Ali Reza Ashrafi; Yao Qing Zhao

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ON 5- AND 6-DECOMPOSABLE FINITE GROUPS

Ali Reza Ashrafi* — Zhao Yaoqing**

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ABSTRACT. A finite group G is called n-decomposable if it is non-simple and each of its non-trivial proper normal subgroups is a union of n distinct conjugacy classes. In this paper, we investigate the structure of non-solvable non-perfect finite group G when G is 5- or 6-decomposable. We prove that G is 5-decomposable if and only if G is isomorphic with $Z_5 \times A_5$, $A_6 \cdot 2_3$ or $\mathrm{Aut}(\mathrm{PSL}(2,q))$ for $q = 7,8$. Also, G is 6-decomposable if and only if G is isomorphic with S_6 or $A_6 \cdot 2_2$. Here, $A_6 \cdot 2_2$ and $A_6 \cdot 2_3$ are non-isomorphic split extensions of the alternating group A_6, in the small group library of GAP [SCHONERT, M. et al.: *GAP, Groups, Algorithms and Programming.* Lehrstuhl für Mathematik, RWTH, Aachen, 1992].

1. Introduction and preliminaries

Let G be a finite group and let $\mathcal{N}_G$ be the set of non-trivial proper normal subgroups of G. An element K of $\mathcal{N}_G$ is said to be n-*decomposable* if K is a union of n distinct conjugacy classes of G. If $\mathcal{N}_G \neq \emptyset$ and every element of $\mathcal{N}_G$ is n-decomposable, then we say that G is n-decomposable.

In [14], Shahryari and Shahabi, independent from Shi and Jing, investigated the structure of finite groups which contain a 2-decomposable subgroup H. In this case, $H \leq G'$, $|H|\big(|H|-1\big)$ divides $|G|$ and H is an elementary abelian normal subgroup of G. Moreover, they proved that, under certain conditions, G is a Frobenius group with kernel H.

In this connection, one might ask about the structure of G if G contains a 3- or 4-decomposable subgroup. In [15], Shahryari and Shahabi studied the structure of finite groups G with a 3-decomposable subgroup H. They proved

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that H is either an elementary abelian subgroup, a metabelian p-group or a Frobenius group with elementary abelian kernel H'. Finally, in [10], Riese and Shahabi determined the structure of finite groups G with a 4-decomposable subgroup.

In [19], Wujie Shi defined the notion of *complete normal subgroup* of a finite group, which we call 2-*decomposable*, and obtained several results about the structure of these groups. In [20], Wang Jing, a student of Wujie Shi, continued his work and defined the notion of *sub-complete normal subgroup* of a group G, which we call 3-*decomposable*, and obtained several results about these groups. The authors wish to express here their gratitude to professor Wujie Shi for pointing out several exact references about this subject.

In [1], Ashrafi and Sahraei characterized the structure of 2-, 3- and 4-decomposable finite groups. Also, they obtained the structure of solvable n-decomposable finite groups. In this paper we continue the study of this problem and classify the non-solvable non-perfect 5- and 6-decomposable finite groups. To do this, we need some deep results of Wujie Shi and Wenze Yang in the field of the quantitative structure of finite groups ([16]). For the motivation of this problem and background material, the reader is encouraged to consult [17] and its references.

Let G be a group. Denote by $\pi_e(G)$ the set of all orders of elements in G. Following Wujie Shi [17], a finite group G is called *EPO-group* if every non-identity element of G has prime order. In [16], Wujie Shi and Wenze Yang discussed finite EPO-group and got an interesting result:

THEOREM 1. (Wujie Shi and Wenze Yang [16]) *The characteristic property of A_5 is:*

(1) *the order of the group contains at least three different prime factors,*
(2) *the order of every non-identity element in the group is a prime.*

COROLLARY. *If G is a non-abelian finite simple group and the order of every non-identity element of G is prime, then G is isomorphic with A_5.*

For the sake of completeness, we give another proof for the previous corollary independent from Theorem 1. This is a proof we received from Professor Victor Danilovich Mazurov in a private communication. The authors wish to thank him for sending this proof. Also, we are very grateful to Professor Wujie Shi for pointing out the Theorem 1 and its reference.

