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DETERMINANTAL REPRESENTATION OF $\{I, J, K\}$ INVERSES AND SOLUTION OF LINEAR SYSTEMS

PREDRAG STANIMIROVIĆ

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ABSTRACT. In this paper we investigate the determinantal representation of $\{1,2\}$, $\{1,2,3\}$ and $\{1,2,4\}$ inverses, introduced in the papers [STANIMIROVIĆ, P.—STANKOVIĆ, M.: *Determinantal representation of weighted Moore-Penrose inverse*, Mat. Vesnik **46** (1994), 41–50] ([19]), [STANIMIROVIĆ, P.—STANKOVIĆ, M.: *Generalized algebraic complement and Moore-Penrose inverse*, Filomat **8** (1994), 57–64], [STANIMIROVIĆ, P.: *General determinantal representation of pseudoinverses and its computation*, Rev. Acad. Cienc. Zaragoza (2) **50** (1995), 41–49] in the light of the recent papers [MIAO, J.: *Reflexive generalized inverses and their minors*, Linear and Multilinear Algebra **35** (1993), 153–163], [PRASAD, K. M.: *Generalized inverses of matrices over commutative ring*, Linear Algebra Appl. **211** (1994), 35–53]. We generalize results of [19], and prove that the determinantal representation developed in [19] is necessary and sufficient for a matrix to be a $\{1,2\}$ inverse. Furthermore, we develop determinantal representation of the solution of a system of linear equations, based on the representation of $\{1,2\}$ inverses introduced.

1. Introduction

Let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ complex matrices, let $\mathbb{C}^n$ be the set of n-dimensional complex vectors, and let $\mathbb{C}_r^{m\times n} = \{X \in \mathbb{C}^{m\times n} : \operatorname{rank}(X) = r\}$. The adjoint matrix of a square matrix B is denoted by $\operatorname{adj}(B)$, and its determinant by $|B|$. The conjugate, transposed and conjugate-transposed matrices of A are denoted by $\overline{A}$, A^T and A^*, respectively. We denote the unit matrix of order k by I_k. The minor of $A \in \mathbb{C}^{m\times n}$ containing rows $\alpha_1, \dots, \alpha_t$ and columns $\beta_1, \dots, \beta_t$ is denoted by $A\begin{pmatrix}\alpha_1 & \dots & \alpha_t\\ \beta_1 & \dots & \beta_t\end{pmatrix} = |A_\beta^\alpha|$, and its algebraic complement is

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defined by

$$A_{ij}\begin{pmatrix}\alpha_1 & \dots & \alpha_{p-1} & i & \alpha_{p+1} & \dots & \alpha_t\\ \beta_1 & \dots & \beta_{q-1} & j & \beta_{q+1} & \dots & \beta_t\end{pmatrix} = (-1)^{p+q} A\begin{pmatrix}\alpha_1 & \dots & \alpha_{p-1} & \alpha_{p+1} & \dots & \alpha_t\\ \beta_1 & \dots & \beta_{q-1} & \beta_{q+1} & \dots & \beta_t\end{pmatrix}.$$

We use the notation $C_r(A)$ to denote the rth compound matrix of $A \in \mathbb{C}_r^{m\times n}$, whose rows are indexed by r-element combinations α of the set $\{1,\dots,m\}$, and whose columns are indexed by r-element subsets β of the set $\{1,\dots,n\}$, and we denote the general (α,β) entry by $|A^{\alpha}_{\beta}|$.

We denote by $A(i \to z)$, $i \in \{1,\dots,n\}$, the matrix obtained from A by replacing its ith column by the vector z.

Consider the following Penrose equations in X:

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$$(AX)^* = AX, \tag{3}$$

$$(XA)^* = XA \tag{4}$$

and the following equation, applicable to square matrices:

$$AX = XA. \tag{5}$$

The set of matrices obeying the conditions contained in a sequence $\mathcal{S}$ of $\{1,2,3,4,5\}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A, and it is denoted by $A^{(\mathcal{S})}$. In particular, for any $A \in \mathbb{C}^{m\times n}$, the *Moore-Penrose inverse* of A, denoted by $A^\dagger$ is the unique $\{1,2,3,4\}$ inverse of A ([12]). In the case $m = n$, the *group inverse* of A, denoted by $A^\#$, is the unique $\{1,2,5\}$ inverse ([7]).

