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PARTIAL LINE GRAPH OPERATOR AND HALF-ARC-TRANSITIVE GROUP ACTIONS

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ABSTRACT. The concept of the partial line graph operator Pl (and its inverse operator Al) on graphs of valency 4 with balanced orientation is developed in order to study transitive permutation groups having a non-self-paired suborbit of length 2 via the corresponding orbital graphs. If G is such a group and X is the orbital graph associated with a suborbit of length 2 of G, which is not self-paired, then X has valency 4 and admits a vertex- and edge- but not arc-transitive action of G. There is a natural balanced orientation of the edge set of X induced and preserved by G. An analysis of the properties of this oriented graph is performed, using operators Pl and Al resulting in some partial results on the point stabilizer of G (in the case when X is connected). Finally, a graphical realization of such actions with nonabelian vertex stabilizers is given, that is, an infinite family of tetravalent graphs admitting a vertex and edge but not arc-transitive action with vertex stabilizer D_8, the dihedral group of order 8, is constructed.

1. Introduction

Throughout this paper, by a *graph* we mean an ordered pair (V, E), where V is a finite nonempty set and E is a symmetric irreflexive relation on V whose transitive closure is the universal relation on V. (Graphs are thus assumed to be finite and connected.) By an *oriented graph* we mean an ordered pair (V, A), where V is a finite nonempty set and A, the set of *arcs*, is an asymmetric relation on V. Furthermore, all groups are assumed to be finite. For a graph X, let $V(X)$, $E(X)$, $A(X)$ and $\operatorname{Aut} X$ denote the respective sets of vertices, edges

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and arcs, and the automorphism group of X. For graph-theoretic and group-theoretic terms not defined here we refer the reader to [1], [2], [4], [13]. A graph is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive*, respectively, if its automorphism group $\operatorname{Aut} X$ acts *vertex-transitively*, *edge-transitively* and *arc-transitively*. If a graph X admits a vertex- and edge- but not arc-transitive action, briefly, a $\frac{1}{2}$-*arc-transitive group action* of a subgroup G of $\operatorname{Aut} X$, we say that X is $(G, \frac{1}{2})$-*arc-transitive*. In particular, if $G = \operatorname{Aut} X$, then X is said to be $\frac{1}{2}$-*arc-transitive*. Furthermore, if H is a vertex stabilizer in the above $\frac{1}{2}$-arc-transitive action of G, we say that X is $(G, \frac{1}{2}, H)$-transitive.

It is the purpose of this paper to develop the concept of the partial line graph operator Pl and its inverse operator Al on graphs of valency 4 with balanced orientation in order to study the structure of transitive permutation groups having a non-self-paired suborbit of length 2 relative to which the corresponding orbital graph is connected (with the emphasis on their point stabilizers); in graph-theoretic language, the structure of graphs of valency 4 admitting a $\frac{1}{2}$-arc-transitive group action. (We refer the reader to [10], [11], [12], [14] for related results on point stabilizers of transitive permutation groups having suborbits of small length, and to [6] for recent research on $\frac{1}{2}$-arc-transitive graphs.) The particular combinatorial point of view adopted here sheds some new light on the structure of 4-valent graphs admitting such group actions with (large) vertex stabilizers and explains why it is that graphs of girth 4, in particular certain Cayley graphs, are the focal point in the above analysis. Theorem 4.1 gives a necessary condition for a group H to be a vertex stabilizer of a $\frac{1}{2}$-arc-transitive action on a 4-valent graph X. Note that H is necessarily a 2-group, say of order 2^h; we call h the G-*height* of X. (Further improvements and generalizations of Theorem 4.1 are obtained in [9].) As one of the by-products, a construction of an infinite family of $\frac{1}{2}$-arc-transitive group actions with point stabilizers isomorphic to the dihedral group D_8, the smallest nonabelian admissible group, is obtained.

In Section 2 the two paired operators Pl and Al are introduced, enabling us to associate with each graph of valency 4 admitting a $\frac{1}{2}$-arc-transitive action of height $h \geq 1$ of a group G, a particular Cayley graph on G (generated by two elements). The study of the properties of these Cayley graphs is the content of Section 3 and provides the machinery needed in the proof of Theorem 4.1 in Section 4. Finally, an infinite family of tetravalent graphs admitting a $\frac{1}{2}$-arc-transitive group action with nonabelian vertex stabilizer D_8 is constructed in Section 4.

2. Pl and Al operators on oriented graphs

Let X be an oriented graph. A path P of X is called *directed* if every vertex of P of valency 2 is the tail of one and the head of the other of its two incident arcs. A *directed cycle* in X is a cycle in which a removal of an arc results in a directed path. An even length cycle C in X is an *alternating cycle* if every other vertex of C is the tail and every other vertex of C is the head of its two incident arcs. By an *edge* of an oriented graph we mean an edge of the underlying undirected graph.

We are now going to introduce two operators on oriented graphs which will play a crucial role in our study of $\frac{1}{2}$-arc-transitive group actions on tetravalent graphs. Let $X = (V, A)$ be an arbitrary oriented graph. The operator Pl is defined as follows. Let the *partial line graph* $Y = \mathrm{Pl}(X)$ of X be the oriented graph with vertex set A such that there exists an arc in Y from $x \in A$ to $y \in A$ in Y if and only if xy is a directed 2-path in X.

If the arc set of Y decomposes into alternating 4-cycles no two of which intersect in more than one vertex, and if the maximum valency of Y is 4 (and so every vertex in Y has valency 2 or 4), we may also introduce the inverse operator Al. Let the vertex set of $\mathrm{Al}(Y)$ be the set of alternating cycles (of length 4) in Y, with two such cycles adjacent in $\mathrm{Al}(Y)$ if and only if they have a common vertex in Y. The orientation of the edges of $\mathrm{Al}(Y)$ is inherited from that of the edges of Y in a natural way. Letting C_v and C_w be the two alternating 4-cycles in Y, corresponding to two adjacent vertices v and w in $\mathrm{Al}(Y)$, we orient the edge $[v, w]$ in $\mathrm{Al}(Y)$ from v to w if and only if the two arcs in Y with the tail in $u \in C_v \cap C_w$ have heads on C_w. Observe that $\mathrm{Al}\big(\mathrm{Pl}(X)\big) = X$ for every balanced oriented graph X of valency 4. Moreover, $\mathrm{Pl}\big(\mathrm{Al}(Y)\big) = Y$ as long as the graph Y has the above assumed properties.

