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Mathematica Slovaca, Vol. 54 (2004), No. 4, 349--368

Persistent URL: http://dml.cz/dmlcz/131211

Math. Slovaca, **54** (2004), No. 4, 349–368

Mathematica Slovaca
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Slovak Academy of Sciences

GENERIC AND STABILITY PROPERTIES OF RECIPROCAL AND PSEUDOGRADIENT VECTOR FIELDS

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(*Communicated by Michal Fečkan*)

ABSTRACT. An analogue of Kupka-Smale theorem for some types of pseudoreciprocal vector fields is proven. Also some stability properties of special pseudoreciprocal vector fields defining second order differential equations are discussed. Pseudogradient vector fields on manifolds are defined and a theorem on the nonexistence of periodic trajectories for this class of vector fields is proven.

1. Introduction

Methods of nonlinear analysis are very often used in various scientific fields to analyse complicated mathematical models having a specific structure. We will study generic and stability properties of so called reciprocal vector fields. Reciprocal vector fields were defined by O. Chua in the paper [2]. These vector fields are of particular importance in nonlinear circuit systems because the state equation of any nonlinear circuit made up of only 2-terminal, and/or reciprocal a-terminal resistors, inductors, and capacitators always gives rise to a pseudoreciprocal vector field. In particular, if only 2-terminal capacitators or only 2-terminal inductors are present in circuit, in addition to reciprocal resistors, then the Jacobi matrix of the associated vector field always assumes the form of a product of two symmetric matrices. This special case corresponds to a pseudogradient vector field.

In this paper an analogue of Kupka-Smale theorem will be proved for some types of pseudoreciprocal vector fields. Also some stability properties of special types of reciprocal vector fields and pseudoreciprocal vector fields will be dis-

2000 Mathematics Subject Classification: Primary 34C05; Secondary 34D10.
Keywords: generic properties, pseudoreciprocal vector field, stability.

This work was supported by the Slovak Grant Agency, Vega Grant number 1/9177/02.

cussed. Pseudogradient vector fields will be defined on manifolds and analogous results to the case of vector fields in $\mathbb{R}^n$ will be obtained.

Now let us recall definitions of reciprocal and pseudoreciprocal vector fields from the paper [2].

DEFINITION 1.1. A matrix is called *reciprocal of order* (p,q) if it has the form

$$\begin{pmatrix} A & C \\ -C^* & D \end{pmatrix},$$

where A and D are matrices of dimensions p and q, respectively.

DEFINITION 1.2. A vector field $f(x)$ on $\mathbb{R}^{2n}$ is called *reciprocal of order* (p,q) if $\mathrm{D}f(x)$ is a reciprocal matrix function of order (p,q) for all $x \in \mathbb{R}^{2n}$.

DEFINITION 1.3. A vector field $f(x)$ on $\mathbb{R}^{2n}$ is called *pseudoreciprocal of order* (p,q) if there exists a nonzero matrix $M(x)$ such that $f(x) = M(x)g(x)$, $x \in \mathbb{R}^{2n}$, where $g(x)$ is a reciprocal vector field of order (p,q).

2. Generic properties of reciprocal matrices

We will use the following notation. Let M_{2n} ($M_{2n}(\mathrm{rec})$) be the set of all $2n \times 2n$ real (reciprocal) matrices and M_{2n}^0 ($M_{2n}^0(\mathrm{rec})$) be the subset of singular matrices.

LEMMA 2.1. *The set* $\mathcal{M}_n = \{A \in M_n :\ A \notin M_n^0\}$ *is open and dense in* M_n.

LEMMA 2.2. *The set* $\mathcal{M}_{2n}(\mathrm{rec}) = \{\mathcal{A} \in M_{2n}(\mathrm{rec}) :\ \mathcal{A} \notin M_{2n}^0(\mathrm{rec})\}$ *is open and dense in* $M_{2n}(\mathrm{rec})$.

Proof. From Lemma 2.1 it follows that $\mathcal{M}_{2n}$ is open in M_{2n}, but $M_{2n}(\mathrm{rec}) \subset M_{2n}$ and $\mathcal{M}_{2n}(\mathrm{rec}) = \mathcal{M}_{2n} \cap M_{2n}(\mathrm{rec})$, so $\mathcal{M}_{2n}(\mathrm{rec})$ is open in $M_{2n}(\mathrm{rec})$.

Now let us show density of $\mathcal{M}_{2n}(\mathrm{rec})$ in $M_{2n}(\mathrm{rec})$. We construct for $\varepsilon > 0$ and arbitrary matrix $\mathcal{B} \in M_{2n}(\mathrm{rec})$ a matrix $\mathcal{C} \in \mathcal{M}_{2n}(\mathrm{rec})$ such that $\|\mathcal{C}-\mathcal{B}\|_{M_{2n}} < \varepsilon$, where $\|\cdot\|_{M_{2n}}$ is a norm in $\mathcal{M}_{2n}(\mathrm{rec})$. Let

$$\mathcal{B} = \begin{pmatrix} A & C \\ -C^* & D \end{pmatrix} \in \mathcal{M}_{2n}(\mathrm{rec}).$$

The assertion of Lemma 2.1 enables us to assume that A is a regular matrix. It is easy to see that if

$$\mathcal{T} = \begin{pmatrix} E & 0 \\ C^*A^{-1} & E \end{pmatrix},$$

then $\det(\mathcal{T}\mathcal{B}) = \det(\mathcal{T})\det(\mathcal{B}) = \det(\mathcal{B})$ and so $\det(\mathcal{B}) = 0$ if and only if $\det(\mathcal{T}\mathcal{B}) = 0$. After some calculation we have:

$$\det(\mathcal{T}\mathcal{B}) = \det(A)\det(C^*A^{-1}C + D).$$

It is enough to show that there exists a matrix D' such that the matrix $C^*A^{-1}C + D'$ is regular, $\|D - D'\|_{M_{2n}} < \varepsilon$ and so

$$\mathcal{C} = \begin{pmatrix} A & C \\ -C^* & D' \end{pmatrix} \in \mathcal{M}_{2n}(\mathrm{rec})$$

with $\|\mathcal{C} - \mathcal{B}\|_{M_{2n}(\mathrm{rec})} < \varepsilon$. From Lemma 2.1 it follows the existence of a regular matrix $M \in M_2$ such that $\|M - (C^*A^{-1}C + D)\|_{M_n} < \varepsilon$. Let

$$D' = [M - C^*A^{-1}C].$$

It is obvious that $\|D - D'\|_{M_{2n}(\mathrm{rec})} < \varepsilon$ and the matrix $C^*A^{-1}C - D'$ is regular. □

3. Generic properties of reciprocal vector fields

DEFINITION 3.1. Let $n, r \in \mathbb{N}$ and $K \subset \mathbb{R}^n$ be a compact set. Then

$$C_b^r(K, \mathbb{R}^n) = \{F \in C^r(K, \mathbb{R}^n) : \ \|F\|_{C_b^r(K)} < \infty\}$$

is a Banach space with the norm

$$\|F\|_{C_b^r(K)} = \max\Big\{ \sup_{x\in K} \|F(x)\|, \sup_{x\in K} \|\,\mathrm{d}F(x)\|, \dots, \sup_{x\in K} \|\,\mathrm{d}^{(r)}F(x)\|\Big\}.$$

