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ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SOME DIFFERENCE EQUATION

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ABSTRACT. Asymptotic properties of solutions of difference equations of the form

$$\Delta^2 x_n = a_n \varphi(x_{n+p+1}, \dots, x_{n+p+k}) + b_n$$

are studied.

By $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ we denote the set of integers, positive integers and real numbers, respectively. Let $p \in \mathbb{Z}$, $k \in \mathbb{N}$. The asymptotic behavior of solutions of a difference equation

$$\begin{gathered} \Delta^2 x_n = a_n \varphi(x_{n+p+1}, \dots, x_{n+p+k}) + b_n \,, \\ n, k \in \mathbb{N}, \quad a_n, b_n \in \mathbb{R}, \quad \varphi \colon \mathbb{R}^k \to \mathbb{R}, \end{gathered} \tag{E}$$

will be investigated.

The results presented here generalize some results of A. Drozdowicz, J. Popenda [2], [3], and J. Migda, M. Migda [4].

By a *solution of the equation* (E) we mean a sequence $x \colon \mathbb{N} \to \mathbb{R}$ for which there exists $q \in \mathbb{N}$ such that the equation (E) is satisfied for all $n \geqslant q$.

The space of all sequences $x \colon \mathbb{N} \to \mathbb{R}$ we denote by SQ. The Banach space of all bounded sequences $x \in$ SQ with the norm $\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}$ we denote by BS.

If $B \subseteq \mathbb{R}$, then B^k denotes the set $B \times B \times \dots \times B \subseteq \mathbb{R}^k$. Similarly, if $c \in \mathbb{R}$, then $c^k = (c, c, \dots, c) \in \mathbb{R}^k$. The standard (Euclidean) metric on $\mathbb{R}^k$ will be denoted by d. We choose a constant $\lambda \in \mathbb{R}$ such that

$$d(t,s) \leqslant \lambda \max\{|t_i - s_i| : i = 1, 2, \dots, k\}$$

for every $t = (t_1, \dots, t_k)$, $s = (s_1, \dots, s_k) \in \mathbb{R}^k$.

If $X \subseteq \mathbb{R}^k$, then $\varphi|_X$ denotes the restriction of the function φ to the set X i.e. $\varphi|_X \colon X \to \mathbb{R}$, $(\varphi|_X)(t) = \varphi(t)$ for any $t \in X$.

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LEMMA. *If the series* $\sum_{n=1}^{\infty} na_n$ *is absolutely convergent,* $r_n = \sum_{i=n}^{\infty} a_i$, *then the series* $\sum_{n=1}^{\infty} r_n$ *is absolutely convergent and* $\sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} na_n$.

THEOREM 1. *Assume that* φ *is continuous, and the series* $\sum_{n=1}^{\infty} na_n$ *is absolutely convergent. Then for any bounded solution* y *of the equation* $\Delta^2 y_n = b_n$ *there exists a solution* x *of* (E) *which possesses the asymptotic behavior*

$$x_n = y_n + o(1).$$

Proof. Assume that y is a bounded solution of the equation $\Delta^2 y_n = b_n$, and Y is the set of values of the sequence y. Choose a number $a > 0$. Let

$$U = \{t \in \mathbb{R}^k : \text{ there exists } s \in Y^k \text{ such that } d(s,t) < \lambda a\}.$$

Since Y^k is a bounded subset of $\mathbb{R}^k$, U is bounded, too. Hence the closure $\bar{U}$ is compact. Therefore φ is uniformly continuous and bounded on $\bar{U}$. Choose $M > 0$ such that $|\varphi(t)| < M$ for any $t \in U$. Let $r_n = \sum_{j=n}^{\infty} |a_j|$ for $n \in \mathbb{N}$. From Lemma, it follows that the series $\sum_{n=1}^{\infty} r_n$ is convergent. Let $\rho_n = \sum_{j=n}^{\infty} r_j$ for $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \rho_n = 0$, there exists $q \geqslant \max\{1, -p\}$ such that $M\rho_n < a$ for any $n \geqslant q$.

