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CERTAIN POLYNOMIAL IDENTITIES AND COMMUTATIVITY OF RINGS

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ABSTRACT. Let $m > 1$, $k > 0$, $s \geq 0$, $n \geq 0$, $t \geq 0$ be non-negative integers and R be an associative ring (may be without unity 1). In the present paper it is shown that R is commutative if and only if it satisfies the property $[x, y] = y^s[x^n, y^m]^k y^t$, for all $x, y \in R$. Commutativity of ring with unity 1 has also been obtained if it satisfies some related polynomial identities. Finally, the result for rings with unity is extended to one sided s-unital rings.

A well-known theorem of Jacobson [4] states that if every element x of a ring R satisfies the relation $x^{n(x)} = x$, where $n(x) > 1$ is a positive integer, then R is commutative. This result at the same time generalizes the theorem of Wedderburn that every finite division ring is commutative and also the result that any Boolean ring is commutative. Further, Herstein [3; Theorem 2] generalized Jacobson's result as follows (signified as Theorem H in sequel): if R is a ring in which for every $x, y \in R$ there exists a positive integer $n = n(x, y) > 1$ such that $[x, y] = [x^n, y]$, then R is commutative.

Inspired by these works we consider the following ring properties:

(P) For all $x, y \in R$, $[x, y] = y^s[x^n, y^m]^k y^t$, where $m > 1$, $k > 0$, $r \geq 0$, $s \geq 0$, $n \geq 0$, $t \geq 0$ are fixed non-negative integers.

(P_1) For all $x, y \in R$, $x^r[x, y] = y^s[x^n, y^m]^k y^t$, where $m > 1$, $k > 0$, $r \geq 0$, $s \geq 0$, $n \geq 0$, $t \geq 0$ are fixed non-negative integers.

(P_2) For all $x, y \in R$, $[x, y]x^r = y^s[x^n, y^m]^k y^t$, where $m > 1$, $k > 0$, $r \geq 0$, $s \geq 0$, $n \geq 0$, $t \geq 0$ are fixed non-negative integers.

We begin with a commutativity theorem for rings with a polynomial identity hypothesis:

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THEOREM 1. *Let R be a ring with unity 1. Then the following statements are equivalent:*

(i) *R is commutative.*
(ii) *R satisfies (P_1).*
(iii) *R satisfies (P_2).*

For the ring R, the symbol $Z(R)$, $N(R)$ and $C(R)$ will denote respectively the centre, the set of nilpotent elements and the commutator ideal. For any $x, y \in R$, as usual $[x,y] = xy - yx$.

In order to facilitate our discussion we state the following well-known lemmas, which are essentially proved in [6; p. 221], [9], [7] respectively.

LEMMA 1. *Let $x, y \in R$ such that $\big[x,[x,y]\big] = 0$. Then $[x^k, y] = kx^{k-1}[x,y]$ for all positive integer k.*

LEMMA 2. *Let R be a ring with unity 1 and $f\colon R \to R$ be a function such that $f(x+1) = f(x)$ holds for all $x \in R$. If there exists a positive integer k such that either $x^k f(x) = 0$ or $f(x)x^k = 0$ for all $x \in R$, then $f(x) = 0$ for all $x \in R$.*

LEMMA 3. *Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \dots, x_n$ with coprime integral coefficients. Then the following are equivalent:*

(i) *For any ring satisfying $f = 0$, $C(R)$ is a nill ideal.*
(ii) *For any prime p, $\big(GF(p)\big)_2$ fails to satisfy $f = 0$.*
(iii) *Every semi-prime ring satisfying $f = 0$ is commutative.*

Now we prove the following:

LEMMA 4. *Let R be a ring satisfying either of the properties (P_1) or (P_2). Then $C(R) \subseteq N(R)$.*

Proof. Let R satisfy the polynomial identity (P_1). We see that $x = e_{11} + e_{12}$, $y = e_{12}$ fail to satisfy this equality in $\big(GF(p)\big)_2$, p a prime. Hence by Lemma 3, we get the required result. On the other hand if R satisfies (P_2), then use similar arguments with the choice of $x = e_{12} + e_{22}$, $y = e_{12}$ to get the required result. $\square$

It is easy to see that every commutative ring R satisfies the properties (P_1) and (P_2). Conversely, if a semi-prime ring R satisfies either of the properties (P_1) or (P_2), then the above lemma together with the Lemma 3, yield that R is commutative. This proves the following:

Theorem 2. *Let R be a semi-prime ring. Then the following statements are equivalent:*

(i) *R is commutative.*
(ii) *R satisfies* (P_1).
(iii) *R satisfies* (P_2).

