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Book Reviews

Mathematica Slovaca, Vol. 53 (2003), No. 4, 427--431

Persistent URL: http://dml.cz/dmlcz/136891

Math. Slovaca, **53** (2003), No. 4, 427–431

Mathematica Slovaca

BOOK REVIEWS

Faure, C.-A. — Frölicher, A.:
MODERN PROJECTIVE GEOMETRY.
Mathematics and its Applications, Vol. 521.
Kluwer Academic Publishers, Dordrecht 2000, 384 pp.
ISBN 0-7923-6525-9

The purpose of this excellent book is the presentation of modern aspects and some recent results which are mainly due to the study of morphisms. The reason why morphisms have not yet been studied much earlier is probably the fact that they are in general partial maps between two point sets. By defining appropriate morphisms between lattices, the well-known isomorphism between projective geometries and lattices of their subspaces is extended to a categorical equivalence. A projective geometry is also determined by the closure operator that associates to an arbitrary set of points the smallest subspace containing it. Defining appropriate morphisms between closure spaces, also this correspondence extends to a categorical equivalence.

The book is accessible for readers with some knowledge of linear algebra and partially ordered sets. The book consists of fourteen chapters. At the end of each chapter there is a section with exercises. Some of them require only the application of results given in the preceding sections, but others introduce additional notions and form complements to the chapter. At the end of the book a list of some open problems can be found.

Chapter 1: **Fundamental Notions of Lattice Theory**
collets some basic facts about complete lattices, atomic and atomistic lattices, meet-continuous lattices, modular and semi-modular lattices, complemented lattices and maximal chains. Zorn's lemma is formulated without proof.

Chapter 2: **Projective Geometries and Projective Lattices**
presents two equivalent characterizations of projective geometries. The first one uses the ternary relation ℓ, "collinear", on the set of points. The other one works with the operator that associates to each couple of points a, b the line through a, b if $a \neq b$ and the singleton $\{a\}$ if $a = b$. The notion of a subspace is introduced. Several examples are presented, the most important among them is the projective geometry associated to a vector space V. The system of all subspaces is characterized as a complete, atomistic, meet-continuous, modular and complemented lattice. In agreement with the well-known Menger-Birkhoff correspondence, the lattice with the latter properties is called a projective lattice. The notions of sub-geometries and quotient geometries (with respect to a subspace) are studied. It is shown that a projective geometry is irreducible if and only if every line contains at least three points.

Chapter 3: **Closure Spaces and Matroids**
shows that a projective geometry G can be described as a set together with a closure operator $C\colon \mathcal{P}(G) \to \mathcal{P}(G)$ satisfying a system of axioms. A system of six axioms is given, the first two, (C1) and (C2), characterize closure spaces. The other two (C3) and (C4) are relevant to a dimension theory. A set M with such a closure operator is called a matroid, and a geometry

if an additional axiom (C5) is also satisfied. Finally, to get a projective geometry, so-called projective axiom (C6) needs to be added. This yields a third characterization of projective geometries. The quotients of closure spaces are described as a generalization of quotients of projective geometries. Also two isomorphism theorems available for projective geometries are generalized, namely $x/F \simeq (x/E)/(x/F)$ and $(E \vee F)/E \simeq F/(E \wedge F)$, for the latter modularity of the interval $[E \wedge F, E \vee F]$ is essential.

In *Chapter 4*: **Dimension Theory**,
the notion of a basis of a subspace of a matroid is defined, which generalizes the notion of linearly independent subset of a vector space. It is shown that every subspace of a matroid admits a basis. Using Zorn's lemma, a transfinite version of the Steinitz Exchange Theorem is proved, which implies that all bases are equipotent, and allows to define the rank of a subspace as the cardinal number of any of its bases.