Let

$$T = \{x \in \mathrm{BS} : x_n = 0 \text{ for } n < q \text{ and } |x_n| \leqslant M\rho_n \text{ for } n \geqslant q\},$$
$$S = \{x \in \mathrm{BS} : x_n = y_n \text{ for } n < q \text{ and } |x_n - y_n| \leqslant M\rho_n \text{ for } n \geqslant q\}.$$

T is a convex and compact subset of the space BS. The mapping $F : T \to S$ defined by $F(x)(n) = x_n + y_n$ is an affine isometry of the set T onto S. Hence S is also convex and compact.

If $x \in S$, $n \geqslant \max\{1, -p\}$, then $(y_{n+p+1}, \dots, y_{n+p+k}) \in Y^k$ and

$$d((x_{n+p+1}, \dots, x_{n+p+k}), (y_{n+p+1}, \dots, y_{n+p+k}))$$
$$\leqslant \lambda \max\{|x_{n+p+i} - y_{n+p+i}| : i = 1, 2, \dots, k\} < \lambda a.$$

It means that $(x_{n+p+1}, \dots, x_{n+p+k}) \in U$ for every $x \in S$ and any $n > \max\{1, -p\}$. Hence $|\varphi(x_{n+p+1}, \dots, x_{n+p+k})| < M$ for every $x \in S$ and any $n > \max\{1, -p\}$. Let $x \in S$. For $n \geqslant \max\{1, -p\}$ let $\tilde{x}_n = a_n \varphi(x_{n+p+1}, \dots, x_{n+p+k})$, $g_n = \sum_{j=n}^{\infty} \tilde{x}_j$.

Then $|\tilde{x}_n| \leqslant M|a_n|$. Hence it follows that

$$|g_n| \leqslant \sum_{j=n}^{\infty} |\tilde{x}_j| \leqslant \sum_{j=n}^{\infty} M|a_j| = Mr_n .$$

Since the series $\sum_{j=1}^{\infty} r_j$ is absolutely convergent, the series $\sum_{j=1}^{\infty} g_j$ is absolutely convergent, too. Let us define the sequence $A(x)$ as follows

$$A(x)(n) = \begin{cases} y_n & \text{for } n < q , \\ y_n + \sum_{j=n}^{\infty} g_j & \text{for } n \geqslant q . \end{cases}$$

If $n \geqslant q$, then $|A(x)(n) - y_n| = \left| \sum_{j=n}^{\infty} g_j \right| \leqslant \sum_{j=n}^{\infty} |g_j|$. But $|g_j| \leqslant Mr_j$. Hence

$$|A(x)(n) - y_n| \leqslant M \sum_{j=n}^{\infty} r_j = M\rho_n .$$

It means that $A(x) \in S$. Hence $A(S) \subseteq S$.

Let $\varepsilon > 0$. Since the function φ is uniformly continuous on U, there exists $\delta > 0$ such that if $s, t \in U$ and $d(s,t) < \lambda\delta$, then $|\varphi(t) - \varphi(s)| < \varepsilon$. Assume $x, z \in S$ and $\|x - z\| < \delta$. If $n \geqslant \max\{1, -p\}$, then

$$d\big((x_{n+p+1}, \dots, x_{n+p+k}), (z_{n+p+1}, \dots, z_{n+p+k})\big) \leqslant \lambda\delta .$$

Let $\tilde{z}_n = a_n \varphi(z_{n+p+1}, \dots, z_{n+p+k})$ and $h_n = \sum_{j=n}^{\infty} \tilde{z}_j$ for $n \in \mathbb{N}$.

Then

$$\|A(x) - A(z)\| = \sup_{n \geqslant p} \left| \sum_{j=n}^{\infty} g_j - \sum_{j=n}^{\infty} h_j \right| \leqslant \sum_{j=p}^{\infty} |g_j - h_j| .$$

But

$$|g_j - h_j| = \left| \sum_{i=j}^{\infty} \tilde{x}_i - \sum_{i=j}^{\infty} \tilde{z}_i \right| \leqslant \sum_{i=j}^{\infty} |\tilde{x}_i - \tilde{z}_i|$$

and

$$|\tilde{x}_i - \tilde{z}_i| = \left| a_i \varphi(x_{i+p+1}, \dots, x_{i+p+k}) - a_i \varphi(z_{i+p+1}, \dots, z_{i+p+k}) \right| \leqslant \varepsilon |a_i| .$$

Hence $|g_j - h_j| \leqslant \varepsilon r_j$. Therefore, $\|A(x) - A(z)\| \leqslant \sum_{j=q}^{\infty} \varepsilon r_j = \varepsilon \rho_q$.