Let us remark that there are instances in this paper when these two operators are also applied to (undirected) graphs. This will occur whenever an accompanying oriented graph is (perhaps tacitly) associated with the undirected graph in question. A typical situation is presented by a tetravalent graph admitting a $\frac{1}{2}$-arc-transitive group action and its two accompanying balanced oriented graphs, or by a Cayley graph arising from a set of non-involutory generators for each of which one of the two possible orientation is prescribed.

Proposition 2.1. *If X is a balanced oriented 4-valent graph, then* $\mathrm{Aut}\,\mathrm{Pl}(X) = \mathrm{Aut}\,X$. *Conversely, let Y be a balanced oriented graph of valency 4 such that the alternating cycles have length 4, no two intersect in more than one vertex, and they decompose the edge set. Then* $\mathrm{Aut}\,\mathrm{Al}(Y) = \mathrm{Aut}\,Y$.

P r o o f. Every automorphism of X permutes the edges of X (that is the edges of the underlying undirected graph) and thus it can be viewed as a permutation of the vertices of $\mathrm{Pl}(X)$. Moreover it maps a directed 2-path onto a directed 2-path and thus it preserves the adjacency in $\mathrm{Pl}(X)$. Hence $\mathrm{Aut}\, X \leq \mathrm{Aut}\,\mathrm{Pl}(X)$. To see that also the reverse inclusion holds, observe that every automorphism of $\mathrm{Pl}(X)$ permutes the alternating 4-cycles of $\mathrm{Pl}(X)$ which correspond to vertices of X, preserving, of course, the adjacency of these cycles. Besides, every automorphism of $\mathrm{Pl}(X)$ preserves the arcs of $\mathrm{Pl}(X)$, that is the directed 2-paths of X, and hence also the arcs of X. Therefore it must be induced by an automorphism of X. Hence $\mathrm{Aut}\,\mathrm{Pl}(X) = \mathrm{Aut}\, X$. To see that the second statement of Proposition 2.1 holds, we only need to take into account the fact that operators Al and Pl are inverses of each other. □

Given a balanced oriented 4-valent graph X, the four arcs incident with a vertex in X give rise to an alternating 4-cycle C in $\mathrm{Pl}(X)$, which may be thought of as the image of that vertex under Pl. As the next step we consider the second image of that vertex to be the subgraph $\mathrm{Pl}(\bar{C})$ of $\mathrm{Pl}^2(X)$, where $\bar{C}$ is the alternating 4-cycle C together with all the incident arcs in $\mathrm{Pl}(X)$. To formalize the notion of the nth image of a vertex it is more convenient to use the Al operator. The *nth image of a vertex* $v \in V(X)$ is the subgraph $U_n = U_n(v)$ of $\mathrm{Pl}^n(X)$ for which $v = \mathrm{Al}^n(U_n)$. It is easy to see that for a fixed n the graph U_n is uniquely determined, that is, it does not depend on X or on the choice of the particular vertex v. An alternative definition of U_n by means of Pl operator reads as follows. Set $U_0 = K_1$. For a given U_n let $\bar{U}_n$ denote the graph formed by adding two ingoing pendant arcs to each vertex of U_n with indegree 0 and by adding two outgoing arcs to each vertex of U_n with outdegree 0. Then $U_{n+1} = \mathrm{Pl}(\bar{U}_n)$.

The lemma below establishes some important properties of the graphs U_n, $n \in \mathbb{Z}^+ \cup \{0\}$. By an *$m$-alternating cycle of reduced length $l = 2n$* we mean a closed walk W in X of the form $P_1Q_1^{-1}P_2Q_2^{-1} \dots P_nQ_n^{-1}$, where P_i and Q_i are directed paths of length m. In particular, 1-alternating cycles are precisely the alternating cycles in X. For a subgraph X' of X let $V^+(X')$ denote the set of all vertices of X' with indegree 0, and similarly, by $V^-(X')$ we denote the set of all vertices of X' with outdegree 0.

LEMMA 2.2. *Let $n \geq 1$ be a positive integer. Then the oriented graph U_n satisfies the following properties:*

(i) *for each $j \in \{0, 1, \dots, n\}$, U_n contains U_j as an induced subgraph;*
(ii) *there are no directed cycles in U_n;*
(iii) *there are n-alternating cycles in U_n of reduced length 4;*
(iv) $|V^+(U_n)| = 2^n = |V^-(U_n)|$ *and* $|V(U_n)| = (n+1)2^n$;
(v) $U_n - V^+(U_n) \cong 2U_{n-1} \cong U_n - V^-(U_n)$.

Proof. To prove (i) it is sufficient to observe that the graphs U_j appear in U_n as jth images of vertices of U_{n-j}.

Next, (ii) follows from the fact that the Al operator produces from a directed cycle in U_j $(j = 1,\dots,n)$ a directed cycle in U_{j-1}. But then, since U_0 contains no directed cycles, there are no directed cycles in U_n either.

To see that (iii) holds, we use induction on n. The statement is clearly valid for $n = 1$. Also it is easy to verify that an n-alternating cycle of reduced length 4 in U_n gives rise to a collection of $(n+1)$-alternating cycles of reduced length 4 in U_{n+1}.

To prove (iv), let $v(n) = |V(U_n)|$, $v^+(n) = |V^+(U_n)|$ and $v^-(n) = |V^-(U_n)|$. Since $U_{n+1} = \mathrm{Pl}(\bar{U}_n)$, the vertices of outdegree 0 in U_{n+1} correspond to pendant outgoing arcs in $\bar{U}_n$. These arcs originate in $V^-(U_n)$, and hence $v^-(n+1) = v^-(n)$. Consequently, $v^-(n) = 2^n$. A similar argument yields $v^+(n) = 2^n$. Since $U_n = \mathrm{Al}(U_{n+1})$, the vertices of U_n are in a $1-1$ correspondence with the alternating 4-cycles in U_{n+1}. Counting the arcs in U_{n+1} we get

$$4v(n) = 2v^+(n+1)+2\big(v(n+1)-v^+(n+1)-v^-(n+1)\big) = 2\big(v(n+1)-v^-(n+1)\big)\,.$$

Taking into account that $v^-(n+1) = 2^{n+1}$ we obtain

$$v(n+1) = 2v(n) + 2^{n+1} \qquad \text{and} \qquad v(0) = 1\,.$$

These two equations determine the function $v(n)$ uniquely. On the other hand, the function $f(n) = 2^n(n+1)$ satisfies the above recursive equation, and so $v(n) = f(n)$.

