho$. Hence the functions $w, w', \dots, w^{(n-1)}$ are monotonic and of constant sign for $t \ge t_1 > t_0 + \rho$. We consider two possibilities, viz., either $\lim\limits_{t\to\infty} w^{(n-1)}(t) = -\infty$ or $\lim\limits_{t\to\infty} w^{(n-1)}(t) = \ell \in \mathbb{R}$. Suppose the former holds. Hence $\lim\limits_{t\to\infty} w(t) = -\infty$. For any $L > \mu$ and for any ε, $0 < \varepsilon < L - \mu$, there exists a $t_2 > t_1$ such that $F(t) < \mu + \varepsilon$ and $w(t) < -L$ for $t \ge t_2$. Hence $y(t) < y(t-\tau)$ for $t \ge t_2$. Thus $y(t)$ is bounded. Consequently, $w(t)$ is bounded, a contradiction. If $\lim\limits_{t\to\infty} w^{(n-1)}(t) = \ell$, then from (4) we obtain

$$\int\limits_{t_1}^{\infty} \Big[\sum_{i=1}^{m} Q_i(t) G\big(y(t-\sigma_i)\big)\Big] \, \mathrm{d}t < \infty\,. \tag{5}$$

Since $w(t)$ is monotonic, then $w(t) > 0$ or $w(t) < 0$ for $t \geq t_3 > t_1$. Let $w(t) > 0$ for $t \geq t_3$. Then $y(t) \geq F^+(t)$ for $t \geq t_3$ and hence

$$\int_{t_3+\rho}^{\infty} \left[\sum_{i=1}^{m} Q_i(t) G\big(F^+(t-\sigma_i)\big)\right] \mathrm{d}t \leq \int_{t_3+\rho}^{\infty} \left[\sum_{i=1}^{m} Q_i(t) G\big(y(t-\sigma_i)\big)\right] \mathrm{d}t < \infty$$

by (5), which is a contradiction to (H_{11}). If $w(t) < 0$ for $t \geq t_3$, then $y(t) \geq F^-(t+\tau)$ and hence

$$\int_{t_3+\rho}^{\infty} \left[\sum_{i=1}^{m} Q_i(t) G\big(F^-(t+\tau-\sigma_i)\big)\right] \mathrm{d}t < \infty$$

by (5), which contradicts (H_{12}).

If $y(t) < 0$ for $t \geq t_0$, then we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$ and

$$\big[x(t) - p(t)x(t-\tau)\big]^{(n)} + \sum_{i=1}^{m} Q_i(t) H\big(x(t-\sigma_i)\big) = \tilde{f}(t)\,, \qquad t \geq 0\,,$$

where $\tilde{f}(t) = -f(t)$ and $H(u) = -G(-u)$. If $\tilde{F}(t) = -F(t)$, then $\tilde{F}^{(n)}(t) = \tilde{f}(t)$, $\tilde{F}(t)$ changes sign, $-\infty < -\mu = \liminf_{t\to\infty} \tilde{F}(t) < 0 < \limsup_{t\to\infty} \tilde{F}(t) = -\lambda < \infty$, $\tilde{F}^+(t) = F^-(t)$ and $\tilde{F}^-(t) = F^+(t)$. Then proceeding as above, we obtain a contradiction. Thus the theorem is proved. □

THEOREM 2.2. *Let $0 \leq p(t) \leq p$, where p is a constant. Let (H_1), (H_3), (H_4), (H_{11}) and (H_{12}) hold. If*

$$\int_{\rho}^{\infty} \left[\sum_{i=1}^{m} Q_i(t) G\big(F^+(t+\tau-\sigma_i)\big)\right] \mathrm{d}t = \infty$$

and

$$\int_{\rho}^{\infty} \left[\sum_{i=1}^{m} Q_i(t) G\big(F^-(t-\sigma_i)\big)\right] \mathrm{d}t = \infty\,,$$

then every bounded solution of equation (1) *oscillates and every unbounded solution of equation* (1) *oscillates or tends to $\pm\infty$ as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1) such that $y(t) > 0$ or $y(t) < 0$ for $t \geq t_0 > T_y$. Suppose that $y(t) > 0$ for $t \geq t_0$. The case $y(t) < 0$ for $t \geq t_0$ may similarly be dealt with. Setting $w(t)$ as in (3) and proceeding as in the proof of Theorem 2.1, we obtain either $\lim_{t\to\infty} w^{(n-1)}(t) = \ell \in \mathbb{R}$, or $\lim_{t\to\infty} w^{(n-1)}(t) = -\infty$. Suppose that the former holds. Then (5) is true. If $w(t) > 0$ for $t \ge t_1 > t_0 + \rho$, then $y(t) \ge F^+(t)$. Proceeding as in the proof of Theorem 2.1, a contradiction to (H_{11}) is obtained due to (5). If $w(t) < 0$ for $t \ge t_1$, then $py(t) > F^-(t+\tau)$ and hence $G(p)G\big(y(t-\sigma_i)\big) \ge G\big(F^-(t+\tau-\sigma_i)\big)$ for $t \ge t_2 > t_1 + \rho$ due to (H_3). This contradicts (H_{12}) in view of (5). If $\lim_{t\to\infty} w^{(n-1)}(t) = -\infty$, then $\lim_{t\to\infty} w(t) = -\infty$ and hence

$$w(t) > -p(t)y(t-\tau) - F(t) > -py(t-\tau) - F(t)$$

implies that $\liminf_{t\to\infty} y(t) = \infty$ due to (H_1). If $y(t)$ is a bounded solution of (1), then a contradiction is obtained; otherwise, $\lim_{t\to\infty} y(t) = \infty$. Thus the proof of the theorem is complete. □

Remark. From Theorems 2.1 and 2.2 it follows that $p > 1$ changes the nature of the unbounded solutions of (1).