It means that A is continuous. By Schauder theorem there exists $x \in S$ such that $A(x) = x$. Then $x_n = y_n + \sum_{j=n}^{\infty} g_j$ for any $n \geqslant q$. Hence it follows that if $n \geqslant q$, then

$$\Delta x_n = y_{n+1} + \sum_{j=n+1}^{\infty} g_j - y_n - \sum_{j=n}^{\infty} g_j = \Delta y_n - g_n .$$

Hence

$$\begin{aligned}\Delta^2 x_n &= \Delta^2 y_n - g_{n+1} + g_n = b_n - \sum_{j=n+1}^{\infty} \tilde{x}_j + \sum_{j=n}^{\infty} \tilde{x}_j \\ &= b_n + \tilde{x}_n = a_n \varphi(x_{n+p+1}, \dots, x_{n+p+k}) + b_n \qquad \text{for} \quad n \geqslant q .\end{aligned}$$

From the convergence of the series $\sum_{j=1}^{\infty} g_j$ it follows that $x_n = y_n + o(1)$. □

COROLLARY 1. *If the series $\sum_{n=1}^{\infty} na_n$, $\sum_{n=1}^{\infty} nb_n$ are absolutely convergent, φ is continuous, then for any $c \in \mathbb{R}$ there exists a solution of* (E) *which converges to c.*

Proof. Let $c \in \mathbb{R}$ and $r_n = \sum_{i=n}^{\infty} b_i$ for $n \in \mathbb{N}$. From Lemma, it follows that the series $\sum_{i=1}^{\infty} r_i$ is convergent. Let $t_n = \sum_{i=n}^{\infty} r_i$ and $y_n = c + t_n$.

Then $\lim_{n\to\infty} t_n = 0$, $\Delta t_n = -r_n$ and $\Delta^2 t_n = \Delta(\Delta t_n) = \Delta(-r_n) = b_n$.

Hence $\Delta^2 y_n = \Delta^2 c + \Delta^2 t_n = b_n$.

Therefore, y is a bounded solution of the equation $\Delta^2 y_n = b_n$. From Theorem 1 it follows that there exists a solution x of (E) such that $x_n = y_n + o(1)$. Obviously, $\lim_{n\to\infty} x_n = c$. □

THEOREM 2. *If the function φ is uniformly continuous and bounded and the series $\sum_{n=1}^{\infty} na_n$ is absolutely convergent, then for any solution y of the equation $\Delta^2 y_n = b_n$ there exists a solution x of* (E) *such that*

$$x_n = y_n + o(1) .$$

Proof. Assume y is a solution of the equation $\Delta^2 y_n = b_n$. Choose $M > 0$ such that $|\varphi(t)| < M$ for any $t \in \mathbb{R}^k$. Similarly as in the proof of Theorem 1 we define $r_n = \sum_{j=n}^{\infty} |a_j|$ and $\rho_n = \sum_{j=n}^{\infty} r_j$. Let

$$\begin{aligned} T &= \{x \in BS : \ |x_n| \leqslant M\rho_n , \ \ n \in \mathbb{N}\} , \\ S &= \{x \in SQ : \ |x_n - y_n| \leqslant M\rho_n , \ \ n \in \mathbb{N}\} . \end{aligned}$$

Let us define the mapping $F\colon T \to S$ by $F(x)(n) = x_n + y_n$. Then the formula $\rho(x,z) = \sup\{|x_n - z_n| : n \in \mathbb{N}\}$ defines a metric on the set S such that F is an isometry of T onto S. T is a convex and compact subset of the space BS. Since S is homeomorphic to T, it follows from Schauder theorem that any continuous mapping $A\colon S \to S$ possesses a fixed point.

For $x \in S$, $n \geqslant \max\{1, -p\}$, let $\tilde{x}_n = a_n \varphi(x_{n+p+1}, \dots, x_{n+p+k})$, $g_n = \sum_{j=n}^{\infty} \tilde{x}_j$ and $A(x)(n) = y_n + \sum_{j=n}^{\infty} g_j$.