In [1] Arghiriade and Dragomir tried to use the method of determinantal inversion in order to get the determinantal representation of the Moore-Penrose inverse of a full rank matrix. In [8], the determinantal representation of the Moore-Penrose inverse of an arbitrary matrix is obtained. The proof is improved in [9]. In [20], we develop an elegant proof, based on a full-rank factorization $A = PQ$. More precisely, we use $A^\dagger = Q^\dagger P^\dagger$ and the well-known results for full rank matrices. In [2], the determinantal representation of the Moore-Penrose inverse over an integral domain is investigated.

THEOREM 1.1. *The element lying on the i-row and j-column of the Moore-Penrose inverse $A^\dagger = \left(a^\dagger_{ij}\right)$ of a given matrix $A \in \mathbb{C}_r^{m\times n}$ can be represented in terms of minors of A as follows:*

$$a^\dagger_{ij} = \frac{\displaystyle\sum_{\substack{1\le\beta_1<\dots<\beta_r\le n\\ 1\le\alpha_1<\dots<\alpha_r\le m}} \overline{A}\begin{pmatrix}\alpha_1 & \dots & i & \dots & \alpha_r\\ \beta_1 & \dots & j & \dots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & i & \dots & \alpha_r\\ \beta_1 & \dots & j & \dots & \beta_r\end{pmatrix}}{\displaystyle\sum_{\substack{1\le\delta_1<\dots<\delta_r\le n\\ 1\le\gamma_1<\dots<\gamma_r\le m}} A\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix} \overline{A}\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}}. \tag{1.1}$$

The determinantal representation of the group inverse is introduced in [21] for complex matrices, and in [12] for matrices over an integral domain.

THEOREM 1.2. *The group inverse* $A^{\#} = \left(a_{ij}^{\#}\right)$ *of* $A \in \mathbb{C}_r^{n\times n}$ *exists if and only if* $\sum\limits_{1\le\gamma_1<\cdots<\gamma_r\le n} A\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \gamma_1 & \dots & \gamma_r\end{pmatrix} \neq 0$, *and possesses the following determinantal representation:*

$$a_{ij}^{\#} = \frac{\sum\limits_{\substack{1\le\alpha_1<\cdots<\alpha_r\le n\\ 1\le\beta_1<\cdots<\beta_r\le n}} A^T\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\le\gamma_1<\cdots<\gamma_r\le n\\ 1\le\delta_1<\cdots<\delta_r\le n}} A^T\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix} A\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}}. \tag{1.2}$$

The determinantal representation of the weighted Moore-Penrose inverse is studied in [19] for complex matrices, and in [14] for matrices over an integral domain.

THEOREM 1.3. *The weighted Moore-Penrose inverse* $A_{M,N}^{\dagger}$ *of* $A \in \mathbb{C}_r^{m\times n}$ *possesses the following determinantal representation:*

$$\left(A_{M,N}^{\dagger}\right)_{ij} = \frac{\sum\limits_{\substack{1\le\alpha_1<\cdots<\alpha_r\le m\\ 1\le\beta_1<\cdots<\beta_r\le n}} \overline{MAN}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\le\gamma_1<\cdots<\gamma_r\le m\\ 1\le\delta_1<\cdots<\delta_r\le n}} \overline{MAN}\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix} A\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}}. \tag{1.3}$$

In [19] we discover the *general determinantal representation* for $\{1,2\}$ inverses of complex matrices. The qualitative improvement is ensured by using the minors of two arbitrary matrices, as well as the minors of the given matrix:

THEOREM 1.4. *Let* $A = PQ$ *be a full-rank factorization of* $A \in \mathbb{C}_r^{m\times n}$ *and let* $W_1 \in \mathbb{C}^{n\times r}$, $W_2 \in \mathbb{C}^{r\times m}$ *be matrices such that*

$$\operatorname{rank}(QW_1) = \operatorname{rank}(Q), \qquad \operatorname{rank}(W_2P) = \operatorname{rank}(P).$$