THEOREM 2.3. *Let $p(t)$ be monotonic decreasing and $-p \le p(t) \le 0$, where $p > 0$ is a constant. Suppose that (H_1'), (H_2)–(H_4) hold. If*

(H_{14}) $\displaystyle\int_\rho^\infty \Big[\sum_{i=1}^m Q_i^*(t)G\big(F^+(t-\sigma_i)\big)\Big]\,dt = \infty$

and

$$\int_\rho^\infty \Big[\sum_{i=1}^m Q_i^*(t)G\big(F^-(t-\sigma_i)\big)\Big]\,dt = \infty,$$

where $Q_i^*(t)$ is same as in (H_{10}),

then every solution of equation (1) *oscillates.*

Proof. If possible, let $y(t)$ be a nonoscillatory solution of (1) with $y(t) > 0$ or $y(t) < 0$ for $t \ge t_0 > T_y$. Let $y(t) > 0$ for $t \ge t_0$. Setting

$$z(t) = y(t) - p(t)y(t-\tau) \tag{6}$$

and $w(t)$ as in (3), we obtain $z(t) > 0$ for $t \ge t_0 + \tau$. Proceeding as in the proof of Theorem 2.1 one obtains either $\lim_{t\to\infty} w^{(n-1)}(t) = -\infty$ or $\lim_{t\to\infty} w^{(n-1)}(t) = \ell \in \mathbb{R}$. If the former holds, then $w(t) < 0$ for $t \ge t_2 > t_1 > t_0 + \rho$ and hence $F(t) > z(t) > 0$, a contradiction to (H_1'). Suppose that the latter holds. Since $w(t)$ is monotonic, we may take $w(t) > 0$ for $t \ge t_3 > t_2$ because $w(t) < 0$ leads to a contradiction as above. Hence $z(t) \ge F^+(t)$ for $t \ge t_3$. The use of

(H_2) and (H_3) yields

$$\begin{aligned}
0 &= w^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\big(y(t-\sigma_i)\big) \\
&\qquad + G\big(-p(t-\sigma)\big)\Big[w^{(n)}(t-\tau) + \sum_{i=1}^{m} Q_i(t-\tau) G\big(y(t-\tau-\sigma_i)\big)\Big] \\
&\geq w^{(n)}(t) + G(p) w^{(n)}(t-\tau) \\
&\qquad + \sum_{i=1}^{m} Q_i^*(t)\Big[G\big(y(t-\sigma_i)\big) + G\big(-p(t-\sigma_i)\big) G\big(y(t-\tau-\sigma_i)\big)\Big] \\
&\geq w^{(n)}(t) + G(p) w^{(n)}(t-\tau) + \delta \sum_{i=1}^{m} Q_i^*(t) G\big(z(t-\sigma_i)\big) \\
&\geq w^{(n)}(t) + G(p) w^{(n)}(t-\tau) + \delta \sum_{i=1}^{m} Q_i^*(t) G\big(F^+(t-\sigma_i)\big)
\end{aligned}$$

for $t \geq t_3 + \rho$, where $\sigma = \min\{\sigma_i :\ 1 \leq i \leq m\}$. Thus

$$\int_{t_3+\rho}^{\infty} \Big[\sum_{i=1}^{m} Q_i^*(t) G\big(F^+(t-\sigma_i)\big)\Big] \, dt < \infty ,$$

which contradicts (H_{14}). The case $y(t) < 0$ for $t \geq t_0$ is treated similarly. This completes the proof of the theorem. □

THEOREM 2.4. *Let $-1 \leq p(t) \leq 0$. Suppose that (H_1'), (H_2), (H_2') and (H_{14}) hold. If*

(H_{15}) $\displaystyle\int_{\rho}^{\infty} \Big[\sum_{i=1}^{m} Q_i^*(t) G\big(-F^-(t-\sigma_i)\big)\Big] \, dt = -\infty ,$

then every solution of (1) *oscillates.*

P r o o f. In view of the proof of Theorem 2.3, it is enough to arrive at a contradiction in the case $\lim_{t\to\infty} w^{(n-1)}(t) = \ell \in \mathbb{R}$ and $w(t) > 0$ for $t \geq t_3$. Since $F^+(t) \leq z(t) \leq y(t) + y(t-\tau)$ for $t \geq t_3$, the use of (H_2) yields

$$\begin{aligned}0 &= w^{(n)}(t) + \sum_{i=1}^{m} Q_i(t)G\big(y(t-\sigma_i)\big) + w^{(n)}(t-\tau) \\ &\quad + \sum_{i=1}^{m} Q_i(t-\tau)G\big(y(t-\tau-\sigma_i)\big) \\ &\ge w^{(n)}(t) + w^{(n)}(t-\tau) + \delta \sum_{i=1}^{m} Q_i^*(t)G\big(y(t-\sigma_i) + y(t-\tau-\sigma_i)\big) \\ &\ge w^{(n)}(t) + w^{(n)}(t-\tau) + \delta \sum_{i=1}^{m} Q_i^*(t)G\big(F^+(t-\sigma_i)\big)\end{aligned}$$

for $t \ge t_3 + \rho$. Hence

$$\int_{t_3+\rho}^{\infty} \left[\sum_{i=1}^{m} Q_i^*(t)G\big(F^+(t-\sigma_i)\big)\right] \mathrm{d}t < \infty ,$$