DEFINITION 3.2. Let $n, r \in \mathbb{N}$, $r \geqslant 1$, and $K \subset \mathbb{R}^{2n}$ be a compact set. Then define the set

$$V_{2n}^r(\mathrm{rec})(K) = \{F \in C_b^r(K, \mathbb{R}^{2n}) : \ F \text{ is the reciprocal vector field of order } (n,n)\}.$$

LEMMA 3.1. *Let $n, r \in \mathbb{N}$, $r \geqslant 1$, and $K \subset \mathbb{R}^{2n}$ be a compact set. Then $V_{2n}^r(\mathrm{rec})(K)$ is a second countable Banach space.*

Proof. It can be shown that $V_{2n}^r(\mathrm{rec})(K)$ is a closed linear subspace of $C_b^r(K, \mathbb{R}^{2n})$. Therefore the set $V_{2n}^r(\mathrm{rec})(K)$ is a Banach space. Since any separable metric space is second countable and $C_b^r(K, \mathbb{R}^{2n})$ is separable ([6]), so $C_b^r(K, \mathbb{R}^{2n})$ is second countable. Therefore $V_{2n}^r(\mathrm{rec})(K)$ is second countable. □

Lemma 3.2. *Let $n \in \mathbb{N}$. Then*

$$M_{2n}^0(\mathrm{rec}) = \bigcup_{i=1}^{l} M_i\,,$$

where M_i is a smooth manifold in $M_{2n}(\mathrm{rec})$ with $\operatorname{codim} M_i \geqslant 1$ *for* $i \in \{1,\dots,l\}$.

P r o o f . It is clear that $M_{2n}^0(\mathrm{rec})$ is an algebraic set. Therefore from Whitney's theorem ([5]) it follows that

$$M_{2n}^0(\mathrm{rec}) = \bigcup_{i=1}^{l} M_i\,,$$

where M_i is a smooth manifold in $M_{2n}(\mathrm{rec})$ for $i \in \{1,\dots,l\}$. The assertion concerning the codimensions is proved in [5; Lemma 3.69], where it is formulated for the set M_n^0. The proof for $M_{2n}^0(\mathrm{rec})$ is analogous to the proof of [5; Lemma 3.69], taking into account that $\dim\bigl(M_{2n}(\mathrm{rec})\bigr) = (2n)^2 - n^2$. □

We will use the notation:

$$\Sigma^0 = \bigl\{(u,v,\mathcal{A}) \in \mathbb{R}^n \times \mathbb{R}^n \times M_{2n}(\mathrm{rec}):\ u=0\,,\ v=0\,,\ \det\mathcal{A} = 0\bigr\}\,,$$

$$\Sigma^i = \bigl\{(u,v,\mathcal{A}) \in \mathbb{R}^n \times \mathbb{R}^n \times M_{2n}(\mathrm{rec}):\ u=0\,,\ v=0\,,\ \mathcal{A} \text{ has a pure imaginary eigenvalue}\bigr\}\,,$$

$$\Sigma = \Sigma^0 \cup \Sigma^i\,.$$

As a direct consequence of Lemma 3.2 we obtain:

Lemma 3.3. *Let $n \in \mathbb{N}$. Then*

$$\Sigma^0 = \bigcup_{i=1}^{l} \Sigma_i^0\,,$$

where

$$\Sigma_i^0 = \bigl\{(u,v,\mathcal{A}) \in \mathbb{R}^n \times \mathbb{R}^n \times M_{2n}(\mathrm{rec}):\ u=0\,,\ v=0\,,\ \mathcal{A} \in M_i\bigr\}\,,$$

M_i are manifolds from Lemma 3.2. and

$$\operatorname{codim}\Sigma_i^0 \geqslant 2n+1 \qquad \textit{for} \quad i \in \{1,\dots,l\}\,.$$

Definition 3.3. Let $n, r \in \mathbb{N}$ and $K \subset \mathbb{R}^{2n}$ be a compact set. Then define the mapping:

$$\rho\colon V_{2n}^{r+1}(\mathrm{rec})(K) \to C^r\bigl(K, \mathbb{R}^{2n} \times M_{2n}(\mathrm{rec})\bigr)\,,$$

$$\rho(F)\colon K \to \mathbb{R}^{2n} \times M_{2n}(\mathrm{rec})\,,$$

$$\rho(F)(x,y) = \bigl(F(x,y), \mathrm{d}F(x,y)\bigr)\,,$$

where $(x,y) \in K$. For $\rho(F)(x,y)$ we use also the notation $\rho_F(x,y)$.

Now we will remind A b r a h a m's transversality theorem.

THEOREM 3.1. *Let B be a Banach space, X be a smooth manifold, $\Sigma = \bigcup_{i\in I} \Sigma_i \subset \mathbb{R}^n$, where Σ_i is smooth manifold for $i \in I$ and $n \in \mathbb{N}$. Then following assertions hold.*

1. *If $K \subset X$ is a compact set and $\rho\colon B \to C^r(X,\mathbb{R}^n)$ is C^1-pseudo-representation $(1 \le r \le \infty)$, then the set $A_{K,\Sigma} = \{F \in B : \ \rho_F \overline{\pitchfork}_K \Sigma\}$ is open in B.*
2. *We suppose that*
   a) *B is second countable Banach space,*
   b) *$r > \max\{0, n-k\}$, where $k = \min_{i\in I} \operatorname{codim} \Sigma_i$,*
   c) *$\rho\colon B \to C^r(X,\mathbb{R}^n)$ is C^r-representation,*
   d) *$\mathrm{ev}_\rho \overline{\pitchfork}_K \Sigma$.*

Then the set $A_\Sigma = \{F \in B : \ \rho_F \overline{\pitchfork}_K \Sigma\}$ is dense in B.

P r o o f. See [5; Theorem 2.114]. □

LEMMA 3.4. *Let $n \in \mathbb{N}$, $K \subset \mathbb{R}^{2n}$ be a compact set and $r > \max\{0, n-k\}$, where $k = \min_{i\in\{1,\dots,l\}} \operatorname{codim} \Sigma_i^0$ and $\Sigma^0 = \bigcup_{i=1}^{l} \Sigma_i^0$. Then the set*

$$S_K^0 = \{F \in V_{2n}^{r+1}(\mathrm{rec})(K) : \ \rho_F \overline{\pitchfork} \Sigma^0\}$$

is open and dense in $V_{2n}^{r+1}(\mathrm{rec})(K)$.

P r o o f. We will use the Abraham transversality theorem. Therefore we consider the evaluation mapping

$$\begin{aligned} \mathrm{ev}_\rho\colon V_{2n}^{r+1}(\mathrm{rec})(K) \times K &\to \mathbb{R}^{2n} \times M_{2n}(\mathrm{rec}) \\ \mathrm{ev}_\rho\big(F(x,y)\big) &= \rho_F\big((x,y)\big)\,. \end{aligned}$$

We know that $V_{2n}^{r+1}(\mathrm{rec})(K)$ is a Banach space and for the proof of openness of the set S_K^0 in $V_{2n}^{r+1}(\mathrm{rec})(K)$ it is enough to show that ρ is a C^1-pseudo-representation, i.e. the evaluation mapping is a C^0-representation and the mapping

$$\begin{aligned} \rho^{(1)}\colon V_{2n}^{r+1}(\mathrm{rec})(K) &\to C^{r-1}\big(T(K), T(\mathbb{R}^{2n} \times M_{2n}(\mathrm{rec}))\big) \\ \rho^{(1)}(F) &= \mathrm{D}\rho_F \end{aligned}$$

is a C^0-representation.