The element lying on the ith row and jth column of $A^{(1,2)} = \left(a_{ij}^{(1,2)}\right)$ *is given by*

$$a_{ij}^{(1,2)} = \frac{\sum\limits_{\substack{1\le\beta_1<\cdots<\beta_r\le n\\ 1\le\alpha_1<\cdots<\alpha_r\le m}} (W_1W_2)^T\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\le\delta_1<\cdots<\delta_r\le n\\ 1\le\gamma_1<\cdots<\gamma_r\le m}} A\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix} (W_1W_2)^T\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}}. \tag{1.4}$$

In [21] we define the *algebraic complement of order* $t \leq \operatorname{rank}(A)$ and the *general determinantal representation of order* t. In this way, in the case $t = r$, we obtain the determinantal representation for all of the generalized inverses mentioned above. Moreover, in [21] we investigate the implementation of the determinantal representation introduced in the programming language C.

THEOREM 1.5. *The determinantal representation of the order* t *of a given matrix* $A \in \mathbb{C}_r^{m\times n}$ *is given by*

$$\frac{\displaystyle\sum_{\substack{1\leq\alpha_1<\cdots<\alpha_t\leq m\\ 1\leq\beta_1<\cdots<\beta_t\leq n}} \overline{R}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_t\\ \beta_1 & \dots & i & \dots & \beta_t\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_t\\ \beta_1 & \dots & i & \dots & \beta_t\end{pmatrix}}{\displaystyle\sum_{\substack{1\leq\gamma_1<\cdots<\gamma_t\leq m\\ 1\leq\delta_1<\cdots<\delta_t\leq n}} \overline{R}\begin{pmatrix}\gamma_1 & \dots & \gamma_t\\ \delta_1 & \dots & \delta_t\end{pmatrix} A\begin{pmatrix}\gamma_1 & \dots & \gamma_t\\ \delta_1 & \dots & \delta_t\end{pmatrix}}, \tag{1.5}$$

where $\operatorname{rank}(R) = r$ *and* $1 \leq t \leq \operatorname{rank}(A)$ *is the largest integer for which the denominator in* (1.5) *is different from zero.*

In the case $R = (W_1W_2)^*$, where the matrices $W_1 \in \mathbb{C}^{n\times r}$ and $W_2 \in \mathbb{C}^{r\times m}$ satisfy $\operatorname{rank}(QW_1) = \operatorname{rank}(Q)$, $\operatorname{rank}(W_2P) = \operatorname{rank}(P)$, Theorem 1.5. reduces to Theorem 1.4. Representation (1.5) is applicable in the case when the rank of the given matrix is unknown. In this case, we start computation with $t = \min\{m, n\}$, and the rank of given matrix is determined during the computation.

For the sake of completeness, we restate the general forms of $\{i, j, k\}$ inverses, the Moore-Penrose and the group inverse.

THEOREM 1.6. *If* $A \in \mathbb{C}_r^{m\times n}$ *has a full-rank factorization* $A = PQ$, $P \in \mathbb{C}_r^{m\times r}$, $Q \in \mathbb{C}_r^{r\times n}$, *and* $W_1 \in \mathbb{C}^{n\times r}$, $W_2 \in \mathbb{C}^{r\times m}$ *are matrices such that* $\operatorname{rank}(QW_1) = \operatorname{rank}(Q)$, $\operatorname{rank}(W_2P) = \operatorname{rank}(P)$, *then:*

$A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^*$ ([3]);
the general solution of (1), (2) *is* $W_1(QW_1)^{-1}(W_2P)^{-1}W_2$ ([16], [18]);
the general solution of (1), (2), (3) *is* $W_1(QW_1)^{-1}(P^*P)^{-1}P^*$ ([16]);
the general solution of (1), (2), (4) *is* $Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2$ ([16]);
$A^\#$ *exists if and only if* QP *is invertible, and* $A^\# = P(QP)^{-2}Q$ ([6]).