which is a contradiction to (H_{14}). Thus the theorem is proved. □

THEOREM 2.5. *Suppose that $p(t)$ changes sign with $-1 \le p(t) \le 1$. If (H_1), (H_2), (H_2'), (H_{12})–(H_{15}) hold, then every solution of* (1) *oscillates.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1) with $y(t) > 0$ or $y(t) < 0$ for $t \ge t_0 > T_y$. We consider the case $y(t) > 0$ for $t \ge t_0$. The case $y(t) < 0$ for $t \ge t_0$ may similarly be dealt with. Setting $w(t)$ as in (3) and $z(t)$ as in (6) and then proceeding as in the proof of Theorem 2.1, one obtains that $w(t)$ is monotonic for large t and either $\lim\limits_{t\to\infty} w^{(n-1)}(t) = -\infty$ or $\lim\limits_{t\to\infty} w^{(n-1)}(t) = \ell \in \mathbb{R}$. If $\lim\limits_{t\to\infty} w^{(n-1)}(t) = -\infty$, then $\lim\limits_{t\to\infty} w(t) = -\infty$. Proceeding as in the proof of Theorem 2.1, we obtain that $y(t)$ is bounded and hence $w(t)$ is bounded, a contradiction. If $\lim\limits_{t\to\infty} w^{(n-1)}(t) = \ell \in \mathbb{R}$, then (5) holds. If $w(t) < 0$ for large t, then proceeding as in the proof of Theorem 2.1 we obtain a contradiction to (H_{12}). Let $w(t) > 0$ for $t > t_2 > t_1 > t_0 + \rho$. Then $F(t) < z(t) \le y(t) + y(t-\tau)$ for $t \ge t_2$. Proceeding as in the proof of Theorem 2.4, we obtain a contradiction to (H_{14}). Thus the proof of the theorem is complete. □

EXAMPLE. Consider

$$\Big[y(t) - \tfrac{2}{3}y(t-2\pi)\Big]'' + \tfrac{4}{9}y^3(t-2\pi) = \tfrac{1}{9}\cos 3t\,, \qquad t \ge 0\,.$$

Hence $F(t) = -\frac{1}{81}\cos 3t$. Clearly, $F(t)$ changes sign with $\liminf_{t\to\infty} F(t) = -\frac{1}{81}$ and $\limsup_{t\to\infty} F(t) = \frac{1}{81}$. Further,

$$F^{+}(t-2\pi) = \begin{cases} 0, & 0 \le 3t \le \frac{\pi}{2}, \\ -\frac{1}{81}\cos 3t, & (4n-3)\frac{\pi}{2} \le 3t \le (4n-1)\frac{\pi}{2}, \\ 0, & (4n-1)\frac{\pi}{2} \le 3t \le (4n+1)\frac{\pi}{2} \end{cases}$$

and

$$F^{-}(t) = \begin{cases} \frac{1}{81}\cos 3t, & 0 \le 3t \le \frac{\pi}{2}, \\ 0, & (4n-3)\frac{\pi}{2} \le 3t \le (4n-1)\frac{\pi}{2}, \\ \frac{1}{81}\cos 3t, & (4n-1)\frac{\pi}{2} \le 3t \le (4n+1)\frac{\pi}{2}, \end{cases}$$

$n = 1, 2, \ldots$ imply that

$$\begin{aligned} \int_{2\pi}^{\infty} Q(t)G\big(F^{+}(t-2\pi)\big)\,\mathrm{d}t &= \frac{4}{9}\int_{2\pi}^{\infty} \big(F^{+}(t-2\pi)\big)^{3}\,\mathrm{d}t \\ &= -\frac{12}{81^4}\sum_{n=4}^{\infty}\int_{(4n-3)\frac{\pi}{2}}^{(4n-1)\frac{\pi}{2}} \cos^3 u\,\mathrm{d}u \\ &= -\frac{12}{81^4}\sum_{n=4}^{\infty}\Big(\frac{-4}{3}\Big) = \infty \end{aligned}$$

and

$$\begin{aligned} \int_{2\pi}^{\infty} Q(t)G\big(F^{-}(t)\big)\,\mathrm{d}t &= \frac{4}{9}\int_{2\pi}^{\infty} \big(F^{-}(t)\big)^{3}\,\mathrm{d}t \\ &> \frac{12}{81^4}\sum_{n=4}^{\infty}\int_{(4n-1)\frac{\pi}{2}}^{(4n+1)\frac{\pi}{2}} \cos^3 u\,\mathrm{d}u \\ &= \frac{12}{81^4}\sum_{n=4}^{\infty}\Big(\frac{4}{3}\Big) = \infty \end{aligned}$$

Similarly, other two conditions of Theorem 2.1 are satisfied. Hence all solutions of the equation oscillate by Theorem 2.1. In particular, $y(t) = \cos t$ is an oscillatory solution.

Remark. We can have similar examples to illustrate other theorems.