$M_{2n}(\mathrm{rec})$ is the linear space. Thus $T_x\big(M_{2n}(\mathrm{rec})\big) \cong M_{2n}(\mathrm{rec})$ for $x \in M_{2n}(\mathrm{rec})$, therefore $T\big(\mathbb{R}^{2n} \times M_{2n}(\mathrm{rec})\big) \cong \mathbb{R}^{2n} \times M_{2n}(\mathrm{rec}) \times \mathbb{R}^{2n} \times M_{2n}(\mathrm{rec})$.

We have to show that the mappings ev_ρ and

$$\mathrm{ev}_{\rho(1)} : V_{2n}^{r+1}(\mathrm{rec})(K) \times K \times \mathbb{R}^{2n} \to \mathbb{R}^{2n} \times M_{2n}(\mathrm{rec}) \times \mathbb{R}^{2n} \times M_{2n}(\mathrm{rec})$$
$$\mathrm{ev}_{\rho(1)}\big(F, (x,y), (u,v)\big) = \big(\mathrm{d}F(x,y)(u,v), \mathrm{d}^2F(x,y)(u,v)\big)\,,$$

are C^0-mappings.

First we show that the mapping ev_ρ is continuous at any $\big(F, (x_1, y_1)\big) \in V_{2n}^{r+1}(\mathrm{rec})(K) \times \mathbb{R}^{2n}$. Let us compute:

$$\begin{aligned}
&\big\|\mathrm{ev}_\rho\big(G, (x_2, y_2)\big) - \mathrm{ev}_\rho\big(F, (x_1, y_1)\big)\big\|_{\mathbb{R}^{2n}\times M_{2n}} \\
&= \big\|\big(G(x_2,y_2) - F(x_1,y_1), \mathrm{d}G(x_2,y_2) - \mathrm{d}F(x_1,y_1)\big)\big\|_{\mathbb{R}^{2n}\times M_{2n}} \\
&= \|G(x_2,y_2) - F(x_2,y_2) + F(x_2,y_2) - F(x_1,y_1)\|_{\mathbb{R}^{2n}} \\
&\qquad + \|\mathrm{d}G(x_2,y_2) - \mathrm{d}F(x_2,y_2) + \mathrm{d}F(x_2,y_2) - \mathrm{d}F(x_1,y_1)\|_{M_{2n}} \\
&\leqslant \|G(x_2,y_2) - F(x_2,y_2)\|_{\mathbb{R}^{2n}} + \|F(x_2,y_2) - F(x_1,y_1)\|_{\mathbb{R}^{2n}} \\
&\qquad + \|\mathrm{d}G(x_2,y_2) - \mathrm{d}F(x_2,y_2)\|_{M_{2n}} + \|\mathrm{d}F(x_2,y_2) - \mathrm{d}F(x_1,y_1)\|_{M_{2n}} \\
&\leqslant 2\|G-F\|_{C_b^r(K)} + \|F(x_2,y_2) - F(x_1,y_1)\|_{\mathbb{R}^{2n}} + \|\mathrm{d}F(x_2,y_2) - \mathrm{d}F(x_1,y_1)\|_{M_{2n}}.
\end{aligned}$$

Now for a given $\varepsilon > 0$ we find $\delta > 0$, such that

$$\big(G, (x_2, y_2)\big) \in V_{2n}^{r+1}(\mathrm{rec})(K) \times \mathbb{R}^{2n}\,, \quad \big\|\big(G, (x_2, y_2)\big) - \big(F, (x_1, y_1)\big)\big\|_{C_b^r(K)\times\mathbb{R}^{2n}} < \delta$$

and the following inequalities hold:

(i) $2\|G - F\|_{C_b^r(K)} < \frac{1}{3}\varepsilon$,

(ii) $\|F(x_2,y_2) - F(x_1,y_1)\|_{\mathbb{R}^{2n}} < \frac{1}{3}\varepsilon$,

(iii) $\|\mathrm{d}F(x_2,y_2) - \mathrm{d}F(x_1,y_1)\|_{M_{2n}} < \frac{1}{3}\varepsilon$.

The inequality (i) holds for $\delta_1 < \frac{1}{6}\varepsilon$. Since $F \in V_{2n}^{r+1}(\mathrm{rec})(K) \subset C_b^{r+1}(K, \mathbb{R}^{2n})$ and $r + 1 \geqslant 2$, there exist $\delta_2 > 0$ a $\delta_3 > 0$ for which (ii), resp. (iii) holds. For $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ we have

$$\big\|\mathrm{ev}_\rho\big(G, (x_2, y_2)\big) - \mathrm{ev}_\rho\big(F, (x_1, y_1)\big)\big\|_{\mathbb{R}^{2n}\times M_{2n}} < \varepsilon\,.$$

Therefore the mapping ev_ρ continuous.

Now we show that $\mathrm{ev}_{\rho(1)}$ is a continuous mapping. Let us compute:

$$\begin{aligned}
&\big\|\mathrm{ev}_{\rho(1)}\big(G,(x_2,y_2),(u_2,v_2)\big) - \mathrm{ev}_{\rho(1)}\big(F,(x_1,y_1),(u_1,v_1)\big)\big\|_{\mathbb{R}^{2n}\times M_{2n}} \\
&\quad = \|\,\mathrm{d}G(x_2,y_2)(u_2,v_2) - \mathrm{d}F(x_1,y_1)(u_1,v_1)\|_{\mathbb{R}^{2n}} \\
&\qquad + \|\,\mathrm{d}^2G(x_2,y_2)(u_2,v_2) - \mathrm{d}^2F(x_1,y_1)(u_1,v_1)\|_{M_{2n}} \\
&\quad \leqslant \|\,\mathrm{d}G(x_2,y_2)(u_2,v_2) - \mathrm{d}F(x_2,y_2)(u_2,v_2)\|_{\mathbb{R}^{2n}} \\
&\qquad + \|\,\mathrm{d}F(x_2,y_2)(u_2,v_2) - \mathrm{d}F(x_2,y_2)(u_1,v_1)\|_{\mathbb{R}^{2n}} \\
&\qquad + \|\,\mathrm{d}F(x_2,y_2)(u_1,v_1) - \mathrm{d}F(x_1,y_1)(u_1,v_1)\|_{\mathbb{R}^{2n}} \\
&\qquad + \|\,\mathrm{d}^2G(x_2,y_2)(u_2,v_2) - \mathrm{d}^2F(x_2,y_2)(u_2,v_2)\|_{M_{2n}} \\
&\qquad + \|\,\mathrm{d}^2F(x_2,y_2)(u_2,v_2) - \mathrm{d}^2F(x_2,y_2)(u_1,v_1)\|_{M_{2n}} \\
&\qquad + \|\,\mathrm{d}^2F(x_2,y_2)(u_1,v_1) - \mathrm{d}^2F(x_1,y_1)(u_1,v_1)\|_{M_{2n}}\,.
\end{aligned}$$

Using the same considerations as above, we obtain that the mapping $\mathrm{ev}_{\rho(1)}$ is continuous. We have shown that S^0_K is open in $V^{r+1}_{2n}(\mathrm{rec})(K)$. Using the second part of the Abraham theorem, we show that S^0_K is dense in $V^{r+1}_{2n}(\mathrm{rec})(K)$.