We now describe the main results of the paper. In the second section we investigate, in detail, the determinantal representation of $\{1,2\}$ inverses of a rectangular matrix, as well as the conditions for its existence. Using Miao's results [11] we generalize the result of Theorem 1.4. More precisely, we prove that the determinantal representation (1.4) is necessary and sufficient for a matrix to be a $\{1,2\}$ inverse of A. The determinantal representations of $\{1,2,3\}$ and $\{1,2,4\}$ inverses are derived from Theorem 1.4. Moreover, the Moore-Penrose, weighted Moore-Penrose inverse and the group inverse can be obtained as special cases of the class of $\{1,2\}$ inverses, according to Theorem 1.6.

In the third section, we point out several characteristic definitions of determinants of rectangular matrices and corresponding $\{1,2\}$ inverses, as special cases of Theorem 1.4.

Finally, in the last section, for a given system of linear equations $Ax = z$, we represent the elements of $A^{(1,2)}z$ as ratios of sums of determinants. In this way, we develop and investigate the determinantal representation of the solution of a system of linear equations, derived in terms of the determinantal representation of $\{1,2\}$ inverses. The well-known representation of the Moore-Penrose solution $A^{\dagger}z$ can be derived in a certain case.

2. Determinantal representations of $\{i, j, k\}$ inverses

In the following theorem we generalize Theorem 1.4.

THEOREM 2.1. *Let $A \in \mathbb{C}_r^{m\times n}$ possesses a full-rank factorization $A = PQ$. The matrix $G = (g_{ij})$ is $\{1,2\}$ inverse of A if and only if g_{ij} is represented in the form*

$$g_{ij} = \frac{\displaystyle\sum_{\substack{1\le\beta_1<\cdots<\beta_r\le n\\ 1\le\alpha_1<\cdots<\alpha_r\le m}} (W_1W_2)^T \begin{pmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix} A_{ji} \begin{pmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix}}{\displaystyle\sum_{\substack{1\le\delta_1<\cdots<\delta_r\le n\\ 1\le\gamma_1<\cdots<\gamma_r\le m}} A\begin{pmatrix} \gamma_1 & \dots & \gamma_r \\ \delta_1 & \dots & \delta_r \end{pmatrix} (W_1W_2)^T \begin{pmatrix} \gamma_1 & \dots & \gamma_r \\ \delta_1 & \dots & \delta_r \end{pmatrix}}, \tag{2.1}$$

where $W_1 \in \mathbb{C}^{n\times r}$, $W_2 \in \mathbb{C}^{r\times m}$ are matrices such that

$$\operatorname{rank}(QW_1) = \operatorname{rank}(Q), \qquad \operatorname{rank}(W_2P) = \operatorname{rank}(P). \tag{2.2}$$

Proof.

$(\Longrightarrow)$: Suppose that $G = A^{(1,2)}$ is an arbitrary $\{1,2\}$ inverse of A. We prove that G possesses the determinantal representation (2.1). So this part of the proof is given in detail. Starting from

$$A^{(1,2)} = W_1(QW_1)^{-1}(W_2P)^{-1}W_2,$$

it is easy to see that $a_{ij}^{(1,2)}$ is equal to

$$a_{ij}^{(1,2)} = \frac{\left(W_1 \operatorname{adj}(QW_1)\cdot \operatorname{adj}(W_2P)W_2\right)_{ij}}{|QW_1||W_2P|}.$$

By applying the Cauchy-Binet theorem, the denominator in the last formula, equal to $|QW_1||W_2P| = |W_2AW_1|$, can be expressed as:

$$\sum_{1\le\alpha_1<\cdots<\alpha_r\le m} P\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ 1 & \dots & r\end{pmatrix} W_2^T\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ 1 & \dots & r\end{pmatrix} \sum_{1\le\beta_1<\cdots<\beta_r\le n} Q\begin{pmatrix}1 & \dots & r\\ \beta_1 & \dots & \beta_r\end{pmatrix} W_1^T\begin{pmatrix}1 & \dots & r\\ \beta_1 & \dots & \beta_r\end{pmatrix}$$
$$= \sum_{\substack{1\le\alpha_1<\cdots<\alpha_r\le m\\ \beta_1\le j_1<\cdots<\beta_r\le n}} (W_1W_2)^T\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ \beta_1 & \dots & \beta_r\end{pmatrix} A\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ \beta_1 & \dots & \beta_r\end{pmatrix}.$$