3. Oscillation of homogeneous equation

This section deals with the oscillation of solutions of equation (2). The results here differ substantially from those in [10], [11]. Different types of sublinear/superlinear G are considered in this paper. We need the following lemmas for our work in the sequel:

LEMMA 3.1. ([4], [5; p. 193]) *Let $y \in C^{(n)}([0,\infty),\mathbb{R})$ be of constant sign. Let $y^{(n)}(t)$ be of constant sign and $\not\equiv 0$ in any interval $[T,\infty)$, $T \ge 0$, and $y(t)y^{(n)}(t) \le 0$. Then there exists a number $t_0 \ge 0$ such that the functions $y^{(j)}(t)$, $j = 1,2,\dots,n-1$, are of constant sign on $[t_0,\infty)$ and there exists a number $k \in \{1,3,\dots,n-1\}$ when n is even or $k \in \{0,2,\dots,n-1\}$ when n is odd such that*

$$y(t)y^{(j)}(t) > 0 \qquad \text{for} \quad j = 0,1,2,\dots,k, \qquad t \ge t_0,$$
$$(-1)^{n+j-1}y(t)y^{(j)}(t) > 0 \qquad \text{for} \quad j = k+1,k+2,\dots,n-1, \quad t \ge t_0.$$

LEMMA 3.2. ([3; p. 46]) *If $q \in C\big([0,\infty),[0,\infty)\big)$ and*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} q(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}},$$

then $x'(t) + q(t)x(t-\tau) \le 0$, $t \ge 0$, cannot have an eventually positive solution and $x'(t) + q(t)x(t-\tau) \ge 0$, $t \ge 0$, cannot have an eventually negative solution.

LEMMA 3.3. ([3; p. 46]) *If q satisfies the conditions of Lemma 3.2, then $x'(t) - q(t)x(t+\tau) \ge 0$, $t \ge 0$, has no eventually positive solution and $x'(t) - q(t)x(t+\tau) \le 0$, $t \ge 0$, has no eventually negative solution.*

LEMMA 3.4. *Suppose that $0 \le p(t) \le p$, where p is a constant and (H_9) holds. If $y(t)$ is a solution of (2) with $y(t) > 0$ for $t \ge t_0 > 0$ and $z(t)$ is set as in (6) for $t \ge t_0 + \tau$, then either*

$$\lim_{t\to\infty} z^{(i)}(t) = -\infty, \qquad i = 0,1,2,\dots,n-1 \tag{7}$$

or

$$\left.\begin{aligned} &(-1)^{n+k}z^{(k)}(t) < 0, \qquad k = 0,1,2,\dots,n-1, \quad t \ge t_1 > t_0 + \rho, \\ &\text{and} \\ &\lim_{t\to\infty} z^{(k)}(t) = 0, \qquad k = 0,1,2,\dots,n-1. \end{aligned}\right\} \tag{8}$$

Proof. From (2) we obtain

$$z^{(n)}(t) = -\sum_{i=1}^{m} Q_i(t)G\big(y(t-\sigma_i)\big) \le 0 \qquad \text{for} \quad t \ge t_0 + \rho.$$

Hence $z, z', \dots, z^{(n-1)}$ are monotonic and are of constant sign for $t \geq t_1 > t_0+\rho$. Further, either $\lim_{t\to\infty} z^{(n-1)}(t) = -\infty$ or $\lim_{t\to\infty} z^{(n-1)}(t) = \ell \in \mathbb{R}$. If the former holds, then $\lim_{t\to\infty} z^{(i)}(t) = -\infty$, $i = 0,1,2,\dots,n-1$. Suppose the latter holds. Then (5) is true. We claim that $\liminf_{t\to\infty} y(t) = 0$. If not, then $y(t) > \alpha > 0$ for $t \geq t_2 > t_1$. Hence

$$G(\alpha) \int_{t_2+\rho}^{\infty} \left(\sum_{i=1}^{m} Q_i(t)\right) dt \leq \int_{t_2+\rho}^{\infty} \left[\sum_{i=1}^{m} Q_i(t) G\big(y(t-\sigma_i)\big)\right] dt < \infty$$

due to (5), a contradiction to (H_9). Thus our claim holds. Consequently, there exists a $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y(t_n) = 0$. Since $z(t)$ is monotonic, $z(t) \leq y(t)$ and $z(t+\tau) > -py(t)$, then $\lim_{t\to\infty} z(t) = 0$ and hence (8) holds. Thus the lemma is proved. □

LEMMA 3.5. *If the range of* $p(t)$ *in Lemma* 3.4 *is replaced by* $0 \leq p(t) \leq 1$, *then only* (8) *holds.*

Proof. If (7) holds, then $\lim_{t\to\infty} z(t) = -\infty$. Since $z(t) < 0$ for large t, then $y(t) < p(t)y(t-\tau) \leq y(t-\tau)$ and hence $y(t)$ is bounded. Consequently, $z(t)$ is bounded, a contradiction. Thus the lemma follows from Lemma 3.4. □

THEOREM 3.6. *Let* $-p \leq p(t) \leq 0$, *where* $p > 0$ *is a constant, and* $p(t)$ *be monotonic decreasing. Let* $\tau < \sigma = \min\{\sigma_i : 1 \leq i \leq m\}$. *If* (H_2)–(H_4), (H_7)(i) *and* (H_{10}) *hold, then every solution of* (2) *oscillates.*

Proof. If possible, let $y(t)$ be a nonoscillatory solution of (2). We may take $y(t) > 0$ for $t \geq t_0 > T_y$ in view of (H_4). Setting $z(t)$ as in (6), we obtain $z(t) > 0$ for $t \geq t_0+\tau$ and either $\lim_{t\to\infty} z^{(n-1)}(t) = -\infty$ or $\lim_{t\to\infty} z^{(n-1)}(t) = \ell \in \mathbb{R}$. If either the former holds or $\ell < 0$, then $z(t) < 0$ for large t, a contradiction. Hence $0 \leq \ell < \infty$. Since $z(t) > 0$ and $z^{(n)}(t) \leq 0$ for $t \geq t_0 + \rho$, then by Lemma 3.1, there exists an integer $k \leq n-1$ and $t_1 > t_0 + \rho$ such that $n-k$ is odd, $z^{(j)}(t) > 0$ for $j = 0,1,\dots,k$ and $z^{(j)}(t)z^{(j+1)}(t) < 0$ for $j = k, k+1, \dots, n-2$ and $t \geq t_1$. By the Taylor series expansion we have, for $t \geq t_1 + r$,