Lemma 5. *Let R be a ring with unity 1 satisfying either of the properties* (P_1) *or* (P_2). *Then $N(R) \subseteq Z(R)$.*

P r o o f. Let R satisfy the property (P_1), and let $a \in N(R)$. Then there exists an integer $q \geq 1$ such that

$$a^{\ell} \in Z(R) \qquad \text{for all} \quad \ell \geq q, \quad q \text{ minimal.} \tag{1}$$

If $q = 1$, then for each such a, the result is obvious. Therefore, assume that $q > 1$. Now replace y by a^{q-1} in (P_1), to get

$$x^r\left[x, a^{q-1}\right] = a^{(q-1)s}\left[x^n, a^{(q-1)m}\right]^k a^{(q-1)t}.$$

Now in view of (1) and the fact that $(q-1)m \geq q$ for $m > 1$, we obtain that $x^r[x, a^{q-1}] = 0$ for all $x \in R$. Using Lemma 2, we find that $[x, a^{q-1}] = 0$, i.e. $a^{q-1} \in Z(R)$. This contradicts the minimality of q in (1). Thus, $q = 1$ and $a \in Z(R)$. Using similar arguments we can prove the result if R satisfies the property (P_2). □

P r o o f o f T h e o r e m 1. Obviously (i) implies (ii) and (iii). Now we show that (ii) implies (i).

If R satisfies the property (P_1), then using Lemmas 4 and 5, we have

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{2}$$

Suppose that $k > 1$. In view of (2), (P_1) can be written as

$$x^r[x,y] = [x^n, y^m]^k y^{s+t}. \tag{3}$$

Now using (2) and (3), we find that for any positive integer β

$$\begin{aligned}
&x^{r+knr+k^2n^2r+\cdots+k^{\beta-1}n^{\beta-1}r}[x,y]\\
&= x^{knr+k^2n^2r+\cdots+k^{\beta-1}n^{\beta-1}r}[x^n, y^m]^k y^{s+t}\\
&= x^{k^2n^2r+\cdots+k^{\beta-1}n^{\beta-1}r}\left(x^{nr}[x^n, y^m]\right)^k y^{s+t}\\
&= x^{k^2n^2r+\cdots+k^{\beta-1}n^{\beta-1}r}\left([x^{n^2}, y^{m^2}]^k y^{m(s+t)}\right)^k y^{s+t}\\
&= x^{k^2n^2r+\cdots+k^{\beta-1}n^{\beta-1}r}[x^{n^2}, y^{m^2}]^{k^2} y^{(s+t)+mk(s+t)}\\
&= x^{k^3n^3r+\cdots+k^{\beta-1}n^{\beta-1}r}[x^{n^3}, y^{m^3}]^{k^3} y^{(s+t)+km(s+t)+k^2m^2(s+t)}\\
&\ \ \vdots\\
&= [x^{n^\beta}, y^{m^\beta}]^{k^\beta} y^{(s+t)+km(s+t)+k^2m^2(s+t)+\cdots+k^{\beta-1}m^{\beta-1}(s+t)}.
\end{aligned}$$

Thus in view of (2) and Lemma 1 the above yields that

$$x^{r+knr+\cdots+k^{\beta-1}n^{\beta-1}r}[x,y]$$
$$=\left(n^{\beta}m^{\beta}x^{n^{\beta}-1}y^{m^{\beta}-1}\right)^{k^{\beta}}[x,y]^{k^{\beta}}y^{(s+t)+km(s+t)+\cdots+k^{\beta-1}m^{\beta-1}(s+t)}.$$