The element in the ith row and jth column of $W_1 \cdot \mathrm{adj}(QW_1)$ is

$$\sum_{k=1}^{r}(W_1)_{ik}\left(\mathrm{adj}(QW_1)\right)_{kj}$$
$$=\sum_{k=1}^{r}(W_1^T)_{ki}\left\{(-1)^{k+j}\sum_{\beta_1<\cdots<\beta_{r-1}} Q\begin{pmatrix}1 & \dots & j-1 & j+1 & \dots & r\\ \beta_1 & \dots & \dots & \dots & \dots & \beta_{r-1}\end{pmatrix} W_1^T\begin{pmatrix}1 & \dots & k-1 & k+1 & \dots & r\\ \beta_1 & \dots & \dots & \dots & \dots & \beta_{r-1}\end{pmatrix}\right\}$$
$$=\sum_{\beta_1<\cdots<\beta_{r-1}}(-1)^jQ\begin{pmatrix}1 & \dots & j-1 & j+1 & \dots & r\\ \beta_1 & \dots & \dots & \dots & \dots & \beta_{r-1}\end{pmatrix}\left\{\sum_{k=1}^{r}(-1)^k(W_1^T)_{ki}W_1^T\begin{pmatrix}1 & \dots & k-1 & k+1 & \dots & r\\ \beta_1 & \dots & \dots & \dots & \dots & \beta_{r-1}\end{pmatrix}\right\}$$

If i is contained in the combination $\beta_1,\dots,\beta_{r-1}$, then

$$\sum_{k=1}^{r}(-1)^k(W_1^T)_{ki}W_1^T\begin{pmatrix}1 & \dots & k-1 & k+1 & \dots & r\\ \beta_1 & \dots & \dots & \dots & \dots & \beta_{r-1}\end{pmatrix}=0.$$

If the set $\{\beta_1,\dots,\beta_{r-1}\}$ does not contain i, then $i=\beta_p$ and the system is denoted by $\beta_1,\dots,\beta_{p-1}$, $\beta_{p+1},\dots,\beta_r$. Now we get the following representation for the (ij)th element of $W_1 \cdot \mathrm{adj}(QW_1)$:

$$\sum_{\beta_1<\cdots<\beta_r}(-1)^jQ\begin{pmatrix}1 & \dots & j-1 & j+1 & \dots & r\\ \beta_1 & \dots & \beta_{p-1} & \beta_{p+1} & \dots & \beta_{r-1}\end{pmatrix}(-1)^pW_1^T\begin{pmatrix}1 & \dots & k-1 & k & k+1 & \dots & r\\ \beta_1 & \dots & \dots & i=\beta_p & \dots & \dots & \beta_r\end{pmatrix}$$
$$=\sum_{\beta_1<\cdots<i<\cdots<\beta_r} W_1^T\begin{pmatrix}1 & \dots & \dots & \dots & r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} Q_{ji}\begin{pmatrix}1 & \dots & \dots & \dots & r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}.$$

Similarly, the element in the ith row and jth column of $\mathrm{adj}(W_2P)\cdot W_2$ is equal to

$$\sum_{1\le\alpha_1<\cdots<\alpha_r\le m} W_2^T\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ 1 & \dots & \dots & \dots & r\end{pmatrix} P_{jk}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ 1 & \dots & \dots & \dots & r\end{pmatrix}.$$

Now, the element lying in the ith row and jth column of the matrix $W_1 \operatorname{adj}(QW_1)\operatorname{adj}(W_2P)W_2$ is

$$\sum_{k=1}^{r}\left\{\sum_{1\le\beta_1<\cdots<\beta_r\le n} W_1^T\begin{pmatrix}1 & \dots & i & \dots & r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} Q_{ki}\begin{pmatrix}1 & \dots & \dots & \dots & r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}\right\}\times$$

$$\times\left\{\sum_{1\le\alpha_1<\cdots<\alpha_r\le m} W_2^T\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ 1 & \dots & \dots & \dots & r\end{pmatrix} P_{jk}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ 1 & \dots & \dots & \dots & r\end{pmatrix}\right\}$$