Finally, we show that $U_{n+1} - F \cong 2U_n$ for $F = V^-(U_{n+1})$. (The case when $F = V^+(U_{n+1})$ is left to the reader.) We use induction on n. Clearly, the statement holds true for $n = 1$. By the induction hypothesis $U_n - F$ consists of two copies of U_{n-1}. The arc set of U_n decomposes into two sets formed by arcs with origins in the two respective copies U^i_{n-1} $(i = 1,2)$ of U_{n-1}. Recall that $U_{n+1} = \mathrm{Pl}(\bar{U}_n)$. Clearly, $\bar{U}^i_{n-1} \subset \bar{U}_n$ and the arc sets of these two subgraphs are disjoint. Moreover, every arc in $\bar{U}^i_{n-1}$ terminates at a vertex of outdegree 2 in $\bar{U}_n$. Hence the corresponding vertices in $U^i_n = \mathrm{Pl}(\bar{U}^i_{n-1})$ also have outdegree 2. Thus $U^1_n \cup U^2_n \subseteq U_{n+1} - V^-(U_{n+1})$. On the other hand, by Lemma 2.2 (iv), we have $|V(U^1_n)|+|V(U^2_n)| = |V(U_{n+1})|-|V^-(U_{n+1})|$. Therefore $U^1_n \cup U^2_n = U_{n+1} - V^-(U_{n+1})$, completing the proof of Lemma 2.2. □

3. Characterizing $\frac{1}{2}$-arc-transitive actions relative to their height

Concepts peculiar to oriented graphs, such as directed and alternating cycles, may be extended to graphs admitting $\frac{1}{2}$-arc-transitive group actions via the orientation of the edge set induced by the corresponding group of automorphisms in the following way.

A $(G, \frac{1}{2})$-arc-transitive graph X of valency 4, where $G \leq \operatorname{Aut} X$, gives rise to two oriented graphs with X as their underlying graph, namely, the two orbital graphs of the action of G on $V(X)$ relative to two paired orbitals of length 2. Let $D_G(X)$ be one of these two graphs fixed from now on. For $u, v \in V(X)$ we shall say that u is the *tail* of (u,v), and that v is the *head* of (u,v) if (u,v) is an arc in $D_G(X)$. We remark that by the G*-orientation* of the edges of X, that is by the orientation induced by the $\frac{1}{2}$-arc-transitive action of G, we shall always mean the corresponding orientation of the edges in $D_G(X)$. A path P in X is a G*-directed path* if it is a directed path in $D_G(X)$. A cycle of X is a G*-directed cycle*, and a G*-alternating cycle*, respectively, provided it is a directed cycle, and an alternating cycle in $D_G(X)$. When the particular group G is clear from the context, the symbol G will sometimes be omitted in the above concepts. It transpires that all G-alternating cycles in X have the same length and form a decomposition of the edge set of X ([5; Proposition 2.4]); half of this length is denoted by $r_G(X)$ and is called the G*-radius* of X. Moreover, any two adjacent G-alternating cycles of X intersect in the same number of vertices. This number, called the G*-attachment number* of X, divides $2r_G(X)$ ([5; Proposition 2.6]).

For a group G and a generating set S of G such that $1 \notin S = S^{-1}$, the *Cayley graph* $\operatorname{Cay}(G,S)$ *of* G *relative to* S has vertex set G and edges of the form $[g, gs]$, $g \in G$, $s \in S$. Note that the group G acts on $\operatorname{Cay}(G,S)$ by left regular action as a regular subgroup of $\operatorname{Aut}\operatorname{Cay}(G,S)$. In this context we will throughout this paper always identify G and any of its subgroups with its left regular action. The accompanied right translation of any subgroup H of G on itself will be denoted by H^*. Let $S = \{a, a^{-1}, b, b^{-1}\}$, where a and b are non-involutory elements of G. Let $\operatorname{Cay}(G; a, b)$ denote the (undirected) graph $\operatorname{Cay}(G,S)$ together with the implicit orientation inherited from the oriented Cayley graph $\operatorname{Cay}(G, \{a,b\})$.

Let X be a graph together with an inherited orientation given via an oriented graph X', whose underlying graph it is. Then let the *partial line graph* $Y = \operatorname{Pl}(X)$ of X be the underlying graph of $\operatorname{Pl}(X')$. In a similar fashion, also the operator Al may be extended to graphs possessing an implicit orientation of their edge sets. Again, these two operators are inverses of each other also for graphs.

PROPOSITION 3.1. *Let X, Y be graphs of valency 4. Then*

(i) *If X is $(G, \frac{1}{2}, H)$-arc-transitive for some $H \leq G \leq \operatorname{Aut} X$ and $|H| > 2$, then $\operatorname{Pl}(X)$ is $(G, \frac{1}{2}, K)$-arc-transitive with G-radius 2 for some $K \leq H$ of index 2 in H. Conversely, if Y is $(G, \frac{1}{2}, K)$-arc-transitive with G-radius 2 and G-attachment number 1 for some $K \leq G \leq \operatorname{Aut} Y$*

such that $|K| \geq 2$, *then* $X = \mathrm{Al}(Y)$ *is* $\left(G, \frac{1}{2}, H\right)$*-arc-transitive for some* $H \leq G$ *such that* $[H : K] = 2$ *and thus* $|H| > 2$.