Proof of the Corollary. Let G be a finite simple non-abelian group such that every non-trivial element of G is of prime order. By Feit-Thompson theorem, the order of G is even and a Sylow 2-subgroup of G is elementary abelian. Moreover, the centralizer of every element of order 2 in G is elementary abelian too. Then, by Brauer-Suzuki-Wall theorem, G is isomorphic with

$\mathrm{PSL}(2,q)$, $q = 2^m$, $m > 1$. This group contains cyclic subgroups of order $q-1$ and $q+1$. By assumption, $q-1$ and $q+1$ are primes. On the other hand, one of these numbers is divisible by 3, so $3 = q \pm 1$, i.e. $q = 4$. Thus G is isomorphic with $\mathrm{PSL}(2,4)$ of order 60. □

Let G be a finite simple group and set $\pi(G) = \{p : p \text{ is a prime and } p \mid |G|\}$. Following D. Gorenstein, a finite simple group G is called a K_3-group if $|\pi(G)| = 3$. For the sake of completeness we mention below the following theorem of Herzog on the structure of simple K_3-groups.

THEOREM 2. (Marcel Herzog [6]) *If G is a simple K_3-group, then G is isomorphic with one of the simple groups A_5, A_6, $U_3(3)$, $U_4(2)$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$ and $\mathrm{PSL}(3,3)$.*

Following [17], we divide the set $\pi_e(G)$ into $\{1\}$, the set $\pi_e'(G)$ consisting of primes and the set $\pi_e''(G)$ consisting of composite numbers. We now state an important result of Shi and Yang, which we will use it in Theorem 6.

THEOREM 3. (Wujie Shi and C. Yang [18]) *Let G be a finite simple group with $|\pi_e''(G)| \le 1$. Then G is one of the following groups:*

(1) Z_p, *p prime,*
(2) $\mathrm{PSL}(2,q)$, $q = 5,7,8,9,11,13$ *or* 16,
(3) $\mathrm{PSL}(3,4)$, $\mathrm{Sz}(8)$,
(4) $\mathrm{PSL}(2,3^n)$, *where* $\frac{3^n-1}{2}$ *and* $\frac{3^n+1}{4}$ *are primes,*
(5) $\mathrm{PSL}(2,2^n)$, *where* 2^n-1 *and* $\frac{2^n+1}{3}$ *are primes.*

Finally, we state a result of [1], which will be used later.

THEOREM 4. (Ashrafi and Sahraei [1]) *Let G be a non-abelian n-decomposable finite group. Then we have:*

(i) *every element of $\mathcal{N}_G$ is maximal and also minimal in $\mathcal{N}_G$,*
(ii) *G is centreless, or n is a prime number and $|Z(G)| = n$,*
(iii) *if K and L are two distinct elements of $\mathcal{N}_G$, then $G = K \times L$,*
(iv) *if K is a solvable element of $\mathcal{N}_G$, then it is elementary abelian,*
(v) *if every element of $\mathcal{N}_G$ is solvable, then $\mathcal{N}_G$ consists of only one element,*
(vi) *G is solvable if and only if G' is abelian; in such a case, $\mathcal{N}_G = \{G'\}$, $G' \cong E(p^r)$, an elementary abelian group of order p^r, and is maximal in G, G is a Frobenius group with kernel G' and its complement is a cyclic group of prime order q with $p^r - 1 = (n-1)q$.*

Throughout this paper, as usual, G' denotes the derived subgroup of G, $Z(G)$ is the centre of G, x^G, $x \in G$, denotes the conjugacy class of G with the representative x, and G is called non-perfect if $G' \neq G$. Also, $\mathrm{SmallGroup}(n,i)$ denotes the ith group of order n in the small group library of GAP, [13]. All groups considered are assumed to be finite. Our notation is standard and taken mainly from [3], [4] and [8].

2. Main results and theorems

The aim of this section is to classify the 5- and 6-decomposable non-solvable non-perfect finite groups. To do this, we need to refer to the conjugacy classes of the projective special linear groups $\mathrm{PSL}(2,2^n)$ and $\mathrm{PSL}(2,3^n)$.