But since commutators are nilpotent, hence we find that

$$x^{r+knr+\cdots+k^{\beta-1}m^{\beta-1}r}[x,y]=0,$$

and by Lemma 2, R is commutative. Henceforth, we shall consider the remaining possibility that $k=1$, which gives

$$x^r[x,y]=y^s[x^n,y^m]y^t \qquad \text{for all} \quad x,y\in R. \tag{4}$$

Now choose positive integer $\alpha = 2^{m+s+t}-2>0$ such that $\alpha x^r[x,y] = (2y)^s\left[x^n,(2y)^m\right](2y)^t - x^r[x,2y] = 0$. Application of Lemma 2, yields that $\alpha[x,y]=0$. Now in view of (2) and Lemma 1, we see that $[x^{\alpha},y]=\alpha x^{\alpha-1}[x,y]$ $=0$ for all $x\in R$, i.e.

$$x^{\alpha}\in Z(R), \qquad \text{for all} \quad x\in R \ (\alpha=2^{m+s+t}-2). \tag{5}$$

Now application of Lemma 1, (2) and (4) several times, yields that

$$\begin{aligned}&\left(1-y^{(m-1)(s+t+m-1)}\right)[x,y^m]x^r\\&=x^r[x,y^m]-y^{(m-1)(s+t+m-1)}my^{m-1}[x,y]x^r\\&=x^r[x,y^m]-my^{(m-1)^2}y^{m-1}y^{(m-1)(s+t)}y^s[x^n,y^m]y^t\\&=x^r[x,y^m]-my^{m(m-1)}y^{ms}[x^n,y^m]y^{mt}\\&=x^r[x,y^m]-y^{ms}[x^n,y^{m^2}]y^{mt}\\&=0.\end{aligned}$$

Apply Lemma 2, to get $\left(1-y^{(m-1)(s+t+m-1)}\right)[x,y^m]=0$. Now replacing x by x^n in the last equation we have $\left(1-y^{(m-1)(s+t+m-1)}\right)[x^n,y^m]=0$, which yields that $\left(1-y^{(m-1)(s+t+m-1)}\right)[x^n,y^m]y^{s+t}=0$, i.e. $\left(1-y^{(m-1)(s+t+m-1)}\right)[x,y]x^r=0$. Again by Lemma 2, we find that $\left(1-y^{(m-1)(s+t+m-1)}\right)[x,y]=0$. This implies that $\left(1-y^{\alpha(m-1)(s+t+m-1)}\right)[x,y]=0$ (α being as in (5)). In view of (5) the last identity can be rewritten as $\left[x,y-y^{\alpha(m-1)(s+t+m-1)+1}\right]=0$, for all $x,y\in R$. Hence by Theorem H, R is commutative.

Using similar arguments, it can be easily shown that (iii) implies (i). □

Remark 1. The existence of non-commutative ring R with R^2 being central rules out the possible generalization of the above theorem for arbitrary rings.

Remark 2. In the proof of Theorem 1, (ii) implies (i) could have been obtained by [8; Theorem]. But, we preferred to provide a direct proof with a view to preparing some ground work for the following theorem, which establishes commutativity of arbitrary rings.

THEOREM 3. *A ring R (may be without unity 1) is commutative if and only if R satisfies the property* (P).

Proof. Obviously, every commutative ring satisfies (P).

Conversely, assume that R satisfies the property (P). A careful scrutiny of the proofs of Lemmas 4 and 5 shows that they are still valid in the present situation, and hence we have

$$C(R) \subseteq N(R) \subseteq Z(R) . \tag{6}$$

If $k > 1$, then for an arbitrary integer β, using similar techniques as used in the proof of Theorem 1 we have

$$[x, y] = [x^{n^\beta}, y^{m^\beta}]^{k^\beta} y^{(s+t)+km(s+t)+\cdots+k^{\beta-1}m^{\beta-1}(s+t)} .$$