$$z(t) = z(t-r) + rz'(t-r) + \frac{r^2}{2!}z''(t-r) + \cdots + \frac{r^k}{k!}z^{(k)}(x) > \frac{r^k}{k!}z^{(k)}(x),$$

where $r > 0$ and $t-r < x < t$. Since $z^{(k)}(t)$ is decreasing, then $z(t) > \frac{r^k}{k!} z^{(k)}(t)$. Another Taylor series expansion yields

$$\begin{aligned} z^{(k)}(t) &= z^{(k)}(t+r) + (-r)z^{(k+1)}(t+r) + \frac{(-r)^2}{2!} z^{(k+2)}(t+r) + \dots \\ &\qquad \dots + \frac{(-r)^{n-k-1}}{(n-k-1)!} z^{(n-1)}(x) \\ &> \frac{r^{n-k-1}}{(n-k-1)!} z^{(n-1)}(t+r) \end{aligned}$$

because $r > 0$, $t < x < t+r$ and $z^{(n-1)}(t)$ is monotonically decreasing. Hence, for $t \geq t_1 + r$,

$$z(t) > \frac{r^{n-1}}{k!(n-k-1)!} z^{(n-1)}(t+r) > \frac{r^{n-1}}{(n-1)!} z^{(n-1)}(t+r). \tag{9}$$

The use of (H_2), (H_3) and (9) yields, for $t \geq t_2 > t_1 + (\sigma - \tau) + \rho$,

$$\begin{aligned} 0 &= z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\big(y(t-\sigma_i)\big) \\ &\qquad + G\big(-p(t-\sigma)\big)\Big[z^{(n)}(t-\tau) + \sum_{i=1}^{m} Q_i(t-\tau) G\big(y(t-\tau-\sigma_i)\big)\Big] \\ &\geq z^{(n)}(t) + G(p) z^{(n)}(t-\tau) \\ &\qquad + \sum_{i=1}^{m} Q_i^*(t)\big[G\big(y(t-\sigma_i)\big) + G\big(-p(t-\sigma_i)\big) G\big(y(t-\tau-\sigma_i)\big)\big] \\ &\geq z^{(n)}(t) + G(p) z^{(n)}(t-\tau) + \delta \sum_{i=1}^{m} Q_i^*(t) G\big(z(t-\sigma_i)\big) \\ &\geq z^{(n)}(t) + G(p) z^{(n)}(t-\tau) + \delta \sum_{i=1}^{m} Q_i^*(t) G\Big(\tfrac{(\sigma_i-\tau)^{n-1}}{(n-1)!} z^{(n-1)}(t-\tau)\Big) \\ &\geq z^{(n)}(t) + G(p) z^{(n)}(t-\tau) + \delta G\Big(\tfrac{(\sigma-\tau)^{n-1}}{(n-1)!} z^{(n-1)}(t-\tau)\Big) \sum_{i=1}^{m} Q_i^*(t). \end{aligned}$$

Hence

$$\delta \sum_{i=1}^{m} Q_i^*(t) + \frac{z^{(n)}(t)}{G(u)} + \frac{G(p) z^{(n)}(t-\tau)}{G(\nu)} \leq 0,$$

where $u = \frac{(\sigma-\tau)^{n-1}}{(n-1)!} z^{(n-1)}(t)$ and $\nu = \frac{(\sigma-\tau)^{n-1}}{(n-1)!} z^{(n-1)}(t-\tau)$ and the fact that $z^{(n-1)}(t)$ is monotonically decreasing is used. Integrating the above inequality we obtain

$$\delta\int\limits_{t_2}^{\infty}\Big(\sum_{i=1}^{m}Q_i^*(t)\Big)\,\mathrm{d}t+\frac{(n-1)!}{(\sigma-\tau)^{n-1}}\int\limits_{c_1}^{\ell}\frac{\mathrm{d}u}{G(u)}+G(p)\frac{(n-1)!}{(\sigma-\tau)^{n-1}}\int\limits_{c_2}^{\ell}\frac{\mathrm{d}\nu}{G(\nu)}\le 0\,,$$

where $c_1=\frac{(\sigma-\tau)^{n-1}}{(n-1)!}z^{(n-1)}(t_2)$ and $c_2=\frac{(\sigma-\tau)^{n-1}}{(n-1)!}z^{(n-1)}(t_2-\tau)$. This leads to a contradiction to (H_{10}) in view of (H_7)(i). Thus the theorem is proved. □

THEOREM 3.7. *Let $-1\le p(t)\le 0$. If (H_2), (H_2'), (H_7) and (H_{10}) hold, then every solution of* (2) *oscillates, where $\tau<\sigma=\min\{\sigma_i:\ 1\le i\le m\}$.*

Proof. Proceeding as in the proof of Theorem 3.6 we obtain (9). Since $z(t)\le y(t)+y(t-\tau)$, then (H_2) and (9) yield, for $t\ge t_2>t_1+(\sigma-\tau)+\rho$,

$$\begin{aligned}0&=z^{(n)}(t)+z^{(n)}(t-\tau)+\sum_{i=1}^{m}Q_i(t)G\big(y(t-\sigma_i)\big)+\sum_{i=1}^{m}Q_i(t-\tau)G\big(y(t-\tau-\sigma_i)\big)\\&\ge z^{(n)}(t)+z^{(n)}(t-\tau)+\delta\sum_{i=1}^{m}Q_i^*(t)G\big(z(t-\sigma_i)\big)\,.\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.6. Thus the proof of the theorem is complete. □