For a vector space the rank of an n-dimensional vector subspace is equal to n, while for a projective geometry or an affine geometry the rank of an n-dimensional subspace is equal to $n+1$. The important theorems are anyhow invariant if the rank is modified by an additive constant. It is shown that the rank provides a distance function on the lattice of subspaces. The classical dimension theorem of projective geometries says that for any subspaces E, F, $r(E \vee F) + r(E \wedge F) = r(E) + r(F)$. For an affine geometry this equation fails in general, but holds for all non-disjoint subspaces. Fourteen equivalent conditions are given. It is shown that for a geometry the following properties are equivalent: the validity of the rank equation for any subspaces, the projective law, the modularity of the lattice of subspaces, the validity of the second isomorphism theorem, and the property that the geometry is projective. The co-rank $\bar{r}(E)$ of a subspace E is defined as the rank of the quotient geometry M/E. Hyper-planes of the projective geometries are characterized in several equivalent manners.

Chapter 5: **Geometries of Degree n**
introduces the following definition: A geometry is of degree n if for any subspaces E, F the rank equation $r(E \vee F) + r(E \wedge F) = r(E) + r(F)$ holds provided that $r(E \wedge F) \geq n$. The geometries of degree 0 are exactly the projective geometries. A geometry of degree n is trivially of degree $n+1$. Within the geometries of degree 1, the projective geometries are characterized by the requirement that parallel lines must be equal, the affine geometries by an axiom requiring that for any line and any point there exists a parallel line containing the point. Möbius geometries are characterized as certain geometries of degree 2. The points of the unit sphere of any pre-Hilbert space form an example of a Möbius geometry. Relations between affine and projective geometries are shown. It is shown that every geometry of degree 1 and rank at least 5 can be embedded into a projective geometry.

Chapter 6: **Morphisms of Projective Geometries**
starts with some generalities on the so-called partial maps. A partial map $f\colon X \to Y$ is a map f from a subset of X, called the domain of f, to Y. The points of X which are not in the domain, form the kernel of f. The sets together with the partial maps form a category denoted by Par. A morphism of projective geometries is defined as a partial map $g\colon G_1 \to G_2$ satisfying simple geometric axioms. Several equivalent definitions are given. It is shown that for a non-constant morphism, the domain cannot be enlarged within G_1. A map between vector spaces $f\colon V \to W$ is called semi-linear if it is additive and satisfies $f(\lambda x) = \sigma(\lambda) f(x)$ for some field homomorphism σ. These mappings are those which induce morphisms $\mathcal{P}(V) \to \mathcal{P}(W)$.

The projective geometries together with their morphisms form a category Proj. The vector spaces together with semi-linear maps form a category Vec, and there is a natural functor $\mathcal{P}\colon$ Vec $\to$ Proj.

Special morphisms, called homomorphisms, are stable under compositions and hence yield a full subcategory Proj_H of Proj.

For the case of a morphism $O_f: \mathcal{P}(V) \to \mathcal{P}(W)$ which is induced by a semi-linear map f with respect to σ, we obtain the result that if f is quasi-linear, i.e., σ is an isomorphism of fields, then $\mathcal{P}f$ is a homomorphism. The converse holds if $\mathcal{P}f$ is nonconstant.

In *Chapter 7*: **Embeddings and Quotient Maps**,
some categorical properties of the category Proj are examined. Sources are considered as families of morphisms of the form $g_v: G \to G_v$, where G does not depend on the index v. Mono-sources and initial sources are considered. Initial sources have the property that a partial map $h: G' \to G$ between two projective geometries is a morphism if and only if $g_v \circ h$ is a morphism for all v.

Monomorphisms and initial morphisms (called embeddings) are introduced as a special cases of morphisms. For an embedding $G' \to G$ the dimension of G' can be greater than that of G. This does not happen for the smaller class of proper embeddings. Still smaller is the class of subspace-embeddings which are inclusions of subspaces (up to an isomorphism). Dual notions of sources, episinks, final sinks and quotient maps are also considered.