In [4; Chap. 2], Collins determines the conjugacy classes of the group $\mathrm{PSL}(2,2^n)$. He proved that the conjugacy classes of this group are:

(i) $\{1\}$,
(ii) one conjugacy class of involutions,
(iii) $\frac{1}{2}(q-2)$ conjugacy classes of elements of orders dividing $q-1$,
(iv) $\frac{1}{2}q$ conjugacy classes of elements of order dividing $q+1$.

In [3; Chap. 20], Berkovich and Zhmud determines the conjugacy classes of the group $\mathrm{SL}(2,q)$ for odd prime powers q. In the following Lemma, using similar methods, we determine the conjugacy classes of the projective special linear group $\mathrm{PSL}(2,q)$ in which $q=3^n$, $\frac{q-1}{2}$ and $\frac{q+1}{4}$ are primes. We need the conjugacy classes of this group for the classification of non-solvable 6-decomposable finite groups.

Lemma 1. *The group $G=\mathrm{PSL}(2,q)=\frac{\mathrm{SL}(2,q)}{Z(\mathrm{SL}(2,q))}$ has exactly $\frac{q-1}{2}+3$ conjugacy classes, as follows:*

(i) $\{Z\}$;
(ii) $(a^iZ)^{\mathrm{PSL}(2,q)}$, $1\le i\le \frac{q-3}{4}$ *of length* $q(q+1)$;
(iii) $\big(b(0,\tau)Z\big)^{\mathrm{PSL}(2,q)}$ *of length* $\frac{1}{2}q(q-1)$;
(iv) $\big(b(\sigma,\tau)Z\big)^{\mathrm{PSL}(2,q)}$ *of length* $q(q-1)$,
(v) $(cZ)^{\mathrm{PSL}(2,q)}$ *and* $(dZ)^{\mathrm{PSL}(2,q)}$ *of length* $\frac{1}{2}(q^2-1)$,

in which,

$$a=\begin{pmatrix} v & 0\\ 0 & v^{-1}\end{pmatrix},\quad c=\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix},\quad d=\begin{pmatrix} 1 & 0\\ v & 1\end{pmatrix},\quad b(\sigma,\tau)=\begin{pmatrix} \sigma & \tau\\ \varepsilon_0\tau & \sigma\end{pmatrix},$$

v denotes a generator of the multiplicative group $GF(q)^\star$, $Z=Z\big(\mathrm{SL}(2,q)\big)$, $\sigma^2-\varepsilon_0\tau^2=1$, and $\varepsilon_0\in GF(q)-GF(q)^2$ is arbitrary.

Proof. First of all, we can see that cZ and dZ have order 3 and are not conjugate in $\mathrm{PSL}(2,q)$. Furthermore, $C_{\mathrm{PSL}(2,q)}(cZ)=C_{\mathrm{PSL}(2,q)}(dZ)=\frac{C_{\mathrm{SL}(2,q)}(c)}{Z}$ and by [3; p. 138, Lemma 4], $C_{\mathrm{SL}(2,q)}(c)$ is the set of matrices

$\begin{pmatrix} \pm1 & 0 \\ \gamma & \pm1 \end{pmatrix}$, where γ runs over the field $GF(q)$. This shows that $|(cZ)^{\mathrm{PSL}(2,q)}| = |(dZ)^{\mathrm{PSL}(2,q)}| = \frac{1}{2}(q^2-1)$.

We now suppose that $xZ \in \langle aZ\rangle$ and $(xZ)^2 \neq Z$. Then with tedious calculations, we can show that a^iZ, $1 \le i \le \frac{q-3}{4}$, has order $\frac{q-1}{2}$ and $C_{\mathrm{PSL}(2,q)}(xZ) = \langle aZ\rangle$. Moreover, if $1 \le i,j \le \frac{q-3}{4}$ are distinct, then a^iZ and a^jZ are non-conjugate elements of the group $\mathrm{PSL}(2,q)$ and we can see that $|(a^iZ)^{\mathrm{PSL}(2,q)}| = q(q+1)$.