THEOREM 3.8. *Let $0\le p(t)\le 1$. If n is odd and if (H_7) and (H_9) hold, then every solution of* (2) *oscillates.*

Proof. Let $y(t)$ be a nonoscillatory solution of (2) with $y(t)>0$ or $y(t)<0$ for $t\ge t_0>T_y$. We consider the case $y(t)>0$ for $t\ge t_0$. The case $y(t)<0$ is similar. Setting $z(t)$ as in (6), we get $z(t)\le y(t)$ for $t\ge t_0+\tau$. Then (8) holds by Lemma 3.5. Since n is odd, then $z(t)>0$ for $t\ge t_1>t_0+\rho$. Taylor series expansion yields, for $t\ge t_1$,

$$\begin{aligned}z(t-r)&=z(t)+(-r)z'(t)+\frac{(-r)^2}{2!}z''(t)+\cdots+\frac{(-r)^{n-1}}{(n-1)!}z^{(n-1)}(x)\\&>\frac{r^{n-1}}{(n-1)!}z^{(n-1)}(t)\,,\end{aligned}\tag{10}$$

because $z^{(n-1)}(t)$ is monotonically decreasing, where $r > 0$, $t - r < x < t$. Hence, for $t \geq t_1$,

$$\begin{aligned} 0 &= z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\big(y(t-\sigma_i)\big) \\ &\geq z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\big(z(t-\sigma_i)\big) \\ &\geq z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\Big(\tfrac{\sigma_i^{n-1}}{(n-1)!} z^{(n-1)}(t)\Big) \\ &\geq z^{(n)}(t) + G\Big(\tfrac{\sigma^{n-1}}{(n-1)!} z^{(n-1)}(t)\Big) \sum_{i=1}^{m} Q_i(t), \end{aligned}$$

where $\sigma = \min\{\sigma_1, \dots, \sigma_m\}$. Proceeding as in the proof of Theorem 3.6 and using (H_7)(i) we obtain a contradiction to (H_9). Hence the theorem is proved. □

Remark. Theorem 3.8 improves [1; Theorem 3]. Moreover, the proof of Theorem 3.8 is simpler than that of Theorem 3. As Theorem 3.8 does not hold for linear G, we have the following theorem.

THEOREM 3.9. *Let* $0 \leq p(t) \leq 1$, n *be odd and* (H_6) *hold. If*

(H_{16}) $\liminf\limits_{t\to\infty} \int\limits_{t-\sigma}^{t} \Big(\sum\limits_{i=1}^{m} Q_i(s)\Big)\, ds > \frac{(n-1)!}{\beta e \sigma^{n-1}}$, *where* $2\sigma = \min\{\sigma_i :\ 1 \leq i \leq m\}$,

then every solution of (2) *oscillates.*

P r o o f. Suppose that $y(t)$ is a nonoscillatory solution of (2) with $y(t) > 0$ for $t \geq t_0 > T_y$. The case $y(t) < 0$ for $t \geq t_0$ may similarly be dealt with. Then $z(t) \leq y(t)$ for $t \geq t_0 + \tau$, where $z(t)$ is same as in (6). We claim that (H_{16}) implies (H_9). Indeed, if (H_9) fails, then

$$0 < \lambda = \int\limits_{0}^{\infty} \Big(\sum_{i=1}^{m} Q_i(t)\Big)\, dt < \infty.$$

Hence

$$\begin{aligned}
&\liminf_{t\to\infty} \int_{t-\sigma}^{t} \left(\sum_{i=1}^{m} Q_i(s)\right) \mathrm{d}s \\
&= \liminf_{t\to\infty} \left[\int_0^t \left(\sum_{i=1}^{m} Q_i(s)\right) \mathrm{d}s - \int_0^{t-\sigma} \left(\sum_{i=1}^{m} Q_i(s)\right) \mathrm{d}s \right] \\
&\le \liminf_{t\to\infty} \int_0^t \left(\sum_{i=1}^{m} Q_i(s)\right) \mathrm{d}s + \limsup_{t\to\infty} \left[- \int_0^{t-\sigma} \left(\sum_{i=1}^{m} Q_i(s)\right) \mathrm{d}s \right] \\
&\le \liminf_{t\to\infty} \int_0^t \left(\sum_{i=1}^{m} Q_i(s)\right) \mathrm{d}s - \liminf_{t\to\infty} \int_0^{t-\sigma} \left(\sum_{i=1}^{m} Q_i(s)\right) \mathrm{d}s \\
&= \lambda - \lambda = 0\,,
\end{aligned}$$

which is a contradiction. Thus (8) holds by Lemma 3.5. Since n is odd, then $z(t) > 0$ for $t \ge t_1 > t_0 + \tau$. Further, (H_6) yields $G\big(z(t)\big) \ge \beta z(t)$ for $t \ge t_2 > t_1$. Proceeding as in the proof of Theorem 3.8, we obtain (10). Hence, for $t \ge t_2 + \rho$,

$$\begin{aligned}
0 &= z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\big(y(t-\sigma_i)\big) \\
&\ge z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\big(z(t-\sigma_i)\big) \\
&\ge z^{(n)}(t) + \beta \sum_{i=1}^{m} Q_i(t) z(t-\sigma_i) \\
&\ge z^{(n)}(t) + \beta \left(\sum_{i=1}^{m} Q_i(t) \right) z(t-2\sigma) \\
&\ge z^{(n)}(t) + \beta \frac{\sigma^{n-1}}{(n-1)!} \left(\sum_{i=1}^{m} Q_i(t) \right) z^{(n-1)}(t-\sigma)\,,
\end{aligned}$$

where the fact that $z(t)$ is decreasing is used. This contradicts Lemma 3.2 due to (H_{16}) because $z^{(n-1)}(t)$ is eventually positive. Thus the theorem is proved. □