Projections of G are homomorphisms $p: G \to G$ satisfying $p \circ p = p$. It is shown that for any projection p the sets $E := \ker p$ and $F := \operatorname{Im} p$ are complementary subspaces and any couple of complementary subspaces E, F are of this form for a unique projection p. In the last section, two so-called factorization systems of the category Proj are described. They imply that every morphism factors in three morphisms of special classes and this decomposition is unique up to isomorphism.

Chapter 8: **Endomorphisms and the Desargues Property**.
It is well known that the Desargues property for a projective geometry G is equivalent with the existence of certain collineations of G. The respective collineations have an axis H (which is a hyperplane of G such that $\phi x = x$ for all $x \in H$), and a center z (which is a point of G such that $\ell(z, x, \phi x)$ for all $x \in G$). The notions axis and center are generalized for the case where $\phi: G \to G$ is any endomorphism. It is shown that an endomorphism ϕ is completely determined by its axis H, its center z and the image $a' = \phi a$ of only one point $a \in \operatorname{Dom} \phi \setminus (H \cup z)$.

A collineation ϕ is called a translation (with axis H) if it has a center in H, and a homothety (with axis H) if it has a center in $G \setminus H$. The translations with axis H form a subgroup T_H of the set C_H of all collineations with axis H.

An irreducible projective geometry is called arguesian with respect to H if $\dim G \geq 2$ and if for any point $z \in G$, $A \in G \setminus (H \cup z)$ and $b \in a * z$ (where $a * z$ denotes the line joining a and z) there exists an endomorphism of G with axis H and center z such that $\phi a = b$. G is called arguesian if it is arguesian for every hyperplane H. It is known that every irreducible projective geometry of dimension ≥ 3 is arguesian. It is shown that G is arguesian if and only if the classical Desargues property (involving pairs of triangles) is satisfied.

Chapter 9: **Homogeneous coordinates**.
On an arguesian projective geometry G, homogeneous coordinates are introduced by means of the homothety group $\mathcal{H}_H^o$ with axis H and an endomorphism α_o with kernel H and constant value o. The homothety group $\mathcal{H}_H^o$ is actually the multiplicative group of a field, on which a vector space V_H^o (or briefly V) can be constructed such that there is an isomorphism $u: \mathcal{P}(V) \to H$. One says that u yields homogeneous coordinates for H. All the vector spaces V_H^o are isomorphic. Conversely, if one has homogeneous coordinates $u: \mathcal{P}(V) \to H$ for a projective geometry H, then by composing it with the inclusion map $i: \mathcal{P}(V) \to V \dot{\cup} \mathcal{P}(V)$ one gets a hyperplane embedding $i \circ u^{-1}$ of H with $\{0\}$ as the complement of the image of H. For an irreducible projective geometry G of dim ≥ 2, the following conditions are equivalent: G is arguesian with respect to some hyperplane H, G is arguesian, G admits a hyperplane embedding. The last section deals with the well-known fact that for an arguesian geometry

the following conditions are equivalent: the classical property of Pappus holds, the homothety fields are commutative, there exist homogeneous coordinates by means of a vector space over a commutative field.

Chapter 10: **Morphisms and Semi-linear Maps**
deals with the Fundamental Theorem of projective geometry: Every non-degenerate morphism g between arguesian geometries is described in homogeneous coordinates by a semi-linear map f. For given coordinates f is unique up to a non-zero constant factor. Moreover, f is quasi-linear if and only if g is a homomorphism. Non-degenerate means that $\operatorname{Im} g$ contains three non-collinear points.

All morphisms between arguesian geometries which are described by semi-linear maps (so-called arguesian morphisms) may be characterized as those that are composites of finitely many non-degenerate morphisms between arguesian geometries. Together with arguesian geometries as objects they form a category Arg. If Vec 3 denotes the category having vector spaces of dimension ≥ 3 as objects and the semi-linear maps as morphisms, one gets a full and dense functor $\mathcal{P}$: Vec 3 $\to$ Arg. Some applications of the Fundamental Theorem are given.