Finally, we consider the elements $b(\sigma,\tau)$ of $\mathrm{SL}(2,q)$ and determine the conjugacy class $\big(b(\sigma,\tau)Z\big)^{\mathrm{PSL}(2,q)}$. We define:

$$D_0 = \{b(\sigma,\tau) : \ \sigma,\tau \in GF(q) \text{ and } \sigma^2 - \varepsilon_0\tau^2 = 1\},$$
$$D_1 = \{c(\sigma,\tau) : \ \sigma,\tau \in GF(q) \text{ and } \sigma^2 - \varepsilon_0\tau^2 = -1\},$$

where $c(\sigma,\tau) = \begin{pmatrix} \sigma & \tau \\ -\tau\varepsilon_0 & -\sigma \end{pmatrix}$. By [3; p. 139, Lemma 11], $D_0 = C_{\mathrm{SL}(2,q)}\big(b(\sigma,\tau)\big)$ and the number of classes $\big(b(\sigma,\tau)\big)^{\mathrm{SL}(2,q)}$ is $\frac{1}{2}(q-1)$. In the simple group $\mathrm{PSL}(2,q)$, we must consider two cases $\sigma = 0$ and $\sigma \neq 0$. For the case $\sigma = 0$, we have $C_{\mathrm{PSL}(2,q)}\big(b(0,\tau)Z\big) = \frac{D_0}{Z} \cup \frac{D_1}{Z}$ and for the case $\sigma \neq 0$, $C_{\mathrm{PSL}(2,q)}\big(b(\sigma,\tau)Z\big) = \frac{D_0}{Z}$. Therefore, we obtain a conjugacy class of length $\frac{1}{2}\big(q(q-1)\big)$ and $\frac{q-3}{4}$ conjugacy classes of lengths $q(q-1)$. This completes the proof. □

By the previous lemma, if $q = 3^n$, $\frac{q-1}{2}$ and $\frac{q+1}{4}$ are primes, then $\pi_e\big(\mathrm{PSL}(2,q)\big) = \left\{1,2,3,\frac{q-1}{2},\frac{q+1}{4},\frac{q+1}{2}\right\}$.

Lemma 2. *Let G be an n-decomposable non-solvable non-perfect finite group, for $n = 5,6$. Then G' is simple.*

Proof. By assumption and Theorem 4, G' is a minimal normal subgroup of G, which is not abelian. So G' is a direct product of k isomorphic non-abelian simple groups, say $H_1,\dots,H_k$. Suppose $k \ge 2$ and p, q are two odd prime divisors of $|H_1|$. Then we can see that $\{1,2,p,q,2p,2q,pq\} \subseteq \pi_e(G')$, which is a contradiction. □

Lemma 3. *Let G be a n-decomposable non-solvable non-perfect finite group and $|\mathcal{N}_G| \ge 2$. Then $|\mathcal{N}_G| = 2$, n is a prime number and $G \cong Z_n \times B$, where B is a non-abelian simple group with exactly n conjugacy classes.*

Proof. Let A and B be elements of $\mathcal{N}_G$. Then by Theorem 4, $G \cong A\times B$. It is easy to see that A and B are simple groups. By [11; p. 88], A and B are the only proper non-trivial normal subgroups of G. So $|\mathcal{N}_G| = 2$. If A and B are non-abelian simple groups, then $G' = G$, a contradiction. Therefore, one of

A or B, say A, is abelian. Since A is simple, n is a prime number and $A \cong Z_n$, proving the lemma. □

Suppose n is a positive integer such that there is a simple group with exactly n conjugacy classes. In this case, we claim that there exists a perfect n-decomposable finite group. To see this, set $G = A \times B$, where A and B are non-abelian simple groups with exactly n conjugacy classes. Then G is a n-decomposable finite group.

In this paper we study only finite non-perfect groups. However, the investigation of non-solvable non-perfect finite groups with exactly one proper non-trivial normal subgroup does not seem to be simple.