Now application of Lemma 1, yields that

$$[x, y] = \left(n^\beta m^\beta x^{n^\beta-1} y^{m^\beta-1}\right)^{k^\beta} [x, y]^{k^\beta} y^{(s+t)+km(s+t)+\cdots+k^{\beta-1}m^{\beta-1}(s+t)} .$$

But, since commutators are nilpotent, the above yields the required result. On the other hand if $k = 1$, then (P) can be written as

$$[x, y] = y^s[x^n, y^m]y^t \qquad \text{for all} \quad x, y \in R . \tag{7}$$

Now, apply similar techniques as used to get (5), we have

$$x^\alpha \in Z(R) \qquad \text{for all} \quad x \in R \ (\alpha = 2^{m+s+t} - 2) . \tag{8}$$

Now using Lemma 1 and (7) repeatedly, we have

$$\begin{aligned}
[x, y^m] - y^{(m-1)(s+t+m-1)}[x, y^m] &= [x, y^m] - y^{(m-1)(s+t+m-1)} m y^{m-1}[x, y] \\
&= [x, y^m] - m y^{m(m-1)} y^{ms}[x^n, y^m] y^{mt} \\
&= [x, y^m] - y^{ms}[x^n, y^{m^2}] y^{mt} \\
&= 0 .
\end{aligned}$$

This yields that $y^s\{[x, y^m] - y^{(m-1)(s+t+m-1)}[x, y^m]\}y^t = 0$, and in view of (7), we find that $[x, y] = y^{(m-1)(s+t+m-1)}[x, y]$. This implies that $[x, y] = y^{\alpha(m-1)(s+t+m-1)}[x, y]$ (α being as in (8)). Now application of (8) gives that $\left[x, y - y^{\alpha(m-1)(s+t+m-1)+1}\right] = 0$, and hence R is commutative by Theorem H. $\square$

Remark 3. Particularly, for $s = 0$, $t = 0$, the above theorem reduces to Theorem 2 of [2].

In view of Remark 1, Theorem 1 cannot be extended to arbitrary rings. However, we extend Theorem 1 to a wider class of rings called s-unital. A ring

R is called *left* (resp. *right*) *s-unital* if $x \in Rx$ (resp. $x \in xR$) for all $x \in R$. A ring R is called *s-unital* if and only if $x \in Rx \cap xR$ for all $x \in R$. If R is s-unital (resp. left or right s-unital), then for any finite subset F of R there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e is called a pseudo-identity of F. Our next result states as follows.

THEOREM 4. *A left (resp. right) s-unital ring R is commutative if and only if R satisfies the property* (P_1) *(resp.* (P_2)*).*

Proof. Every commutative left (resp. right) s-unital ring satisfies the property (P_1) (resp. (P_2)).

Conversely, let R be a left (resp. right) s-unital ring satisfying (P_1) (resp. (P_2)), and y be arbitrary element of R. Choose an element $e \in R$ such that $ey = y$ (resp. $ye = y$). Replace x by e, to get

$$e^r[e,y] = y^s[e^n,y^m]^k y^t \qquad (\text{resp. } [e,y]e^r = y^s[e^n,y^m]^k y^t)\,.$$

If $k = 1$ this yields that $y = y\bigl(e + y^{s+t+m-1} - y^{s+m-1}e^n y^t\bigr) \in yR$ (resp. $y = \bigl(e - y^s e^n y^{m+t-1} + y^{s+t+m-1}\bigr)y \in Ry$), since $m > 1$. On the other hand if $k > 1$, then we find that $e^r[e,y] = 0$ (resp. $[e,y]e^r = 0$), i.e. $y = ye \in yR$ (resp $y = ey \in Ry$) for all $y \in R$. Therefore, R is right (resp. left) s-unital. Thus, R is s-unital, and by Proposition 1 of [4], we can assume that R has unity 1. Hence, R is commutative by Theorem 1.

Remark 4. Notice that in the above theorems, the exponents in the [illegible] conditions are assumed to be 'global'. It would be interesting to further [illegible] alize these results for the case when they are assumed to be 'local' [illegible] i.e. they depend on the pair of elements x, y for their values

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