THEOREM 3.10. *Let $1 \le p(t) \le p$, where $p > 0$ is a constant. Let n be odd and $\tau > \sigma^* = \max\{\sigma_i :\ 1 \le i \le m\}$. If (H_8) and (H_9) hold, then every solution of (2) oscillates.*

Proof. If possible, let $y(t)$ be a nonoscillatory solution of (2). Let $y(t) > 0$ for $t \ge t_0 > T_y$. The case $y(t) < 0$ for $t \ge t_0$ may similarly be dealt with. Then either (7) holds or (8) holds by Lemma 3.4, where $z(t)$ is defined by (6). If (7) holds, then $z^{(j)}(t) < 0$ for $t \ge t_1 > t_0$, $0 \le j \le n-1$. By the Taylor series expansion we have, for $t \ge t_1 + r$,

$$z(t) = z(t-r) + rz'(t-r) + \frac{r^2}{2!}z''(t-r) + \cdots + \frac{r^{n-1}}{(n-1)!}z^{(n-1)}(x),$$

where $t - r < x < t$ and $r > 0$. Since $z^{(n-1)}(t)$ is monotonically decreasing, then $z(t) < \frac{r^{n-1}}{(n-1)!}z^{(n-1)}(t-r)$. Further, $z(t) > -py(t-\tau)$ for $t \ge t_1$ implies that $y(t) > -\frac{1}{p}z(t+\tau)$. Hence, for $t \ge t_1 + \rho$,

$$\begin{aligned}
0 &= z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t)G\big(y(t-\sigma_i)\big) \\
&\ge z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t)G\big(-\tfrac{1}{p}z(t+\tau-\sigma_i)\big) \\
&\ge z^{(n)}(t) + G\big(-\tfrac{1}{p}z(t+\tau-\sigma^*)\big)\sum_{i=1}^{m} Q_i(t) \\
&\ge z^{(n)}(t) + G\Big(-\tfrac{(\tau-\sigma^*)^{n-1}}{p(n-1)!}z^{(n-1)}(t)\Big)\sum_{i=1}^{m} Q_i(t)
\end{aligned}$$

that is,

$$\sum_{i=1}^{m} Q_i(t) + \frac{1}{G(u)}z^{(n)}(t) \le 0,$$

where $u = -\frac{(\tau-\sigma^*)^{n-1}}{p(n-1)!}z^{(n-1)}(t)$. Hence

$$\int_{t_2}^{\infty}\Big(\sum_{i=1}^{m} Q_i(t)\Big)\,\mathrm{d}t \le \frac{p(n-1)!}{(\tau-\sigma^*)^{n-1}}\int_{c}^{\infty}\frac{\mathrm{d}u}{G(u)},$$

where $t_2 > t_1 + \rho$ and $c = -\frac{(\tau-\sigma^*)^{n-1}}{p(n-1)!}z^{(n-1)}(t_2)$. This contradicts (H_9) due to (H_8). Hence (8) holds. Consequently, (5) is true. Since n is odd, then $z(t) > 0$ for $t \ge t_1$ and hence $y(t) > p(t)y(t-\tau) \ge y(t-\tau)$. Thus $\liminf_{t\to\infty} y(t) > 0$. This contradicts (H_9) in view of (5). Hence the proof of the theorem is complete. □

THEOREM 3.11. *Let $1 \le p(t) \le p$, where $p > 0$ is a constant. Let n be odd, $\tau > \sigma^* = \max\{\sigma_i : 1 \le i \le m\}$ and* (H_5) *hold. If*

(H_{17}) $\liminf\limits_{t\to\infty} \int\limits_{t-\delta}^{t} \Big(\sum\limits_{i=1}^{m} Q_i(s)\Big)\,\mathrm{d}s > \frac{p(n-1)!}{e\,\alpha(\tau-\sigma^*-\delta)^{n-1}}$, *where $0 < \delta < \tau - \sigma^*$,*

then every solution of (2) *oscillates.*

Proof. We may note that (H_{17}) implies (H_9). Proceeding as in the proof of Theorem 3.10, we obtain $z(t) < \frac{r^{n-1}}{(n-1)!} z^{(n-1)}(t-r)$ for $t \ge t_1 + r$ when (7) holds. Further, $y(t) > -\frac{1}{p} z(t+\tau)$ for $t \ge t_1$. From (7) it follows that $z(t) \to -\infty$ as $t \to \infty$. Hence $G(z(t)) > \alpha z(t)$ for $t \ge t_2 > t_1 + \rho$. Hence, for $t \ge t_3 > t_2 + \rho$,

$$\begin{aligned} 0 &= z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\big(y(t-\sigma_i)\big) \\ &\ge z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\big(-\tfrac{1}{p} z(t+\tau-\sigma_i)\big) \\ &\ge z^{(n)}(t) - \frac{\alpha}{p} \sum_{i=1}^{m} Q_i(t) z(t+\tau-\sigma_i) \\ &\ge z^{(n)}(t) - \frac{\alpha}{p} z\big(t+\tau-\sigma^*\big) \sum_{i=1}^{m} Q_i(t) \\ &\ge z^{(n)}(t) - \frac{\alpha(\tau-\sigma^*-\delta)^{n-1}}{p(n-1)!} z^{(n-1)}(t+\delta) \sum_{i=1}^{m} Q_i(t), \end{aligned}$$

which contradicts Lemma 3.3 in view of (H_{17}) because $z^{(n-1)}(t) < 0$ for $t \ge t_3$. If (8) holds, we arrive at a contradiction as in the proof of Theorem 3.10. Thus the theorem is proved. □