THEOREM 5. *A non-solvable non-perfect group G is 5-decomposable if and only if G is isomorphic with $Z_5 \times A_5$, $A_6 \cdot 2_3$ or $\mathrm{Aut}(\mathrm{PSL}(2,q))$ for $q = 7, 8$.*

Proof. It is a well-known fact that A_5 is the only non-abelian finite simple group with exactly five conjugacy classes. Using this fact, if $|\mathcal{N}_G| = 2$, then by Lemma 3, $G \cong Z_5 \times A_5$, as desired. Therefore, we can assume that G has exactly one proper non-trivial normal subgroup, i.e. G'. By Lemma 2, G' is simple and we can see that $3 \le |\pi(G')| \le 4$ and $|G : G'| = p$, where p is prime. If $|\pi(G')| = 4$, then G' is a simple EPO-group and by Corollary, $G' \cong A_5$, which is a contradiction. Thus $|\pi(G')| = 3$ and G' is a K_3-group. Now by Theorem 2 and this fact that G' is a union of five G-conjugacy classes, G' is isomorphic with A_5, A_6, $\mathrm{PSL}(2,7)$ or $\mathrm{PSL}(2,8)$. If $p \notin \pi(G')$, then $G \cong Z_p \propto G'$. Suppose $\varphi\colon Z_p \to \mathrm{Aut}(G')$ is the homomorphism which determines the semidirect product. Since $|\pi(G')| = |\pi(\mathrm{Aut}(G'))|$ and $p \notin \pi(G')$, the image of a generator of Z_p by φ must be identity. This shows that the homomorphism φ is trivial and $G \cong Z_p \times G'$, which is a contradiction. Therefore, $p \in \pi(G')$. Now, our main proof will consider a number of cases.

Case $G' \cong A_5$. In this case, $|G : G'| = p \in \pi(G') = \{2, 3, 5\}$ and so $|G| = 120, 180, 300$. Using the character table of the groups of these orders, stored in GAP, we can see that there are two finite groups SmallGroup(120, 34) and SmallGroup(120, 35) of order 120 whose derived subgroup are isomorphic with A_5. But SmallGroup(120, 35) $\cong Z_2 \times A_5$, which is a contradiction. On the other hand, SmallGroup(120, 34) $\cong S_5$ and A_5 is a 4-decomposable subgroup of S_5, which contradicts our assumptions. Using similar arguments, the cases $|G| = 180$ and $|G| = 300$ lead to a contradiction.

Case $G' \cong A_6$. In Table I, we calculate the conjugacy classes of A_6. By this table, G' has exactly seven conjugacy classes of elements of order 1, 2, 3, 4 and 5. Since G' is a union of five G-conjugacy classes, two classes of elements of order three and two classes of elements of order five in G' must be fused in G. This shows that there are some conjugacy classes of lengths, 1, 45, 80, 90 and 144 in G. Consider an element x of order 5 in G'. Then

$|G| = |x^G| \cdot |C_G(x)| = 144 \cdot 5t$ for some positive integer t. But $|G : G'|$ is a prime number, so $t = 1$ and $|G| = 720$. Using the small group library of GAP, we can see that there are four groups of this order with a derived subgroup isomorphic with A_6. These are S_6, $Z_2 \times A_6$, SmallGroup(720, 764) and SmallGroup(720, 765). $Z_2 \times A_6$ have two proper non-trivial normal subgroups and A_6 is a union of six S_6-conjugacy classes. So $G \cong$ SmallGroup(720, 764) $= A_6 \cdot 2_2$ or SmallGroup(720, 765) $= A_6 \cdot 2_3$. Our calculations show that A_6 is a union of six $A_6 \cdot 2_2$-conjugacy classes, while $A_6 \cdot 2_3$ satisfies the assumptions of our theorem.

Case $G' \cong \mathrm{PSL}(2,7)$. In Table I, we calculate the conjugacy classes of $\mathrm{PSL}(2,7)$. By this table, G' has exactly six conjugacy classes of elements of order 1, 2, 3, 4 and 7. Since G' is a union of five G-conjugacy classes, two classes of elements of order seven in G' must be fused in G. This shows that there exists a G-conjugacy class of length 48. Consider an element x of order 7 in G'. Then it is easy to see that $|G| = |x^G| \cdot |C_G(x)| = 48 \cdot 7t$ for some positive integer t. But $|G : G'|$ is a prime number, so $t = 1$ and $|G| = 336$. Using the small group library of GAP we can see that there are two groups of this order with a derived subgroup isomorphic with $\mathrm{PSL}(2,7)$. These are $Z_2 \times \mathrm{PSL}(2,7)$ and $\mathrm{Aut}(\mathrm{PSL}(2,7))$. Our calculations in Table I show that $\mathrm{Aut}(\mathrm{PSL}(2,7))$ is a solution for the problem.