The assumptions a), b) of the Abraham theorem are satisfied.

Now we show that also the assumptions c), d) of this theorem are satisfied.

Let us verify the assumption c). We prove that $\rho\colon V^{r+1}_{2n}(\mathrm{rec})(K) \to C^r(K,\mathbb{R}^{2n})$ is a C^r-representation. We have proved that the mapping ev_ρ is continuous. Now we show that ev_ρ is a C^1-mapping. One can see that:

$$\begin{aligned}
&\mathrm{Dev}_\rho\big(F,(x,y)\big)\big(G,(u,v)\big) \\
&= \lim_{t\to 0}\frac{1}{t}\big(\mathrm{ev}_\rho\big(F{+}tG,(x{+}tu,y{+}tv)\big) - \mathrm{ev}_\rho\big(F,(x,y)\big)\big) \\
&= \lim_{t\to 0}\frac{1}{t}\big((F{+}tG)(x{+}tu,y{+}tv){-}F(x,y),\mathrm{d}(F{+}tG)(x{+}tu,y{+}tv){-}\,\mathrm{d}F(x,y)\big) \\
&= \lim_{t\to 0}\frac{1}{t}\big(F(x{+}tu,y{+}tv){-}F(x,y),\mathrm{d}F(x{+}tu,y{+}tv){-}\,\mathrm{d}F(x,y)\big) \\
&= \lim_{t\to 0}\frac{1}{t}\big(tG(x{+}tu,y{+}tv),t\,\mathrm{d}G(x{+}tu,y{+}tv)\big) \\
&= \big(\mathrm{d}F(x,y)(u,v){+}G(x,y),\mathrm{d}^2F(x,y)(u,v){+}\,\mathrm{d}G(x,y)(u,v)\big)\,.
\end{aligned}$$

Since $F,G \in V^{r+1}_{2n}(\mathrm{rec})(K) \subset C^{r+1}_b(K,\mathbb{R}^{2n})$, so Dev_ρ is a continuous mapping and ev_ρ is a C^1-mapping. The assertion for $r>1$ can be proved by induction.

Let us verify d). It is sufficient to prove that $\mathrm{ev}_\rho \pitchfork \Sigma^0$. So we have to show that

$$\mathrm{Image}\big(\mathrm{Dev}_\rho\big(F,(x,y)\big)\big) + T_{\mathrm{ev}_\rho(F,(x,y))}\Sigma^0 = \mathbb{R}^{2n}\times M_{2n}(\mathrm{rec})$$

for any $\big(F,(x,y)\big) \in V^{r+1}_{2n}(\mathrm{rec})(K)\times\mathbb{R}^{2n}$, for which $\mathrm{ev}_\rho\big(F,(x,y)\big)\in\Sigma^0$. Now we show that for any $w\in\mathbb{R}^{2n}$, $\mathcal{A}\in M_{2n}(\mathrm{rec}) = \begin{pmatrix} A & C \\ -C^* & D\end{pmatrix}$, $A,C,D\in M_n$, there exists a $G \in V_{2n}^{r+1}(\mathrm{rec})(K)$ such that

$$\mathrm{Dev}_\rho\big(F,(x,y)\big)(G,0,0) = (w,\mathcal{A}).$$

Let us compute:

$$\begin{aligned}
&\mathrm{Dev}_\rho\big(F,(x,y)\big)(G,0,0)\\
&\quad= \lim_{t\to 0}\frac{1}{t}\big(\mathrm{ev}_\rho\big(F+tG,(x,y) - \mathrm{ev}_\rho\big(F,(x,y)\big)\big)\big)\\
&\quad= \lim_{t\to 0}\frac{1}{t}\big((F+tG)(x,y) - F(x,y), \mathrm{d}(F+tG)(x,y) - \mathrm{d}F(x,y)\big)\\
&\quad= \lim_{t\to 0}\frac{1}{t}\big(tG(x,y), t\,\mathrm{d}G(x,y)\big)\\
&\quad= \big(G(x,y), \mathrm{d}G(x,y)\big).
\end{aligned}$$

Therefore we need to find a map G such that

$$\begin{aligned}
G(x,y) &= w,\\
\mathrm{d}G(x,y) &= \mathcal{A}.
\end{aligned}$$

Obviously, the map G, defined as

$$G(p,q) = \begin{pmatrix} A & C\\ -C^* & D\end{pmatrix}\begin{pmatrix} p\\ q\end{pmatrix} + w - \begin{pmatrix} A & C\\ -C^* & D\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}$$

for $(p,q) \in K \subset \mathbb{R}^{2n}$, satisfies these conditions. Thus we have proved that $\mathrm{ev}_\rho \,\overline{\pitchfork}\, \Sigma^0$. □

LEMMA 3.5. *Let* $n \in \mathbb{N}$, $K \subset \mathbb{R}^{2n}$ *be a compact set and* $r > \max\{0, n-k\}$, *where* $k = \min\limits_{j\in\{1,\dots,s\}} \operatorname{codim}\Sigma_j^i$ *and* $\Sigma^i = \bigcup\limits_{j=1}^{s}\Sigma_j^i$. *Then the set*

$$S_K^i = \big\{F \in V_{2n}^{r+1}(\mathrm{rec})(K) :\ \rho_F \,\overline{\pitchfork}\, \Sigma^i\big\}$$

is open and dense in $V_{2n}^{r+1}(\mathrm{rec})(K)$.

P r o o f. The proof is analogous to the proof of Lemma 3.4, because it can be shown that the set Σ^i has the same properties as Σ^0. □

THEOREM 3.2. *Let* $n \in \mathbb{N}$, $K \subset \mathbb{R}^{2n}$ *be a compact and* $r > \max\{0, n-k\}$, *where* $k = \min\{k_1, k_2\}$ *and*

$$k_1 = \min_{j\in\{1,\dots,l\}} \operatorname{codim}\Sigma_j^0 \qquad \textit{and} \qquad \Sigma^0 = \bigcup_{j=1}^{s}\Sigma_j^0,$$

$$k_2 = \min_{j\in\{1,\dots,s\}} \operatorname{codim}\Sigma_j^i \qquad \textit{and} \qquad \Sigma^i = \bigcup_{j=1}^{s}\Sigma_j^i.$$

Then the set

$$S_K = \{F \in V_{2n}^{r+1}(\mathrm{rec})(K) : \ \rho_F \,\overline{\pitchfork}\, \Sigma\}$$

is open and dense in $V_{2n}^{r+1}(\mathrm{rec})(K)$.