$$=\sum_{\substack{\alpha_1<\cdots<\alpha_r\\ \beta_1<\cdots<\beta_r}} (W_1W_2)^T\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}\cdot\sum_{k=1}^{r} P_{jk}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ 1 & \dots & \dots & \dots & r\end{pmatrix} Q_{ki}\begin{pmatrix}1 & \dots & \dots & \dots & r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}$$

$$=\sum_{\substack{1\le\alpha_1<\cdots<\alpha_r\le m\\ 1\le\beta_1<\cdots<\beta_r\le n}} (W_1W_2)^T\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}.$$

The denominator in (2.1) is $|QW_1||W_2P|$. Consequently, the conditions (2.2) ensure the regularity of the matrices QW_1 and W_2P, which implies that the denominator in (2.1) is non zero.

($\Longleftarrow$): Conversely, suppose that the matrix $G \in \mathbb{C}^{n\times n}$ has representation (2.1), where the matrices W_1 and W_2 satisfy the conditions (2.2). We prove that G is a reflexive g-inverse of A. For this purpose, we restate here the following result from [11]: A matrix $G = (g_{ij})$ is a reflexive g-inverse of $A \in \mathbb{R}_r^{m\times n}$ if and only if

$$g_{ij} = \sum_{\substack{1\le\beta_1<\cdots<i<\beta_r\le n\\ 1\le\alpha_1<\cdots<j<\alpha_r\le m}} \lambda_{\alpha,\beta}\cdot A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix},$$

where $\lambda_{\alpha,\beta} \in \mathbb{R}$, satisfy

$$\sum_{\substack{1\le\beta_1<\cdots<\beta_r\le n\\ 1\le\alpha_1<\cdots<\alpha_r\le m}} \lambda_{\alpha,\beta}\cdot A\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ \beta_1 & \dots & \beta_r\end{pmatrix} = 1,$$

and the rank of the matrix $\Lambda = (\lambda_{\alpha,\beta})$ is $\operatorname{rank}(\Lambda) = 1$.

Consider the following real numbers:

$$\lambda_{\alpha,\beta} = \frac{(W_1W_2)^T\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ \beta_1 & \dots & \beta_r\end{pmatrix}}{\displaystyle\sum_{\substack{1\le\delta_1<\cdots<\delta_r\le n\\ 1\le\gamma_1<\cdots<\gamma_r\le m}} (W_1W_2)^T\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix} A\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}}. \tag{2.3}$$

According to the definition of the rth compound matrix, the matrix $\Lambda = (\lambda_{\alpha,\beta})$ can be expressed as:

$$\Lambda = \frac{C_r\left((W_1W_2)^T\right)}{\sum\limits_{\substack{1\le\delta_1<\cdots<\delta_r\le n\\ 1\le\gamma_1<\cdots<\gamma_r\le m}} (W_1W_2)^T\left(\begin{smallmatrix}\gamma_1 & \cdots & \gamma_r\\ \delta_1 & \cdots & \delta_r\end{smallmatrix}\right) A\left(\begin{smallmatrix}\gamma_1 & \cdots & \gamma_r\\ \delta_1 & \cdots & \delta_r\end{smallmatrix}\right)}. \tag{2.4}$$

The conditions (2.2) ensure the existence of the expressions (2.3) and (2.4), because $|W_2AW_1| \neq 0$. It is trivial to verify $\operatorname{rank}(W_1W_2) = r$. Indeed, $\operatorname{rank}(W_1W_2) \geq \operatorname{rank}(W_2AW_1) = r$. On the other hand, it is evident that $\operatorname{rank}(W_1W_2) \leq r$. Now we obtain

$$\operatorname{rank}(\Lambda) = \operatorname{rank}\left(C_r\left((W_1W_2)^T\right)\right) = 1\,.$$

Also, in view of (2.3), one can verify the following:

$$\sum_{\substack{1\le\beta_1<\cdots<i<\beta_r\le n\\ 1\le\alpha_1<\cdots<j<\alpha_r\le m}} \lambda_{\alpha,\beta}\cdot A\begin{pmatrix}\alpha_1 & \cdots & \alpha_r\\ \beta_1 & \cdots & \beta_r\end{pmatrix} = 1\,.$$

Consequently, using Miao's result, we conclude that G represents a reflexive g-inverse of A. □

Remark 2.1. Theorem 2.1 gives a characterization of all $\{1,2\}$-inverses. Selecting appropriate values for the matrices W_1 and W_2 which satisfy the conditions (2.2), we derive the well-known determinantal representations of the Moore-Penrose, weighted Moore-Penrose and the group inverse.