Case $G' \cong \mathrm{PSL}(2,8)$. By Table I, G' has exactly nine conjugacy classes of elements of order 1, 2, 3, 7 and 9. Since G' is a union of five G-conjugacy classes, three classes of elements of order seven and three classes of elements of order nine in G' must be fused in G. This shows that there exists a G-conjugacy class of length 168. Consider an element x of order 9 in G'. It is easy to see that $|G| = |x^G| \cdot |C_G(x)| = 168 \cdot 9t$ for some positive integer t. So $|G : G'| = 3$ and $|G| = 1512$. Using GAP software ([13]), we can see that there are two groups of this order with a derived subgroup isomorphic with $\mathrm{PSL}(2,8)$. These are $Z_2 \times \mathrm{PSL}(2,8)$ and $\mathrm{Aut}(\mathrm{PSL}(2,8))$. Our calculations in Table I show that $\mathrm{Aut}(\mathrm{PSL}(2,8))$ is a solution for the problem. This completes the proof. □

In the following theorem we apply Lemmas 1, 2 and 3 to classify the non-solvable non-perfect 6-decomposable finite groups. We have:

THEOREM 6. *A non-solvable non-perfect finite group* G *is* 6*-decomposable if and only if* G *is isomorphic with* S_6 *or* $A_6 \cdot 2_2$.

Proof. By Lemma 3, G has exactly one proper non-trivial normal subgroup, i.e. G', and by Lemma 2, G' is simple. By assumption, $|\pi(G')| = 3, 4, 5$. If $|\pi(G')| = 5$, then G' has at least five G-conjugacy classes of elements of prime orders. This shows that every element of G' has a prime order and by Corollary, $G' \cong A_5$. But A_5 has exactly five conjugacy classes, and so it cannot be 6-decomposable, which is a contradiction. Thus $|\pi(G')| = 3, 4$.

Using similar argument as in Theorem 5, we can show that $|G : G'| = p$, p is prime and $p \in \pi(G')$. If $|\pi(G')| = 3$, then by Theorem 2, G' is isomorphic with A_6, PSL(2,7) or PSL(2,8). Suppose $G' \cong A_6$ and x is an element of order five in G'. By Table I, $|G| = 5 \cdot 144t = 360 \cdot 2t$, and so $p = 2$. Using conjugacy classes of A_6 and computations with GAP, we can see that S_6 and SmallGroup(720, 764) have exactly one proper non-trivial normal subgroup $G' \cong A_6$ and G' is a union of six conjugacy classes of G. We now assume that $G' \cong \mathrm{PSL}(2,7)$. Since $\pi(G') = \{2,3,7\}$, $|G| = 336, 504, 1176$. In these cases, our computations with GAP shows that there is no finite group G which satisfies the conditions of the theorem. Finally, suppose that $G' \cong \mathrm{PSL}(2,8)$. Using Table I, we can see that either two classes of lengths 72 and all of classes of lengths 56 or two classes of lengths 56 and all of classes of lengths 72 must be fused in G. In any case, using a similar argument as in Theorem 5, we can obtain a contradiction.

We now assume that $|\pi(G')| = 4$. By Lemma 2, $\pi_e(G') = \{1, 2, p, s, r, a\}$, where p, s and r are primes and a is a composite number. By assumption and Theorem 3, G is either isomorphic with $\mathrm{PSL}(2, 3^n)$, where $\frac{3^n-1}{2}$ and $\frac{3^n+1}{4}$ are primes, or with $\mathrm{PSL}(2, 2^n)$, where $2^n - 1$ and $\frac{2^n+1}{3}$ are primes, or with Sz(8). In our proof, we consider a number of cases.