P r o o f . If we use the notations from Lemma 3.4 and Lemma 3.5, we have

$$S_K = S_K^0 \cap S_K^i .$$

Since the sets S_K^0 and S_K^i are open and dense in $V_{2n}^{r+1}(\mathrm{rec})(K)$ and $V_{2n}^{r+1}(\mathrm{rec})(K)$ is a Banach space, the set S_K is also open and dense in $V_{2n}^{r+1}(\mathrm{rec})(K)$. □

DEFINITION 3.4. The equilibrium point x_0 of the system $\dot{x} = f(x)$ is called *hyperbolic* if the Jacobi matrix $J_f(x_0)$ has not an eigenvalue with zero real part.

COROLLARY 3.1. *Let* $n \in \mathbb{N}$, $K \subset \mathbb{R}^{2n}$ *be a compact set and* $\mathrm{int}(K) \neq \emptyset$. *Let the assumptions of Theorem* 3.1 *hold and let* $F = (F_1, F_2) \in S_K \subset V_{2n}^{r+1}(K)$. *Then the system*

$$\dot{x} = F_1(x,y)\,, \qquad \dot{y} = F_2(x,y)$$

has only hyperbolic singular points in $\mathrm{int}(K)$.

P r o o f . From definition of the set Σ it is clear that equilibrium point $(x,y) \in \mathbb{R}^{2n}$ is not hyperbolic singular point if and only if $\rho_F(x,y) \in \Sigma$. So, it is enough to show that $(\rho_F)^{-1}(\Sigma) = \emptyset$.

Since $F \in S_K$, so $\rho_F \,\overline{\pitchfork}\, \Sigma$. Therefore

$$\rho_F \,\overline{\pitchfork}\, \Sigma_j^0 \qquad \text{for} \quad j \in \{1, \dots, l\}$$

and

$$\rho_F \,\overline{\pitchfork}\, \Sigma_j^i \qquad \text{for} \quad j \in \{1, \dots, s\}.$$

From [3; Chap. 2, Theorem 4.4] it follows that $(\rho_F)^{-1}(\Sigma_j^0)$ is a submanifold of $\mathbb{R}^{2n}$ and

$$\mathrm{codim}\big((\rho_F)^{-1}(\Sigma_j^0)\big) \geq 2n+1 \qquad \text{for} \quad j \in \{1, \dots, l\}.$$

Therefore

$$(\rho_F)^{-1}(\Sigma_j^0) = \emptyset \qquad \text{for} \quad j \in \{1, \dots, l\}$$

and so

$$(\rho_F)^{-1}(\Sigma^0) = \bigcup_{j=1}^{l} (\rho_F)^{-1}(\Sigma_j^0) = \emptyset .$$

Analogously one can show that $(\rho_F)^{-1}(\Sigma^i) = \emptyset$ and so we have

$$(\rho_F)^{-1}(\Sigma) = (\rho_F)^{-1}(\Sigma^0) \cup (\rho_F)^{-1}(\Sigma^i) = \emptyset .$$

□

4. Second order ODE's defining reciprocal vector fields

We shall study a special class of reciprocal vector fields on $\mathbb{R}^n \times \mathbb{R}^n$ which represents a second order ODE on $\mathbb{R}^n$.

LEMMA 4.1. *The system*

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= g(x,y)\,, \end{aligned} \tag{4.1}$$

where $x, y \in \mathbb{R}^n$ and the point $(0,0) \in \mathbb{R}^n \times \mathbb{R}^n$ is an equilibrium point, is reciprocal if and only if $g(x,y) = -x + Dy + h(y)$, where $D \in M_n$ and $h(y) = o(\|y\|)$.

Proof. The Jacobi matrix of the system (4.1) has the form:

$$\begin{pmatrix} 0 & E \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix}$$

and so the reciprocity condition for the system (4.1) is

$$g_x(x,y) = -E\,.$$

Therefore, if the point $(0,0) \in \mathbb{R}^n \times \mathbb{R}^n$ is an equilibrium point, then

$$g(x,y) = -x + Dy + h(y)\,,$$

where $h(y) = o(\|y\|)$. The converse assertion is obviously valid. □

For planar system we can prove next assertion.

LEMMA 4.2. *If $h \in C^1(\mathbb{R},\mathbb{R})$ and $D + \mathrm{d}h(y) \neq 0$ for any $y \in \mathbb{R}$, then the system* (4.1) *does not have periodic trajectories.*

Proof. Since

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} y \\ -x + Dy + h(y) \end{pmatrix},$$

so

$$\operatorname{div} f(x,y) = D + \mathrm{d}h(y) \neq 0$$

and the assertion follows from the Bendixon criterion. (See e.g. [4; Theorem 1.8.2]). □

LEMMA 4.3. *If $h \in C^1(\mathbb{R},\mathbb{R})$ and $Dy^2 + yh(y) < 0$ for all $y \in \mathbb{R}$, then the point $(0,0)$ is a global attractor of the system* (4.1)*, i.e., if $\big(x(t), y(t)\big)$ is arbitrary solution of the system* (4.1)*, then $\lim\limits_{t\to\infty} \big(x(t), y(t)\big) = (0,0)$.*

Proof. Let

$$H(x,y) = \frac{1}{2}(x^2 + y^2)\,.$$

If $\big(x(t), y(t)\big)$ is a solution of the system (4.1), then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(H\big(x(t), y(t)\big)\big) &= x(t)\dot{x}(t) + y(t)\dot{y}(t)\\ &= x(t)y(t) + y(t)\big(-x(t) + Dy(t) + h\big(y(t)\big)\big)\\ &= D\big(y(t)\big)^2 + y(t)h\big(y(t)\big) < 0\end{aligned}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(H\big(x(t), y(t)\big)\big) < 0$$

for all $t \in \mathbb{R}$. This yields that the mapping H is decreasing along any trajectory of the system (4.1). Therefore the point $(0,0)$ is the global attractor of the system (4.1). □

Remark 4.1. If $h \in C^1(\mathbb{R},\mathbb{R})$ and $Dy^2 + yh(y) > 0$ for any $y \in \mathbb{R}$, then the point $(0,0)$ is the global repeller of the system (4.1), i.e., if $\big(x(t), y(t)\big)$ is an arbitrary solution of the system (4.1), then the function $\big(x^2(t) + y^2(t)\big)^{\frac{1}{2}}$ is increasing.

Remark 4.2. If $h \in C^1(\mathbb{R},\mathbb{R})$ and $Dy^2 + y\,\mathrm{d}h(y) \neq 0$ for any $y \in \mathbb{R}$, then the system (4.1) has not periodic trajectories.

LEMMA 4.4. *The eigenvalues of the linearisation matrix at point* $(0,0) \in \mathbb{R}^n \times \mathbb{R}^n$ *of the system* (4.1) *has the following form:*

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}\,,$$

where μ *are eigenvalues of matrix* D.