THEOREM 3.12. *Suppose that $0 \le p(t) \le 1$. If n is even, $\tau < \sigma = \min\{\sigma_i : 1 \le i \le m\}$ and* (H_7) *and* (H_9) *hold, then every solution of* (2) *oscillates.*

Proof. Let $y(t)$ be a nonoscillatory solution of (2) with $y(t) > 0$ for $t \ge t_0 > T_y$. From Lemma 3.5 it follows that (8) holds, where $z(t)$ is given by (6). Since n is even, then $z(t) < 0$, $z'(t) > 0, \dots, z^{(n-1)}(t) > 0$ for $t \ge t_1 > t_0 + \rho$. Further,

$$z(t-r) = z(t) + (-r)z'(t) + \frac{(-r)^2}{2!} z''(t) + \cdots + \frac{(-r)^{n-1}}{(n-1)!} z^{(n-1)}(x),$$

where $r > 0$ and $t - r < x < t$, implies that $z(t-r) < \frac{(-r)^{n-1}}{(n-1)!} z^{(n-1)}(t)$ for $t \ge t_1$. Since $y(t) > -z(t+\tau)$ for $t \ge t_1$, then

$$\begin{aligned} 0 &= z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\big(y(t-\sigma_i)\big) \\ &\ge z^{(n)}(t) + \sum_{i=1}^{m} Q_i(t) G\big(-z(t+\tau-\sigma_i)\big) \\ &\ge z^{(n)}(t) + G\big(-z(t-\sigma+\tau)\big) \sum_{i=1}^{m} Q_i(t) \\ &\ge z^{(n)}(t) + G\Big(\tfrac{(\sigma-\tau)^{n-1}}{(n-1)!} z^{(n-1)}(t)\Big) \sum_{i=1}^{m} Q_i(t) \end{aligned}$$

for $t \ge t_2 > t_1 + \rho$. Since $z^{(n-1)}(t) \to 0$ as $t \to \infty$, then integrating the above inequality from t_2 to ∞ yields a contradiction to (H_9) due to (H_7) (i). A similar contradiction is obtained if $y(t) < 0$ for $t \ge t_0$. Hence the proof of the theorem is complete. □

Following theorems may be proved using the techniques employed in the above theorems.

THEOREM 3.13. *Let $0 \le p(t) \le 1$, n be even and $\tau < \sigma = \min\{\sigma_i : 1 \le i \le m\}$. If (H_6) holds and*

$$\liminf_{t\to\infty} \int_{t-c}^{t} \Big(\sum_{i=1}^{m} Q_i(s)\Big)\, \mathrm{d}s > \frac{(n-1)!}{\beta\, \mathrm{e} (\sigma-\tau-c)^{n-1}},$$

where $0 < c < \sigma - \tau$, then every solution of (2) oscillates.

THEOREM 3.14. *Let $1 \le p(t) \le p$, n be even and $\tau < \sigma = \min\{\sigma_i : 1 \le i \le m\}$. If (H_7) and (H_9) hold, then every bounded solution of (2) oscillates.*

4. Summary

We have observed that the behaviour of the forcing term $f(t)$ greatly influences the nature of solutions of (1). It is not known how the solutions of (1) would behave when $f(t)$ is such that $0 \le \liminf_{t\to\infty} F(t) < \limsup_{t\to\infty} F(t) \le \infty$ or $-\infty \le \liminf_{t\to\infty} F(t) < \limsup_{t\to\infty} F(t) \le 0$, where $F \in C^{(n)}\big([0,\infty), \mathbb{R}\big)$ with $F^{(n)}(t) = f(t)$ and $p(t) > 0$. We may note that this condition can be reduced to (H_1) if $\limsup_{t\to\infty} F(t) < \infty$ or $\liminf_{t\to\infty} F(t) > -\infty$. This can be reduced to (H_1') otherwise. In Theorems 2.1–2.5, the conditions on Q_i, $1 \le i \le m$, are so strong that the superlinearity or sub-linearity of G does not matter. We expect to weaken these conditions. Further, we note that these conditions are sufficient. It would be interesting to obtain conditions which are necessary as well as sufficient for oscillation of all solutions of (1) when F satisfies (H_1) or (H_1'). No result is known for (1) if $p(t)$ changes sign but not necessarily $-1 \le p(t) \le 1$.

It is interesting to notice that the range of $p(t)$, the nature of n and superlinearity/sublinearity of G are closely related in the results concerning (2). We have no result for superlinear G when $0 \le p(t) \le 1$ or $-p \le p(t) \le 0$ irrespective of n odd or even, where p is any positive scalar. No result for (2) is known if $p(t)$ changes sign with or without $-1 \le p(t) \le 1$. The conditions imposed on $Q_i(t)$, $1 \le i \le m$, in Theorems 3.6–3.14 are sufficient.

In [12], equations (1) and (2) are studied for $n = 1$. For $n = 1$ or $n \ge 2$, similar results may be obtained for (1)/(2) when $Q_i(t) \le 0$, $1 \le i \le m$. It seems that no result is known for $Q_i(t)$ changing sign.

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