(ii) *If* X *is* $\left(G, \frac{1}{2}, \mathbb{Z}_2\right)$*-arc-transitive for some nonabelian subgroup* $G \leq \operatorname{Aut} X$, *then there exist non-involutory generators* a *and* b *of* G *such that* $(ab^{-1})^2 = 1$ *and* $\mathrm{Pl}(X) \cong \mathrm{Cay}(G; a, b)$. *Conversely, if* Y *is such a Cayley graph, then* $\mathrm{Al}(Y)$ *is* $\left(G, \frac{1}{2}, \mathbb{Z}_2\right)$*-arc-transitive.*

Proof. To prove the first part of (i) assume that X is $\left(G, \frac{1}{2}, H\right)$-arc-transitive with $H \leq G \leq \operatorname{Aut} X$ and $|H| > 2$. It follows from Proposition 2.1 that G is a subgroup of the automorphism group of $Y = \mathrm{Pl}(X)$ and that it preserves the orientation of Y induced by X. Since G acts transitively on edges of X, it acts transitively on vertices of Y. Let xy and xz be two directed 2-paths in X with the arc x in common. To see that G acts transitively on edges of Y it is sufficient to realize that the assumption $|H| > 2$ implies the existence of an element of G fixing the arc x and interchanging the arcs y and z in X. Since Y has twice as many vertices as X, it follows that $[H : K] = 2$, where K is a vertex stabilizer in the action of G on Y.

To prove the second part of (i) assume that $Y = \mathrm{Pl}(X)$ is $\left(G, \frac{1}{2}, K\right)$-arc-transitive for some $K \leq G \leq \operatorname{Aut} Y$. It follows from Proposition 2.1 that $G \leq \operatorname{Aut} X$ and that G preserves the G-orientation of the edges of X induced by Y. Since G acts transitively on G-alternating 4-cycles and vertices of Y, it acts transitively on vertices and edges of X, respectively. Thus X is $\left(G, \frac{1}{2}\right)$-arc-transitive. Since X has half the number of vertices of Y, a vertex stabilizer of the action of G must have twice as many elements as K.

To prove the first part of (ii), we conclude as in (i) that G acts on $Y = \mathrm{Pl}(X)$ as a group of automorphisms and preserves the orientation of the edges in Y induced by X. Since G acts transitively on edges of X, the action of G on Y is transitive on vertices. Since Y has twice as many vertices as X and since the stabilizer of the action on X is isomorphic to $\mathbb{Z}_2$, we conclude that G acts regularly on the set of vertices of Y, and so Y is a Cayley graph of G. Let xy and xz be two directed 2-paths in X. The $\frac{1}{2}$-arc-transitivity implies the existence of automorphisms a, b mapping the arc x onto y and z, respectively. Clearly, both a and b are non-involutory. Thus $Y \cong \mathrm{Cay}(G; a, b)$. Besides, ab^{-1} fixes the head of x. Hence, $(ab^{-1})^2 = 1$.

To prove the converse statement, let $Y = \mathrm{Cay}(G; a, b)$ satisfy the assumptions. The relation $(ab^{-1})^2 = 1$ gives rise to a decomposition of the set of edges into G-alternating 4-cycles. Moreover, the existence of two G-alternating 4-cycles in Y intersecting in two vertices forces G to be abelian, contradicting the assumption. Thus $X = \mathrm{Al}(Y)$ is well defined. As above we deduce that G acts on X as an orientation preserving group of automorphisms. Since the action of G on Y is transitive on G-alternating 4-cycles and on vertices, the action on X is transitive on vertices and edges of X, in other words, X is $(G, \frac{1}{2})$-arc-transitive. Since X has half the number of vertices of Y, it follows that a vertex stabilizer of the action of G on X is isomorphic to $\mathbb{Z}_2$. □

The above result shows that 4-valent graphs of radius 2, and more generally of girth 4, are of crucial importance for the understanding of $\frac{1}{2}$-arc-transitive group actions. These graphs were extensively studied in [8], where it was proved that, apart from a few exceptional families, such graphs have either only alternating 4-cycles or only directed 4-cycles. In the latter case the height of the action must be 1. We say that a $(G, \frac{1}{2})$-arc-transitive graph with G-radius 2 is *G-genuine* if every 4-cycle in X is a G-alternating 4-cycle. It may be seen that the operator Pl preserves genuinity whereas by performing the operator Al no 4-cycle which is not alternating is created, thus preserving genuinity as long as the radius is still 2 (see also [8; Theorem 4.1]).

By making an additional assumption on the genuinity of the graphs in question, it is possible to say how the full automorphism group of a graph behaves with respect to the operator Al.

PROPOSITION 3.2. *Let Y be a graph of valency 4, let $X = \mathrm{Al}(Y)$, and let G be the largest subgroup of $\mathrm{Aut}\,X$ such that X is $(G, \frac{1}{2})$-arc-transitive and Y is G-genuine. Then $\mathrm{Aut}\,X \geq \mathrm{Aut}\,Y$. Moreover, if the G-radius of X is 2, then either*

(i) $\mathrm{Aut}\,X = G = \mathrm{Aut}\,Y$ *and X and Y are $\frac{1}{2}$-arc-transitive;*

or

(ii) $\mathrm{Aut}\,X = \mathrm{Aut}\,Y$, $[\mathrm{Aut}\,X : G] = 2$ *and X and Y are arc-transitive.*

P r o o f . By the genuinity of Y, every 4-cycle in Y is G-alternating. Therefore every automorphism of Y permutes the set of G-alternating 4-cycles and preserves their adjacency. Hence every automorphism of Y can be seen as a permutation of the vertex set of X, preserving adjacency of vertices. Therefore it is an automorphism of X. So $\mathrm{Aut}\,X \geq \mathrm{Aut}\,Y$.

Assume now that the G-radius of X is 2. By the comments preceding the statement of this proposition, a 4-cycle in X is necessarily G-alternating. We now prove the equality $\mathrm{Aut}\,X = \mathrm{Aut}\,Y$. First, since the G-alternating 4-cycles in X decompose the edge set of X, an automorphism α of X induces a permutation of the set of these cycles which either preserves or reverses the orientation of all the edges on a given cycle. Moreover, the behaviour of α on two adjacent G-alternating 4-cycles in X is uniform, that is, the orientation is either preserved or reversed by α. We conclude that α either preserves the G-orientation in X or it reverses this orientation. But then α maps a directed 2-path into a directed 2-path in X, implying that the action of α on Y preserves adjacency in Y, and so $\alpha \in \mathrm{Aut}\,Y$. Thus $\mathrm{Aut}\,X \leq \mathrm{Aut}\,Y$ and then the equality holds.

Now assume that $G \neq \operatorname{Aut} X$. Then there must exist an automorphism γ of X which reverses the G-orientation in X. Obviously, $\gamma^2 \in G$. It follows from the above discussion that $\operatorname{Aut} X = \langle G, \gamma \rangle$. Hence $[\operatorname{Aut} X : G] = 2$, and moreover, X is arc-transitive in this case. □

Note that the above result is quite useful when one wants to construct $\frac{1}{2}$-arc-transitive graphs of valency 4 with large vertex stabilizer (see [7]).