P r o o f. It is clear that the linearisation matrix of the system (4.1) is the matrix

$$B = \begin{pmatrix} 0 & E\\ -E & D\end{pmatrix} \tag{4.2}$$

and its determinant is equal 1, i.e. this matrix is regular. Now let us solve the characteristic equation:

$$\det\begin{pmatrix} -\lambda E & E\\ -E & D - \lambda E\end{pmatrix} = 0\,. \tag{4.3}$$

After some calculations and taking into account that the matrix (4.2) is regular, one can see that the equation (4.3) can be rewritten as

$$\det\Big(D - \big(\lambda + \tfrac{1}{\lambda}\big)E\Big) = 0\,.$$

Now it is clear that $\mu = \left(\lambda + \frac{1}{\lambda}\right)$ is an eigenvalue of the matrix D, where λ is an eigenvalue of the matrix B. This means that any eigenvalue μ of the matrix D defines two eigenvalues

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

of the matrix D. □

COROLLARY 4.1. *If the matrix D is regular, then the matrix* (4.2) *has not eigenvalues with zero real part.*

COROLLARY 4.2. *If the matrix D is regular, then the point $(0,0) \in \mathbb{R}^n \times \mathbb{R}^n$ is the hyperbolic equilibrium point of the system* (4.1).

DEFINITION 4.1. The equilibrium point x_0 of the system $\dot{x} = f(x)$ is called *sink* (*source*) if the Jacobi matrix $J_f(x_0)$ has all eigenvalues with negative (positive) real parts.

COROLLARY 4.3. *If the matrix D has all eigenvalues with negative* (*positive*) *real parts, then the point $(0,0) \in \mathbb{R}^n \times \mathbb{R}^n$ is the sink* (*source*) *of the system* (4.1).

LEMMA 4.5 (WAŻEWSKI INEQUALITY). *If $D \in M_n$ and there exists $a, b \in \mathbb{R}$ such that $a \leqslant \Re(\mu) \leqslant b$ for any eigenvalue μ of the matrix D, then there exists a base $\mathcal{B}$ of $\mathbb{R}^n$ such that*

$$a\|x\|_{\mathcal{B}}^2 \leqslant \langle Dx, x\rangle_{\mathcal{B}} \leqslant b\|x\|_{\mathcal{B}}^2,$$

where $\|x\|_{\mathcal{B}} = \sqrt{\langle x, x\rangle_{\mathcal{B}}}$.

Proof. See [5; Lemma 3.82]. □

LEMMA 4.6. *Let the reciprocal system have the form*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + Dy + \varepsilon f(y), \end{aligned} \tag{4.4}$$

where $x, y \in \mathbb{R}^n$, $f(y) = o(\|y\|) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, ε is a parameter and $D \in M_n$ such that $\Re(\mu) < -\alpha < 0$ for any eigenvalue μ of the matrix D. Then for any compact set $K \subset \mathbb{R}^{2n}$ the point $(0,0)$ is the attractor of the system (4.4) *for $\varepsilon \in \left(0, \varepsilon_0(K)\right)$ with respect to K, i.e. if $\left(x(t), y(t)\right)$ is a solution of* (4.4) *with $\left(x(0), y(0)\right) \in K$, then $\lim\limits_{n\to\infty} \left(x(t), y(t)\right) = (0,0)$.*

Proof. Let $r \in \mathbb{R}$ be such that $K \subset B(0,r) = \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \|x\|_{\mathcal{B}} + \|y\|_{\mathcal{B}} \leqslant r\right\}$. We will show that there exists such $C > 0$ that for y, $\|y\|_{\mathcal{B}} \leqslant r$, there holds

$$\|f(y)\|_{\mathcal{B}} \leqslant C\|y\|_{\mathcal{B}}, \qquad C > 0.$$

Since $f(y) = o(\|y\|) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$,

$$f_k(y) - f_k(0) = \int_0^1 \mathrm{d}f_k(ty)y \; \mathrm{d}t = \int_0^1 \sum_{i=1}^n \frac{\partial}{\partial y_i} f_k(ty) y_i \; \mathrm{d}t$$

$$= \sum_{i=1}^n y_i \int_0^1 \frac{\partial}{\partial y_i} f_k(ty) \; \mathrm{d}t , \qquad f_k(0) = 0 ,$$

for $k \in \{1, \dots, n\}$. Therefore we obtain

$$f_k(y) = \sum_{i=1}^n y_i \int_0^1 \frac{\partial}{\partial y_i} f_k(ty) \; \mathrm{d}t .$$

The functions

$$\int_0^1 \frac{\partial}{\partial y_i} f_k(ty) \; \mathrm{d}t , \qquad i, k \in \{1, \dots, n\} ,$$

are continuous and therefore there exist constants $C_i^k(r) > 0$ such that

$$\sup_{y \in B(0,r)} \int_0^1 \frac{\partial}{\partial y_i} f_k(ty) \; \mathrm{d}t \leqslant C_i^k(r) .$$

Using the previous inequality we obtain that

$$|f_k(y)| = \left| \sum_{i=1}^n y_i \int_0^1 \frac{\partial}{\partial y_i} f_k(ty) \; \mathrm{d}t \right|$$

$$\leqslant \sum_{i=1}^n |y_i| \left| \int_0^1 \frac{\partial}{\partial y_i} f_k(ty) \; \mathrm{d}t \right| \leqslant \sum_{i=1}^n |y_i| C_i^k(r)$$

$$\leqslant n \max_{i \in \{1,\dots,n\}} |y_i| \max_{i \in \{1,\dots,n\}} C_i^k(r) = \|y\|_{\max} n \max_{i \in \{1,\dots,n\}} C_i^k(r) ,$$

where $\| \cdot \|_{\max}$ is the maximum norm on $\mathbb{R}^n$. Let

$$C^k(r) = n \max_{i \in \{1,\dots,n\}} C_i^k(r) .$$

The equivalence of all norms on $\mathbb{R}^n$ yields the existence of constants $q_1 > 0$, $q_2 > 0$ such that

$$q_1 \|y\|_{\mathcal{B}} \leqslant \|y\|_{\max} \leqslant q_2 \|y\|_{\mathcal{B}}$$

and so we have

$$|f_k(y)| \leqslant \|y\|_{\max} C^k(r) \leqslant q_2 \|y\|_{\mathcal{B}} C^k(r)$$

for $k \in \{1, \dots, n\}$. Therefore

$$\|f(y)\|_{\mathcal{B}} \leqslant \frac{1}{q_1} \big\| \big(f_1(y), \dots, f_n(y)\big) \big\|_{\max} \leqslant \frac{q_2}{q_1} \|y\|_{\mathcal{B}} \max_{k \in \{1, \dots, n\}} C^k(r) .$$

If

$$C = \frac{q_2}{q_1} \max_{k \in \{1, \dots, n\}} C^k(r) ,$$

then

$$\|f(y)\|_{\mathcal{B}} \leqslant C \|y\|_{\mathcal{B}} .$$

Now we define the mapping

$$\begin{aligned} &V \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \\ &V(x, y) = \langle x, x \rangle_{\mathcal{B}} + \langle y, y \rangle_{\mathcal{B}} . \end{aligned}$$