Case $G' \cong \mathrm{PSL}(2, 2^n)$, where $2^n - 1$ and $\frac{2^n+1}{3}$ are primes. Suppose $q = 2^n$ and U is a Sylow 2-subgroup of G'. Then by [4], $N_{\mathrm{PSL}(2,2^n)}(U) = HU$, where $|H| = 2^n - 1 = p$. So G' has exactly p Sylow 2-subgroups. Since $2^n - 1 = p$ is prime, G' has exactly $\frac{1}{2}(p-1) - 1$ conjugacy classes of elements of order p. Suppose $x \in G'$ has order p. Then $|C_{\mathrm{PSL}(2,2^n)}(x)| = p$ and $|x^{\mathrm{PSL}(2,2^n)}| = q(q+1)$. Since every element of order p of $\mathrm{PSL}(2, 2^n)$ should fuse in G, G has a conjugacy class of length $\frac{1}{2}q(q+1)(q-4)$. Thus $|G| = |\mathrm{PSL}(2,2^n)| \cdot \frac{1}{2}(q-4)t$ for some positive integer t. Therefore, $\frac{1}{2}(q-4) = 2^{n-1} - 2$ is prime. This shows that $n = 3$, which is a contradiction.

Case $G' \cong \mathrm{PSL}(2, 3^n)$, where $\frac{3^n-1}{2}$ and $\frac{3^n+1}{4}$ are primes. Consider the conjugacy classes of G' of elements of order p and apply Lemma 1. We can see that G has a conjugacy class of length $\frac{1}{4}q(q-3)(q+1)$. Therefore, $\frac{1}{4}(q-3) = 1$ or it is a prime number. This shows that $\frac{1}{4}(q-3) \in \left\{1, 2, 3, \frac{q-1}{2}, \frac{q+1}{4}\right\}$, which is impossible.

Case $G' \cong$ Sz(8). In this case $|G'| = 29120$ and by Table I, the elements of order 4, the elements of order 7 and elements of order 13 must be fused in G. This shows that $|G : G'| = 3$ and $|G| = 87360$. Since $|G'|$ is not divisible by 3, $G \cong Z_3 \propto_{\varphi} \mathrm{Sz}(8)$, where φ is a homomorphism from Z_3 into Sz(8). Obviously, φ is not trivial. By [5], Aut(Sz(8)) has two conjugacy classes $3A$ and $3B = 3A^{-1}$ of elements of order three. So $Z_3 \propto_{\varphi} \mathrm{Sz}(8) \cong \mathrm{Aut}(\mathrm{Sz}(8))$.

Now, by Table I, Sz(8) is the union of 7 conjugacy classes of G', which is a contradiction. This completes the proof. □

Table I
The fusion map of some K_3-groups.

A_6-Classes	1a	2a	3a	3b	4a	5a	5b		
Class lengths	1	45	40	40	90	72	72		
Fusion into S_6	1A	2A	3A	3B	4A	5A	5A		
A_6-Classes	1a	2a	3a	3b	4a	5a	5b		
Class lengths	1	45	40	40	90	72	72		
Fusion into $A_6 \cdot 2_2$	1A	2A	3A	3A	4A	5A	5B		
A_6-Classes	1a	2a	3a	3b	4a	5a	5b		
Class lengths	1	45	40	40	90	72	72		
Fusion into $A_6 \cdot 2_3$	1A	2A	3A	3A	4A	5A	5A		
$PSL(2,7)$-Classes	1a	2a	3a	4a	7a	7b			
Class lengths	1	21	56	42	24	24			
Fusion into Aut(PSL(2,7))	1A	2A	3A	4A	7A	7A			
PSL(2,8)-Classes	1a	2a	3a	7a	7b	7c	9a	9b	9c
Class lengths	1	63	56	72	72	72	56	56	56
Fusion into Aut(PSL(2,8))	1A	2A	3A	7A	7A	7A	9A	9A	9A
Sz(8)-Classes	1a	2a	4a	4b	5a	7a	7b	7c	13a
Class lengths	1	455	1820	1820	5824	4160	4160	4160	2240
Fusion into Aut(Sz(8))	1A	2A	4A	4B	5A	7A	7A	7A	13A
Sz(8)-Classes	13b	13c							
Class lengths	2240	2240							
Fusion into Aut(Sz(8))	13A	13A							

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* *Department of Mathematics*
Faculty of Science
University of Kashan
Kashan
IRAN

Institute for Studies in Theoretical Physics and Mathematics
Tehran
IRAN

E-mail: ashrafi@kashanu.ac.ir

** *Department of Mathematics and Physics*
Wuyi University
Jiangmen
Guangdong 529020
P. R. CHINA