Let X be the Cayley graph $X = \operatorname{Cay}(G; a, b)$ of a group G generated by two non-involutory generators a and b. We say that X *satisfies the property* $\operatorname{Cyc}(h)$ for some integer $h \geq 1$ if a system of h irreducible relations of the form $T_i U_i^{-1} V_i W_i^{-1} = 1$, $i = 1, 2, \ldots, h$, is satisfied in G, where T_i, U_i, V_i, W_i are words of length i consisting of letters a and b, but not containing their inverses a^{-1} and b^{-1}.

It is easily seen that the lexicographic products $C_n[K_2^c]$, $n \geq 3$, are the only 4-valent graphs admitting a $\frac{1}{2}$-arc-transitive group action with respect to which the radius and the attachment number both equal 2. Thus the condition on the G-attachment number in Proposition 3.1(i) can be replaced by the condition $Y \not\cong C_n[K_2^c]$, $n \geq 3$. In view of this fact any graph of the form $\operatorname{Pl}^j(C_n[K_2])$, where $n \geq 3$ and $0 \leq j \leq n-1$, will be called *degenerate*.

THEOREM 3.3.

(i) *Let X be a $(G, \frac{1}{2})$-arc-transitive 4-valent graph with G-height $h \geq 1$ for some subgroup $G \leq \operatorname{Aut} X$. Then there exist non-involutory generators a and b of G such that $\operatorname{Pl}^h(X) \cong \operatorname{Cay}(G; a, b)$ and $\operatorname{Pl}^h(X)$ contains U_h as an induced subgraph, and in particular, $\operatorname{Pl}^h(X)$ satisfies the property $\operatorname{Cyc}(h)$.*

(ii) *Conversely, let G be a group generated by two non-involutory elements a and b such that $Y = \operatorname{Cay}(G; a, b)$ is not degenerate and contains U_h as an induced subgraph for some positive integer h. Then $\operatorname{Al}^h(Y)$ is a $(G, \frac{1}{2})$-arc-transitive graph of G-height h.*

P r o o f . To prove (i) we first apply the operator Pl on the graph X successively $h-1$ times. By Proposition 3.1(i), the graph $\operatorname{Pl}^{h-1}(X)$ is $(G, \frac{1}{2}, \mathbb{Z}_2)$-arc-transitive. By Proposition 3.1(ii), there exist non-involutory generators a and b of G such that the graph $Y = \operatorname{Pl}^h(X)$ is isomorphic to the Cayley graph $\operatorname{Cay}(G; a, b)$. Moreover, Y (being equal to $\operatorname{Pl}^h(X)$) contains an nth image of every vertex of X. Finally, Lemma 2.2(iii) implies that the condition $\operatorname{Cyc}(h)$ is satisfied in Y.

We now prove (ii). Since Y contains U_h as an induced subgraph for some $h \geq 1$, its edge set decomposes into G-alternating 4-cycles. Since Y is not degenerate, Proposition 3.1(ii) implies that $\operatorname{Al}(Y)$ is a $(G, \frac{1}{2}, \mathbb{Z}_2)$-arc-transitive graph. Since $\operatorname{Al}(Y)$ contains U_{h-1} as an induced subgraph and is not degenerate, we can repeat the procedure provided $h > 1$. Using Proposition 3.1(i) at each step we derive that $\operatorname{Al}^h(Y)$ is $(G, \frac{1}{2})$-arc-transitive of height h. □

Let us discuss the degenerate graphs in more detail. The graph $X_n = C_n[K_2^c]$, $n \geq 3$, may be thought of as being formed from a cycle C of length n by replacing each vertex v of C by two vertices v_0 and v_1 and joining u_i to v_j by an edge if and only if the vertices u and v are adjacent in C. If we choose one of the two directed orientations of C, then this orientation induces an orientation of X_n in the obvious way. Clearly, the stabilizer of a vertex u in X_n contains all the transpositions $\tau_v = (v_0, v_1)$, where $v \neq u$. Moreover, these transpositions preserve the prescribed orientation and they generate a group $H = \langle\{\tau_v : v \in V(C) - u\}\rangle$ isomorphic to $\mathbb{Z}_2^{n-1}$. Furthermore, there exists a rotary automorphism ρ of X_n mapping every vertex to one of its two successors. Now, it is easy to realize that X_n is a $\left(G_n, \frac{1}{2}, \mathbb{Z}_2^{n-1}\right)$-arc-transitive graph. By Proposition 3.1, the graph $\mathrm{Pl}^j(X_n)$ is $\left(G_n, \frac{1}{2}, \mathbb{Z}_2^{n-1-j}\right)$-arc-transitive for $0 \leq j \leq n-2$. The automorphisms in G_n can be identified with the elements of the semidirect product $H_n = \mathbb{Z}_2^n \rtimes \mathbb{Z}_n$. It may be checked that H_n can be generated by two non-involutory elements a and b satisfying the relations $(a^i b^{-i})^2 = 1$ for $i = 1, \dots, n-1$. Indeed, if we set $a = (\varepsilon_1, 1)$ and $b = (\varepsilon_2, 1)$, where ε_2 is the image of ε_1 under the action of $1 \in \mathbb{Z}_n$ on $\mathbb{Z}_2^n$, then a direct computation yields the required relations. Set $Y = \mathrm{Cay}(H_n; a, b)$. By Proposition 3.1, the graph $Z = \mathrm{Al}^j(Y)$, $j \in \{1, \dots, n-1\}$, is a $\left(H_n, \frac{1}{2}, \mathbb{Z}_2^j\right)$-arc-transitive graph. In fact, one can prove that Z is isomorphic to $\mathrm{Pl}^{n-1-j}(X_n)$.

We may now give a characterization of $\frac{1}{2}$-arc-transitive group actions of height $h \in \{1, 2, 3\}$.