Now we show that for an arbitrary compact set $K \subset \mathbb{R}^n \times \mathbb{R}^n$ there exists an $\varepsilon_0(K) > 0$ such that arbitrary solution $\varphi(t) = \big(x(t), y(t)\big)$ of the system (4.4) with $\varepsilon \in \big(0, \varepsilon_0(K)\big)$ and $\varphi(0) = (x_0, y_0) \in K$, is the function $V\big(\varphi(t)\big)$ decreasing. So the point $(0,0)$ is the attractor. Let us compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V\big(\varphi(t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t} V\big(x(t), y(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t} \langle x(t), x(t) \rangle_{\mathcal{B}} + \frac{\mathrm{d}}{\mathrm{d}t} \langle y(t), y(t) \rangle_{\mathcal{B}} \\ &= 2\big[\langle x(t), \dot{x}(t) \rangle_{\mathcal{B}} + \langle y(t), \dot{y}(t) \rangle_{\mathcal{B}}\big] \\ &= 2\big[\langle x(t), y(t) \rangle_{\mathcal{B}} + \langle y(t), -x(t) + Dy(t) + \varepsilon f(y(t)) \rangle_{\mathcal{B}}\big] \\ &= 2\big[\langle y(t), Dy(t) \rangle_{\mathcal{B}} + \langle y(t), \varepsilon f(y(t)) \rangle_{\mathcal{B}}\big] . \end{aligned}$$

Since $\Re(\mu) < -\alpha$ for each eigenvalue μ of the matrix D, using the Ważewski inequality we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V\big(\varphi(t)\big) &\leqslant 2\big[-\alpha \|y(t)\|_{\mathcal{B}}^2 + \langle y(t), \varepsilon f(y(t)) \rangle_{\mathcal{B}}\big] \\ &\leqslant 2\big[-\alpha \|y(t)\|_{\mathcal{B}}^2 + \|y(t)\|_{\mathcal{B}} \|\varepsilon f(y(t))\|_{\mathcal{B}}\big] \\ &\leqslant 2\big[-\alpha \|y(t)\|_{\mathcal{B}}^2 + \varepsilon C \|y(t)\|_{\mathcal{B}}^2\big] . \end{aligned}$$

This yields

$$\frac{\mathrm{d}}{\mathrm{d}t} V\big(\varphi(t)\big) < 0$$

for $\varepsilon < \frac{\alpha}{C}$, $t \in \mathbb{R}$, i.e the function $V\big(\varphi(t)\big)$ is decreasing on $\mathbb{R}$. □

LEMMA 4.7. *Let the reciprocal system have the form*

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -x + Dy + f(y)\,, \end{aligned} \tag{4.5}$$

where $x, y \in \mathbb{R}^n$, $f(y) = o(\|y\|) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and* $D \in M_n$ *such that* $\Re(\mu) < -\alpha < 0$ *for any eigenvalue* μ *of the matrix* D. *Then there exists* $r \in \mathbb{R}$ *such that, for the set* $B(0,r) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \|x\|_{\mathcal{B}} + \|y\|_{\mathcal{B}} < r\}$, *the point* $(0,0)$ *is the attractor.*

Proof. Define the mapping

$$\begin{aligned} &V\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\,, \\ &V(x,y) = \langle x, x\rangle_{\mathcal{B}} + \langle y, y\rangle_{\mathcal{B}}\,. \end{aligned}$$

We will show that there exists $r \in \mathbb{R}$ such that for arbitrary solution

$$\varphi(t) = \big(x(t), y(t)\big)$$

of (4.5) with $\varphi(0) = (x_0, y_0) \in B(0,r)$ the function $V(\varphi(t))$ is decreasing. In the proof of Lemma 4.6 we have shown that

$$\frac{\mathrm{d}}{\mathrm{d}t} V\big(x(t), y(t)\big) \leqslant 2\big[-\alpha\|y(t)\|_{\mathcal{B}}^2 + \|f(y(t))\|_{\mathcal{B}}\|y(t)\|_{\mathcal{B}}\big] \tag{4.6}$$

and also that

$$f_k(y) = \sum_{i=1}^{n} y_i g_i^k(y)\,, \qquad \text{where} \quad g_i^k(y) = \int_0^1 \frac{\partial}{\partial y_i} f_k(ty)\,\mathrm{d}t$$

is a continuous function for $i, j \in \{1, \dots, n\}$ and

$$g_i^k(0) = 0 \qquad \text{for} \quad i, j \in \{1, \dots, n\}\,.$$

Since $f = o(\|y\|)$, so

$$\frac{\partial}{\partial y_i} f_k(0) = 0\,,$$

and therefore

$$\int_0^1 \frac{\partial}{\partial y_i} f_k(0)\,\mathrm{d}t = 0\,.$$

From these properties of $g_i^k(y)$ there follows the existence of $r_i^k \in \mathbb{R}$ for an arbitrary $\varepsilon > 0$ such that

$$g_i^k(y) < \varepsilon \qquad \text{for all } y \text{ with } \|y\|_{\mathcal{B}} < r_i^k\,.$$

If

$$r = \min_{i,j\in\{1,\dots,n\}} r_i^k ,$$

then

$$g_i^k(y) < \varepsilon$$

for all $i,k \in \{1,\dots,n\}$ and all y with $\|y\|_{\mathcal{B}} < r$. Similarly as in the proof of Lemma 4.6 one can show that

$$\|f(y)\|_{\mathcal{B}} < \frac{q_1}{q_2}\varepsilon\|y\|_{\mathcal{B}}$$

for all y with $\|y\|_{\mathcal{B}} < r$. Using (4.6) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V\big(x(t),y(t)\big) \leqslant 2\Big[-\alpha\|y(t)\|_{\mathcal{B}}^2 + \frac{q_1}{q_2}\varepsilon\|y(t)\|_{\mathcal{B}}^2\Big]$$

for all y with $\|y\|_{\mathcal{B}} < r$. If

$$\varepsilon < \frac{|\alpha|q_2}{q_1},$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}V\big(x(t),y(t)\big) < 0 \qquad \text{for all} \quad t\in\mathbb{R}.$$

Therefore the point $(0,0)\in\mathbb{R}^n\times\mathbb{R}^n$ is the attractor of the set $B(0,r)$. □

Now we will study more general reciprocal systems.

LEMMA 4.8. *Let the reciprocal system have the form*

$$\begin{aligned} \dot{x} &= By\,,\\ \dot{y} &= -B^*x + Dy + f(y)\,, \end{aligned} \tag{4.7}$$

where $x,y\in\mathbb{R}^n$, $B,D\in M_n$ are regular matrices and $f(y) = o\big(\|y\|\big)$. If there does not exist $w\in\mathbb{R}^n$ with $\|w\| = 1$ such that $\langle w, Dw\rangle = 0$ and $\langle w, B^*Bw\rangle < 0$, then the point $(0,0)\in\mathbb{R}^n\times\mathbb{R}^n$ is the hyperbolic equilibrium point of the system (4.7).