The rest of the proof is analogous to the second part of the proof of Theorem 1. □

Corollary 2. *If the series $\sum_{n=1}^{\infty} na_n$, $\sum_{n=1}^{\infty} nb_n$ are absolutely convergent, φ is uniformly continuous and bounded, then for any numbers $c, d \in \mathbb{R}$ there exists a solution x of* (E) *such that*

$$x_n = cn + d + o(1).$$

Proof. Let $c, d \in \mathbb{R}$ and $r_n = \sum_{i=n}^{\infty} b_i$, $u_n = \sum_{i=n}^{\infty} r_i$ and $y_n = cn + d + u_n$ for $n \in \mathbb{N}$. Since $\Delta^2(cn + d) = 0$, then, similarly as in the proof of Corollary 1, one can conclude that y is a solution of the equation $\Delta^2 y_n = b_n$. The assertion holds since $u_n = o(1)$ and by Theorem 2. □

Example. Let $a_n = \frac{1}{n^2}$, $b_n = 0$ for any $n \in \mathbb{N}$ and let φ be a constant function equal to 1. We will show the equation (E) has no solutions of the form $x_n = y_n + o(1)$, where y_n denotes a solution of the equation $\Delta^2 y_n = b_n$. Let $s_n = a_1 + a_2 + \dots + a_n$, $s = \sum_{i=1}^{\infty} a_i$ and $r_n = \sum_{i=n}^{\infty} a_i$.

Assume $x_n = y_n + o(1)$ is a solution of (E) and $\Delta^2 y_n = b_n = 0$. Then $y_n = bn + c$ for some $b, c \in \mathbb{R}$. It means that $x_n = bn + c + z_n$, $z_n = o(1)$. Since $\Delta^2 x_n = a_n$, $a_n = \Delta^2(bn + c + z_n) = \Delta^2 z_n$. Let $u_n = \Delta z_n$. Then $\Delta u_n = \Delta^2 z_n = a_n$. Hence $u_n = u_1 + a_1 + \dots + a_{n-1} = u_1 + s_{n-1}$. Since $z_n = o(1)$, $u_n = \Delta z_n = o(1)$. Hence $0 = \lim_{n\to\infty} u_n = u_1 + \lim_{n\to\infty} s_{n-1} = u_1 + s$, which implies $u_n = -s + s_{n-1} = -r_n$.

Since $u_n = \Delta z_n$, it holds that $z_n = z_1 + \sum_{i=1}^{n-1} u_i = z_1 - \sum_{i=1}^{n-1} r_i$, and since $r_1 + r_2 + r_3 + \dots = a_1 + 2a_2 + 3a_3 + \dots = 1 + \frac{1}{2} + \frac{1}{3} + \dots = \infty$, we obtain $\lim_{n\to\infty} z_n = z_1 - \infty = -\infty$.

However, this contradicts the hypothesis $z_n = o(1)$.

THEOREM 3. *If the series* $\sum_{n=1}^{\infty} na_n$ *is absolutely convergent and the function* $\varphi|[a,\infty)^k$ *is uniformly continuous and bounded for some* $a \in \mathbb{R}$, *then for any solution* y *of the equation* $\Delta^2 y_n = b_n$ *which diverges to infinity there exists a solution* x *of* (E) *such that*

$$x_n = y_n + o(1).$$

Proof. Assume $\Delta^2 y_n = b_n$ and $\lim_{n\to\infty} y_n = \infty$. Choose $M > 0$ such that $|\varphi(t)| < M$ for any $t \in [a,\infty)^k$. Similarly as in the proof of Theorem 1 we define r_n and ρ_n. Choose $q \geqslant \max\{1, -p\}$ such that $y_n \geqslant a + M\rho_1$ for any $n \geqslant q$. Let

$$S = \{x \in \mathrm{SQ} : \ x_n = y_n \text{ for } n < q \text{ and } |x_n - y_n| \leqslant M\rho_n \text{ for } n \geqslant q\}.$$

The set S possesses the fixed point property (as in the proof of Theorem 2).

If $x \in S$ and $n \geqslant q$, then $x_n \geqslant y_n - M\rho_n \geqslant a + M\rho_1 - M\rho_n \geqslant a$. Hence $(x_{n+p+1}, \dots, x_{n+p+k}) \in [a,\infty)^k$ for any $x \in S$, $n \geqslant q$. The rest of the proof is analogous to the proof of Theorem 1. □

COROLLARY 3. *If the series* $\sum_{n=1}^{\infty} na_n$, $\sum_{n=1}^{\infty} nb_n$ *are absolutely convergent, there exists such* $a \in \mathbb{R}$ *that the function* $\varphi|[a,\infty)^k$ *is uniformly continuous and bounded, then for any* $c > 0$ *and* $d \in \mathbb{R}$ *there exists a solution* x *of* (E) *for which* $x_n = cn + d + o(1)$.