COROLLARY 3.4. *Let X be a $(G, \frac{1}{2})$-arc-transitive graph of height $1 \leq h \leq 3$. Then there exist non-involutory generators a and b such that $\mathrm{Pl}^h(X) \cong \mathrm{Cay}(G; a, b)$ and a, b satisfy the following relations:*

(i) $ab^{-1}ab^{-1} = 1$ *if* $h = 1$;

(ii) $ab^{-1}ab^{-1} = a^2b^{-2}a^2b^{-2} = 1$ *if* $h = 2$;

(iii) $ab^{-1}ab^{-1} = a^2b^{-2}a^2b^{-2} = a^3b^{-3}a^3b^{-3} = 1$

or

$ab^{-1}ab^{-1} = a^2b^{-2}a^2b^{-2} = a^3b^{-3}a^3b^{-1}a^{-1}b^{-1} = 1$ *if* $h = 3$.

Proof. In view of Theorem 3.3 (i) it is sufficient to prove that if the Cayley graph $Y = \mathrm{Cay}(G; a, b)$, where a and b are non-involutory generators of G, contains U_h, $h \in \{1, 2, 3\}$, as a subgraph, then a and b satisfy relations (i), (ii) and (iii). The case $h = 1$ is trivial. To prove the cases $h = 2$ and $h = 3$, we inspect all possible assignments of arcs of the graphs U_2 and U_3 by elements a and b. Taking into account that every G-alternating 4-cycle can be colored by a and b in a unique way (up to interchanging a with b), and using the automorphisms of U_2 and U_3, we get a unique coloring of U_2, and two colorings of U_3 (see Figure 1). The relations can be now derived from the colored graphs. □

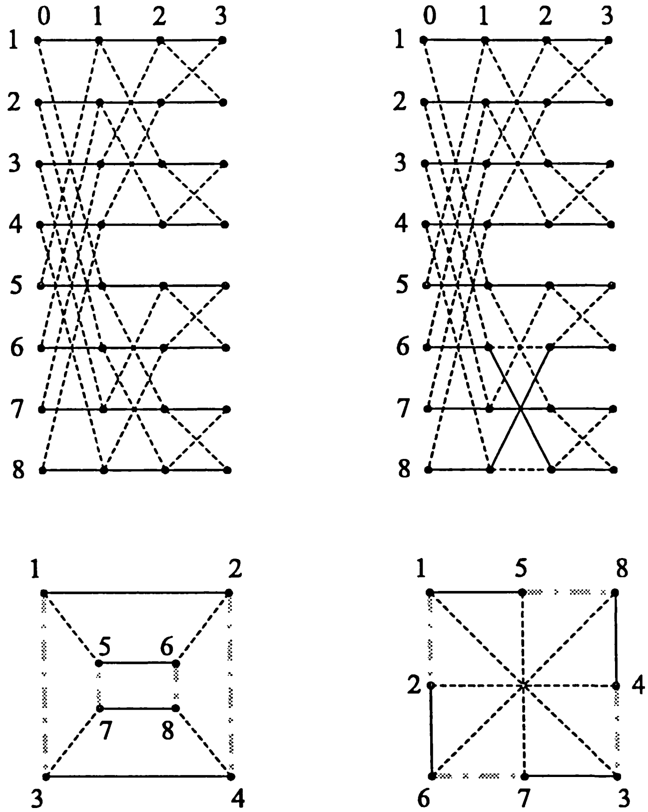

FIGURE 1. Two colorings of U_3 and the associated Cayley graphs — the edges are oriented from "left" to "right".

4. On the vertex stabilizer

The theorem below gives a necessary condition for a group to be a vertex stabilizer of a $\frac{1}{2}$-arc-transitive action on a connected 4-valent graph.

THEOREM 4.1. *Let G be a group acting $\frac{1}{2}$-arc-transitively on a 4-valent graph X with vertex stabilizer H. Then there exists an integer $h \geq 1$ such that*

(i) *H is generated by h involutions $\tau_1, \dots, \tau_h$;*
(ii) *for each $i \in \{0, \dots, h-1\}$ and each $j \in \{1, \dots, h-i\}$, the subgroup $\langle \tau_{i+1}, \dots, \tau_{i+j} \rangle$ has order 2^j;*

(iii) *for any* $i \in \{0,\dots,h-1\}$, $k \in \{0,\dots,i\}$ *and* $j \in \{1,\dots,h-i\}$, *the subgroups* $\langle \tau_{i+1},\dots,\tau_{i+j}\rangle$ *and* $\langle \tau_{k+1},\dots,\tau_{k+j}\rangle$ *are isomorphic.*

Proof. Clearly, due to $\frac{1}{2}$-arc-transitivity, H is a 2-group, say of order 2^h for some integer $h \geq 1$. By Theorem 3.3(i) there exist non-involutory generators a and b of G such that the graph $\mathrm{Pl}^h(X)$ is isomorphic to the Cayley graph $\mathrm{Cay}(G,a,b)$. In what follows, we shall restrict our analysis to a copy of U_h arising as an hth image of a given vertex v in X such that $H = G_v$. Apart from the oriented structure in U_h we also have a coloring of its arcs induced by the two generators a and b. The respective colors will be called *red* and *blue*.