Proof. We have to show that the Jacobi matrix of the system (4.7) at the point $(0,0)\in\mathbb{R}^n\times\mathbb{R}^n$ has not eigenvalues with zero real parts. The Jacobi matrix has the form

$$\begin{pmatrix} 0 & B\\ -B^* & D\end{pmatrix},$$

and since B,D are regular matrices, it is clear that

$$\det\begin{pmatrix} 0 & B\\ -B^* & D\end{pmatrix} \neq 0\,.$$

Therefore $\lambda = 0$ is not an eigenvalue of the Jacobi matrix and thus the matrix

$$\begin{pmatrix} E & 0 \\ \frac{1}{\lambda}B^* & E \end{pmatrix}$$

is well defined and regular for any eigenvalue λ of the Jacobi matrix. The equation

$$\det \begin{pmatrix} -\lambda E & B \\ B^* & D - \lambda E \end{pmatrix} = 0$$

is equivalent to the equation

$$\det \begin{pmatrix} E & 0 \\ \frac{1}{\lambda}B^* & E \end{pmatrix} \begin{pmatrix} -\lambda E & B \\ B^* & D - \lambda E \end{pmatrix} = 0 .$$

After some calculations we can see that it is equivalent to the equation

$$\det\bigl(\lambda^2 E - \lambda D - B^* B\bigr) = 0 .$$

The number λ is a solution of this equation if and only if there exists $w \in \mathbb{R}^n$, $\|w\| \neq 0$, such that

$$\bigl(\lambda^2 E - \lambda D - B^* B\bigr) w = 0 . \tag{4.8}$$

We can choose w with $\|w\| = 1$. From (4.8) it follows that

$$\bigl\langle w, (\lambda^2 E - \lambda D - B^* B) w \bigr\rangle = \lambda^2 \langle w, w\rangle - \lambda \langle w, Dw\rangle - \langle w, B^* B w\rangle = 0 ,$$

i.e.

$$\lambda^2 - \lambda \langle w, Dw\rangle - \langle w, B^* B w\rangle = 0 . \tag{4.9}$$

The eigenvalues of the Jacobi matrix have to satisfy (4.9) for some $w \in \mathbb{R}^n$ with $\|w\| = 1$. If there exists an eigenvalue with zero real part, then there exists a $w \in \mathbb{R}^n$ with $\|w\| = 1$ such that

$$\langle w, Dw\rangle = 0$$

and also

$$\langle w, B^* B w\rangle < 0 .$$

It is a contradiction with the assumption. □

LEMMA 4.9. *Let the reciprocal system have the form* (4.7). *If the matrix D is negatively definite* (D *is positively definite*) *and*

$$\langle w, Dw\rangle^2 - 4\langle w, B^* B w\rangle \leqslant 0$$

for an arbitrary $w \in \mathbb{R}^n$ with $\|w\| = 1$, then the point $(0,0) \in \mathbb{R}^n \times \mathbb{R}^n$ is the sink (*source*) *of this system.*

Proof. From the proof of Lemma 4.8 it follows that eigenvalues of the Jacobi matrix satisfy equation (4.9) and so

$$\lambda_{1,2} = \frac{\langle w, Dw\rangle \pm \sqrt{\langle w, Dw\rangle^2 - 4\langle w, B^* B w\rangle}}{2} , \tag{4.20}$$

where $w \in \mathbb{R}^n$, $\|w\| = 1$. Therefore, if the assumptions are satisfied, then all eigenvalues of Jacobi matrix are negative (positive). Therefore the point $(0,0) \in \mathbb{R}^n \times \mathbb{R}^n$ is the sink (source) of this system. □

5. Pseudogradient vector fields on manifolds

In this part we will use the notification and results from differential topology, which can be found in [3; Chap. 1]. For convenience of reader we just recall some definitions.

DEFINITION 5.1. A vector field $F \in V^r(X)$, where $0 \leqslant r \leqslant \infty$, is called the *gradient vector field* if there exists a mapping $G\colon X \to \mathbb{R}$ such that $F(x) = \nabla G(x)$ for all $x \in X$.

Denote by $L(T_x(X), T_x(X))$ the set of all linear mappings of $T_x(X)$ to $T_x(X)$ and

$$L(X,X) = \bigcup_{x \in X} L(T_x(X), T_x(X)).$$

DEFINITION 5.2. If $A \in L(T_x(X), T_x(X))$. Then A is called *positively (negatively) definite* if $\langle v, Av\rangle_x > 0$ ($\langle v, Av\rangle_x < 0$) for $v \in T_x(X)$, $v \neq 0_x$.

DEFINITION 5.3. A vector field $F \in V^r(X)$, where $0 \leqslant r \leqslant \infty$, is called the *pseudogradient* if there exists $A \in L(X,X)$ and a mapping $G\colon X \to \mathbb{R}$ such that $F(x) = A(x)\nabla G(x)$ for all $x \in X$.

Now we can formulate theorem about the non existence of periodic trajectories of a pseudogradient vector field on a manifold.

THEOREM 5.1. *Let X be a smooth manifold and $F \in V_r(X)$, where $0 \leqslant r \leqslant \infty$, is a pseudogradient vector field such that $F(x) = A(x)\nabla G(x)$, where $A(x) \in L(T_x(X), T_x(X))$ is positively definite for all $x \in X$. Then the vector field F does not have periodic trajectories.*

Proof. Let $\phi\colon \mathbb{R} \to X$ be a periodic integral curve passing through the point $x_0 \in \mathbb{R}$, i.e.

$$\begin{aligned} \phi(0) &= x_0, \\ \mathrm{D}_t\,\phi(e_t) &= A(\phi(t))\nabla G(\phi(t)) \qquad \text{for all} \quad t \in \mathbb{R}. \end{aligned}$$

We will show that the mapping $G \circ \phi\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing. Let us compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[G \circ \phi](t) &= \big(\mathrm{D}_{\phi(t)}\, G \circ D_t\phi\big)(e_t) \\ &= \mathrm{D}_{\phi(t)}\, G\big(A(\phi(t))\nabla G(\phi(t))\big) \\ &= \big\langle \nabla G\big(\phi(t)\big), A\big(\phi(t)\big)\nabla G\big(\phi(t)\big)\big\rangle_{\phi(t)} \geqslant 0 \qquad \text{for all} \quad t \in \mathbb{R}. \end{aligned} \tag{5.1}$$

Thus the mapping $G \circ \phi\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing.

Since ϕ is a periodic integral curve, there exists $t_0 \in \mathbb{R}$ such that

$$\phi(0) = \phi(t_0)\,.$$

Thus

$$G\big(\phi(0)\big) = G\big(\phi(t_0)\big)\,.$$

Therefore $G \circ \phi$ is nondecreasing and we obtain

$$G\big(\phi(0)\big) = G\big(\phi(t)\big) \qquad \text{for} \quad 0 \leqslant t \leqslant t_0\,.$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t}[G \circ \phi](t) = 0\,,$$

i.e.

$$\big\langle \nabla G\big(\phi(t)\big), A\big(\phi(t)\big)\nabla G\big(\phi(t)\big)\big\rangle_{\phi(t)} = 0 \qquad \text{for} \quad 0 \leqslant t \leqslant t_0\,. \tag{5.2}$$

Form (5.2) and the fact that A is positively definite it follows that

$$\nabla G\big(\phi(t)\big) = 0 \qquad \text{for} \quad 0 \leqslant t \leqslant t_0\,.$$

Therefore

$$F(x_0) = A(x_0)\nabla G(x_0) = 0\,,$$

and so the point x_0 is an equilibrium point and $\phi(t) = x_0$ for all $t \in \mathbb{R}$. Now it is clear that $\phi(t)$ is not periodic integral curve. □

COROLLARY 5.1. *Let X be a smooth manifold and $F \in V_r(X)$, where $0 \leqslant r \leqslant \infty$, is a pseudogradient vector field such that $F(x) = A(x)\nabla G(x)$, where $A(x) \in L\big(T_x(X), T_x(X)\big)$ is negatively definite for all $x \in X$. Then the vector field F does not have periodic integral curves.*

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Received April 10, 2003

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