THEOREM 4. *If the series* $\sum_{n=1}^{\infty} na_n$ *is absolutely convergent, the function* $\varphi|(-\infty,a]^k$ *is uniformly continuous and bounded for some* $a \in \mathbb{R}$, *then for any solution* y *of the equation* $\Delta^2 y_n = b_n$ *which diverges to* $-\infty$ *there exists a solution* x *of* (E) *such that*

$$x_n = y_n + o(1).$$

Proof. One can prove this theorem similarly as Theorem 3. □

The following corollary is a consequence of Theorem 4 and Corollary 3.

COROLLARY 4. *If the series* $\sum_{n=1}^{\infty} na_n$, $\sum_{n=1}^{\infty} nb_n$ *are absolutely convergent,* $a > 0$, $H = (-\infty,-a]^k \cup [a,\infty)^k$, *and the function* $\varphi|_H$ *is absolutely continuous and bounded, then for any nonzero* $c \in \mathbb{R}$ *and any* $d \in \mathbb{R}$ *there exists a solution* x *of* (E) *for which* $x_n = cn + d + o(1)$.

THEOREM 5. *Assume φ is bounded, the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, a sequence (x_n) is a solution of* (E) *and $s_n = b_1 + b_2 + \cdots + b_n$ for any $n \in \mathbb{N}$. Then:*

(1) *if (s_n) is bounded from above, then (x_n/n) is bounded from above,*
(2) *if (s_n) is bounded from below, then (x_n/n) is bounded from below,*
(3) *if (s_n) is bounded, then (x_n/n) is bounded,*
(4) *if (s_n) is convergent, then (x_n/n) is convergent,*
(5) *if $\lim_{n\to\infty} s_n = \infty$, then $\lim_{n\to\infty} (x_n/n) = \infty$,*
(6) *if $\lim_{n\to\infty} s_n = -\infty$, then $\lim_{n\to\infty} (x_n/n) = -\infty$.*

Proof. For $n \geqslant \max\{1, -p\}$ let $g_n = a_n \varphi(x_{n+p+1}, \dots, x_{n+p+k})$, $t_n = g_1 + g_2 + \cdots + g_n$.

The function φ is bounded and the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, so the series $\sum_{n=1}^{\infty} g_n$ is absolutely convergent. Hence the sequence (t_n) is convergent. Since $\Delta^2 x_n = g_n + b_n$, we obtain

$$\Delta x_n - \Delta x_1 = \Delta^2 x_1 + \Delta^2 x_2 + \cdots + \Delta^2 x_{n-1} = t_{n-1} + s_{n-1}. \qquad (*)$$

Hence it follows that if (s_n) is bounded from above, then (Δx_n) is bounded from above. Assume $\Delta x_n \leqslant M$ for some $M > 0$ and any $n \in \mathbb{N}$. Then

$$x_n - x_1 = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_{n-1} \leqslant (n-1)M \leqslant nM.$$

Hence (x_n/n) is bounded from above if (s_n) is bounded from above.

Similarly, one can show (2). Assertion (3) follows immediately from (1) and (2). Now, assume (s_n) is convergent. Then by $(*)$ the sequence (Δx_n) is also convergent. If $\lim_{n\to\infty} \Delta x_n = c$, then $\lim_{n\to\infty} (\Delta x_n/\Delta n) = \lim_{n\to\infty} \Delta x_n = c$. Hence by Stolz theorem ([1; Theorem 1.7.9]), $\lim_{n\to\infty} (x_n/n) = c$. Similarly, if $\lim_{n\to\infty} s_n = \infty$, then $\lim_{n\to\infty} \Delta x_n = \infty$ and by Stolz theorem $\lim_{n\to\infty} (x_n/n) = \infty$. Analogously, if $\lim_{n\to\infty} s_n = -\infty$, then $\lim_{n\to\infty} (x_n/n) = -\infty$. □

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