Since $H = G_v$, it follows that H fixes U_h setwise. In view of Lemma 2.2(ii), every monochromatic connected component in U_h is a (directed) path joining a vertex in $V^+ = V^+(U_h)$ with a vertex in $V^- = V^-(U_h)$. Let $\mathcal{B}$ and $\mathcal{R}$ denote the respective sets of all such blue and red paths. Clearly, H permutes elements of $\mathcal{B}$ as well as those of $\mathcal{R}$. Since H is a subgroup of the group G acting regularly on $\mathrm{Pl}^h(X)$, the action of H on each of these two sets is semiregular. But $|H| = 2^h = |V^+| = |\mathcal{B}| = |\mathcal{R}|$, and so both of these actions are regular. We may think of the vertices of U_h as arranged in a $(2^h, h+1)$-array with the rows corresponding to the 2^h blue paths and the $h+1$ columns $\mathcal{C}_i$, $i = 0,1,\dots,h$, consisting of all the vertices at distance i along directed monochromatic paths from the set $\mathcal{C}_0 = V^+$. Clearly, the sets $\mathcal{C}_i$ are precisely the orbits of H on $V(U_h)$. We will now study the regular action of H on $\mathcal{B}$ (rows of the array) in detail. In order to do that, we define an associated graph Y as follows. The vertex set of Y is $\mathcal{B}$ with two blue paths in $\mathcal{B}$ being adjacent in Y if and only if there exists a red arc in U_h joining them in U_h. Note that if x is a red arc joining a blue path P to a blue path Q, then there exists a unique red arc y joining Q to P. The corresponding four vertices form a G-alternating 4-cycle in U_h. (Clearly, on each G-alternating 4-cycle the colors alternate too, see Figure 2 for $h = 3$.) Thus each edge in Y is associated with a pair of red arcs whose heads are in the same column $\mathcal{C}_i$ for some $i \in \{1,2,\dots,h\}$. We may thus define a coloring of the edge set of Y by assigning color $i \in \{1,2,\dots,h\}$ to all of the edges in Y arising from pairs of red arcs with heads in $\mathcal{C}_i$. Clearly, each vertex of Y is incident with an edge of each color, thus giving rise to an edge decomposition of Y. Recall that the group H is regular on the vertex set of Y. Besides, it preserves G-alternating 4-cycles in U_h, and so it preserves adjacency of vertices in Y. Moreover, since the columns $\mathcal{C}_i$ are orbits of the action of H on U_h, colors of the edges in Y are also preserved by H. In fact, the orbits of the action of H on $E(Y)$ are precisely the color classes. Note that since U_h is connected, the graph Y is connected too. It follows that Y is a Cayley graph of a group isomorphic to H, with respect to the generators τ_i, $i \in \{1,\dots,h\}$, given by the coloring of the edges of Y. More precisely, $\tau_i(u) = w$ if and only if $[u,w]$ is an edge in Y colored by i. In particular, all of these generators are distinct involutions because, if $\tau_i = \tau_j$ for some $i < j$, then by the definition of τ_i and τ_j we have the identity $ab^{j-i} = b^{j-i}a$. However, this is impossible because the existence of U_h in $\mathrm{Pl}^h(X) \cong \mathrm{Cay}(G; a, b)$ implies, by Lemma 2.2(v), that all the words in a and b (not containing their inverses) of length at most h are distinct. This proves (i).

The proof of (ii) is now easily at hand. Applying Lemma 2.2(v) it may be seen that a removal of the column C_h from U_h gives rise to two copies of U_{h-1}. In the graph Y this operation corresponds to the removal of all the edges colored with color h, yielding two connected components. Thus there must exist a subgroup of index 2 in H which is generated by elements $\tau_1, \ldots, \tau_{h-1}$. This subgroup acts regularly on each of the two copies of U_{h-1} above. By repeating this operation $h-(i+j)$ times, we get $|\langle\tau_1, \ldots \tau_{i+j}\rangle| = 2^{i+j}$. Let us consider a connectivity component Z of $U_h - \{C_{i+j+1}, \ldots, C_h\}$, that is the graph obtained from U_h by a removal of the columns $C_{i+j+1}, \ldots, C_h$. Note that Z is isomorphic to U_{i+j}. Now let Y' be the subgraph of Y corresponding to Z. It follows that the subgroup $\langle\tau_1, \ldots, \tau_{i+j}\rangle$ acts regularly on Y'. Applying Lemma 2.2(v) we again have that $U_{i+j} - C_1$ consists of two copies of U_{i+j-1}. In Y' the corresponding operation consists of removing all edges colored with color 1. Consequently, Y' splits into isomorphic connectivity components. By repeating this operation i times, we obtain $|\langle\tau_{i+1}, \ldots, \tau_{i+j}\rangle| = 2^j$.

Finally, to prove (iii), let us denote by $H_{i,j}$ and by $H_{k,j}$ the respective subgroups $\langle\tau_{i+1}, \ldots, \tau_{i+j}\rangle$ and $\langle\tau_{k+1}, \ldots, \tau_{k+j}\rangle$ and prove that $H_{i,j} \cong H_{k,j}$. Note that Lemma 2.2(v) implies

$$\begin{aligned} U_h - \{C_1, \ldots, C_i, C_{i+j+1}, \ldots, C_h\} &\cong 2^{h-j} U_j \\ &\cong U_h - \{C_1, \ldots, C_k, C_{k+j+1}, \ldots, C_h\}. \end{aligned}$$

Let us choose connectivity components $U_j' \subseteq U_h - \{C_1, \ldots, C_i, C_{i+j+1}, \ldots, C_h\}$ and $U_j'' \subseteq U_h - \{C_1, \ldots, C_k, C_{k+j+1}, \ldots, C_h\}$. Then U_j' and U_j'' are jth images of vertices u' and u'' of the $(G, \frac{1}{2})$-arc-transitive graph $\mathrm{Al}^j(\mathrm{Pl}^h(X))$. As above we derive that the groups $H_{i,j}$ and $H_{k,j}$ act regularly on the first columns U_j' and U_j'', respectively. Thus $H_{i,j} \cong G_{u'} \cong G_{u''} \cong H_{k,j}$, completing the proof of Theorem 4.1. □

In view of Theorem 4.1 let us now revise the case of height $h \in \{1, 2, 3\}$ discussed in Corollary 3.4. Clearly, $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are the only groups satisfying conditions (i) and (ii) in Theorem 4.1 for $h = 1$ and $h = 2$, respectively. Let $h = 3$. If G_v is abelian, then clearly $G_v \cong \mathbb{Z}_2^3$. If G_v is not abelian, then, by [3; p. 134], it is either dihedral or the group of quaternions. However, Theorem 4.1 excludes the latter possibility, because the group of quaternions does not contain three involutions. On the other hand, condition (ii) is satisfied for $\tau_1 = (1, 3)$, $\tau_2 = (1, 3)(2, 4)$ and $\tau_3 = (1, 2)(3, 4)$. Thus the group $\langle\tau_1, \tau_2, \tau_3\rangle \cong D_8$ satisfies the assumptions of Theorem 4.1. We conclude that either $G_v \cong \mathbb{Z}_2^3$ or $G_v \cong D_8$. The comments below and the construction given in Section 5 show that both possibilities do occur. To summarize, we have the following corollary.