Chapter 11: **Duality**
starts with some elementary results on duals of vector spaces. Then the dual G^* of a projective geometry G is defined. It is shown that if G is irreducible and arguesian, the same holds for G^*. The notion of pairing between two projective geometries G_1 and G_2 is introduced in terms of two partial maps $g_1 : G_1 \to G_2^*$ and $g_2 : G_2 \to G_1^*$ satisfying three simple conditions. It turns out that g_1 and g_2 determine each other and they are homomorphisms. A pairing is called duality if one requires that g_1 and g_2 have empty kernels. The couple $J: G \to G^{**}$, where J is the canonical embedding, and $\mathrm{Id}_{G^*} : G^* \to G^*$ form a duality between G and G^*. The general results for dualities imply that J is a subspace embedding which is an isomorphism if $\dim G < \infty$.

The map $G \to G^*$ can be extended to a contravariant functor $*$: $\mathrm{Proj}\,H \to \mathrm{Proj}\,H$, where $\mathrm{Proj}\,H$ is the category of projective geometries together with homomorphisms. By means of the Fundamental Theorem it is shown that pairings between arguesian geometries are described in homogeneous coordinates by sesqui-linear forms and dualities by non-singular sesqui-linear forms.

Chapter 12: **Related Categories**
establishes a correspondence between projective geometries and projective lattices, which can be extended to an equivalence of categories. This equivalence, moreover, is the restriction of a more general one, between simple closure spaces and complete lattices.

In the second part of this chapter, morphisms between affine geometries are studied. The main result is an improved version of the so-called fundamental theorem of affine geometry, which characterizes non-degenerate morphisms between two vector spaces, considered as affine geometries.

In *Chapter 13*: **Lattices of Closed Subspaces**,
projective geometries with an additional structure consisting of a set of distinguished subspaces, called the closed subspaces, are considered. A motivation comes from topological vector spaces, where closed subspaces of the associated projective geometry are subspaces in the form $\mathcal{P}(W)$, where W is a closed subspace. Projective geometries with the property that if E is a closed subspace, then also $a \vee E$ is a closed subspace, are so-called Mackey geometries. Any topological vector space over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ fulfils this condition. A correspondence between Mackey geometries and two different types of lattices is established:

(1) the projective lattices equipped with an appropriate closure operator,
(2) the complete atomistic lattices which are upper and lower semi-modular.

This correspondence is extended into an equivalence of three categories. A particular case of dualized geometries is studied, where the set of closed subspaces is determined by the set of closed hyperplanes.

Chapter 14: **Orthogonality**
deals with a particular case of dualized geometries, when the set of closed hyperplanes is given by a polarity. The so-obtained orthogeometries can be equivalently described by an orthogonality relation $\perp$. A typical example is the projective geometry associated to a vector space equipped with a nonsingular reflexive sequi-linear form. Moreover, one can choose either an alternating bilinear form or a hermitian form. The two types of lattices that correspond to orthogeometries are called orthosystems and ortholattices, respectively. Orthogonal morphisms between orthogeometries are studied, a typical example is the morphism induced by an orthogonal sesqui-linear map. Under some additional assumptions this semi-linear map is a quasi-linear isometry. This yields a generalization of the well-known theorem of Wigner for pre-Hilbertian spaces. A well-known result on Hilbertian spaces says that a pre-Hilbertian space V is a Hilbertian space if and only if for every vector subspace $W \subset V$ with $W = W^{\perp\perp}$ one has $V = W + W^{\perp}$. An ortholattice satisfying this additional property is called a Hilbertian lattice. These lattices together with the corresponding Hilbertian geometries and propositional systems (i.e., orthosystems satisfying the orthomodular law) are described in the final section.

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