COROLLARY 4.2. *Let X be a $(G, \frac{1}{2})$-arc-transitive graph for some $G \le \operatorname{Aut} X$ of G-height $h \in \{1,2,3\}$ and let $v \in V(X)$. Then $G_v \cong \mathbb{Z}_2$ if $h = 1$, $G_v \cong \mathbb{Z}_2^2$ if $h = 2$, and $G_v \cong \mathbb{Z}_2^3$ or $G_v \cong D_8$ if $h = 3$.*

We remark that $\frac{1}{2}$-arc-transitive group actions with abelian stabilizers are completely characterized in [9]. It is proved there that, given two abstract groups $H \le G$, where H is abelian, there exists a 4-valent $(G, \frac{1}{2}, H)$-arc-transitive graph X if and only if there exist $a, b \in G$ and an integer $h \ge 1$ such that the following conditions are satisfied

(j) $G = \langle a, b \rangle$,
(jj) $\sigma_i = a^i b^{-i}$ is an involution for $i = 1, \dots, h$,
(jjj) $H = \langle \sigma_1, \dots, \sigma_h \rangle$ is not a normal subgroup of G.

Note that in particular, H must be elementary abelian. Using the above result we can construct infinite families of $\frac{1}{2}$-arc-transitive group actions on finite 4-valent graphs with vertex stabilizers isomorphic to an elementary abelian group of arbitrarily large order (see [9]). Moreover $\frac{1}{2}$-arc-transitive 4-valent graphs with elementary abelian stabilizers of arbitrarily large height are constructed in [7].

5. An infinite family with nonabelian stabilizer D_8

The aim of this section is to construct an infinite family of $(G, \frac{1}{2}, D_8)$-arc-transitive graphs. The construction is based on Theorem 3.3(i), which reduces the problem to the task of constructing two non-involutory group elements a and b satisfying an appropriate set of relations. More precisely, for every $n \equiv 0 \pmod 3$ we shall construct a group G_n generated by two non-involutory elements a and b satisfying identities

$$(ab^{-1})^2 = 1\,, \qquad (a^2b^{-2})^2 = 1\,, \qquad a^3b^{-3}a^3b^{-1}a^{-1}b^{-1} = 1\,. \tag{1}$$

The elements a and b are chosen in such a way that $Y_n = \operatorname{Cay}(G_n; a, b)$ contains a copy of U_3. By Theorem 3.3(ii) we then have that the graph $X_n = \operatorname{Al}^3(Y_n)$ is $(G_n, \frac{1}{2})$-arc-transitive with G-height 3. Combining together Corollary 3.4 and the comments at the end of the previous section we have that the stabilizer in G_n of a vertex of X_n is isomorphic to D_8.

Let us now define the two elements a and b which generate the group G_n. We do this by representing them as permutations acting on the set $\mathbb{Z}_8 \times \mathbb{Z}_n$, where $n \equiv 0 \pmod 3$ in the following way:

$$(i,j)^a = \big(f_j(i), j+1\big) \quad \text{and} \quad (i,j)^b = \big(g_j(i), j+1\big), \qquad i \in \mathbb{Z}_8,\ j \in \mathbb{Z}_n,$$

where the functions $f_j(i)$ and $g_j(i)$ are defined in Tables 1 and 2 below.

Table 1. The function $f_j(i)$.

$f_j(i)$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
$j \equiv 0 \pmod 3$	0	5	2	3	4	1	6	7
$j \equiv 1 \pmod 3$	0	1	2	3	6	5	4	7
$j \equiv 2 \mod 3$	0	1	3	2	4	5	6	7

Table 2. The function $g_j(i)$.

$g_j(i)$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
$j \equiv 0 \pmod 3$	4	1	6	7	0	5	2	3
$j \equiv 1 \pmod 3$	2	3	0	1	4	7	6	5
$j \equiv 2 \pmod 3$	1	0	2	3	5	4	7	6

Since $(i,j)^a = (i, j+3)^a$ and $(i,j)^b = (i, j+3)^b$, in order to prove that the identities (1) hold true, it is sufficient to verify that the permutations $\alpha = (ab^{-1})^2$, $\beta = (a^2b^{-2})^2$, and $\gamma = a^3b^{-3}a^3b^{-1}a^{-1}b^{-1}$ fix each of the 24 ordered pairs (i,j), $i \in \mathbb{Z}_8$ and $j \in \mathbb{Z}_3$. Indeed, this is the case and therefore the group G_n satisfies the required properties.

Let us remark that the above definition of the elements a and b as permutations on $\mathbb{Z}_8 \times \mathbb{Z}_n$ comes from an appropriate coloring of arcs of the oriented graph U_3 (see Figure 2). Define a (colored and oriented) base graph B_n with vertex set $\mathbb{Z}_8 \times \mathbb{Z}_n$ and with arcs joining (i,j) to $\big(f_j(i), j+1\big)$ and to $\big(g_j(i), j+1\big)$. In other words the graph B_n is obtained by consecutive glueing of $\frac{n}{3}$ copies of U_3 colored as in Figure 2. It may be seen that the Cayley graph Y_n covers B_n in such a way that the coloring is preserved. To prove that U_3 is a subgraph of Y_n, it suffices to show that an arbitrary copy of U_3 in B_n lifts to disjoint unions of U_3 in Y_n. We omit the details.

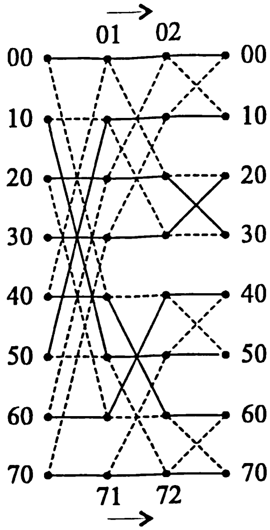

FIGURE 2. The coloring of U_3 defining the group $G_3 = \langle a, b : (ab^{-1})^2 = (a^2b^{-2}) = a^3b^{-3}a^3b^{-1}a^{-1}b^{-1} = 1, \dots \rangle$.

The smallest group G_3 in the above family is generated by the following two permutations

$$a = (00, 01, 02)(10, 51, 52, 50, 11, 12)(20, 21, 22, 30, 31, 32)(40, 41, 62, 60, 61, 42)(70, 71, 72),$$

$$b = (00, 41, 42, 50, 51, 72, 60, 21, 02, 10, 11, 32, 30, 71, 52, 40, 01, 22, 20, 61, 62, 70, 31, 12),$$

where (i, j) is identified with ij for all $(i, j) \in \mathbb{Z}_8 \times \mathbb{Z}_3$. The group G_3 has 1008 elements and hence the associated 4-valent $(G_3, \frac{1}{2}, D_8)$-arc-transitive graph $\mathrm{Al}^3(Y_3)$ has 126 vertices.

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