From Theorem 2.1, in the case $W_1 = Q^*$, $W_2 = P^*$, we obtain the representation of the Moore-Penrose inverse, restated in Theorem 1.1.

Also, the substitutions $W_1 = (QN)^*$, $W_2 = (MP)^*$ lead to the known determinantal representation of the weighted Moore-Penrose inverse, restated in Theorem 1.4.

When $m = n$, $W_1 = P$, $W_2 = Q$ we obtain the representation of the group inverse, and the results from Theorem 1.2. From the conditions (2.2), we conclude that $A^{\#}$ exists if and only if QP is invertible, which is the well-known Cline characterization of the group inverse (see [6]).

Moreover, if the matrices W_1, W_2 satisfy

$$(W_1W_2)^T\begin{pmatrix}\alpha_1 & \cdots & \alpha_r\\ \beta_1 & \cdots & \beta_r\end{pmatrix} = 1\,,$$

we obtain the definitions of the determinant and generalized inverses introduced in [10] and [22].

If the matrices W_1, W_2 satisfy

$$(W_1W_2)^T\begin{pmatrix}\alpha_1 & \cdots & \alpha_r\\ \beta_1 & \cdots & \beta_r\end{pmatrix} = (-1)^{\alpha_1+\cdots+\alpha_r+\beta_1+\cdots+\beta_r}\,,$$

we obtain the definition of determinant and generalized inverses discovered in [17].

From Theorem 2.1. and Theorem 1.6. we obtain the following characterization of $\{1,2,3\}$ and $\{1,2,4\}$-inverses:

COROLLARY 2.1. *If $A = PQ$ is a full rank factorization of $A \in \mathbb{C}_r^{m\times n}$ and $W_1 \in \mathbb{C}^{n\times r}$, $W_2 \in \mathbb{C}^{r\times m}$ are selected such that the conditions* (2.2) *are satisfied, then $a_{ij}^{(1,2,3)} \in A^{(1,2,3)}$ and $a_{ij}^{(1,2,4)} \in A^{(1,2,4)}$ if and only if $a_{ij}^{(1,2,3)}$ and $a_{ij}^{(1,2,4)}$ are represented as follows:*

$$a_{ij}^{(1,2,3)} = \frac{\displaystyle\sum_{\substack{1\le\beta_1<\cdots<\beta_r\le n\\ 1\le\alpha_1<\cdots<\alpha_r\le m}} (W_1P^*)^T\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}}{\displaystyle\sum_{\substack{1\le\delta_1<\cdots<\delta_r\le n\\ 1\le\gamma_1<\cdots<\gamma_r\le m}} A\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}(W_1P^*)^T\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}};$$

$$a_{ij}^{(1,2,4)} = \frac{\displaystyle\sum_{\substack{1\le\beta_1<\cdots<\beta_r\le n\\ 1\le\alpha_1<\cdots<\alpha_r\le m}} (Q^*W_2)^T\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}}{\displaystyle\sum_{\substack{1\le\delta_1<\cdots<\delta_r\le n\\ 1\le\gamma_1<\cdots<\gamma_r\le m}} A\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}(Q^*W_2)^T\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}}.$$

Comparing the results of Theorem 2.1. and the results contained in [15], it seems interesting to state the following problems:

PROBLEM 2.1. To find alternative solutions for the matrix Λ, if it is possible.

PROBLEM 2.2. Effective determinantal representation of $\{1\}$ inverses of $A = (a_{ij})$ can be developed by finding all possible solutions for the elements $\lambda_{\alpha,\beta}$ of the matrix Λ, from the following system [15]:

$$a_{ij}\left(\sum_{\substack{1\le\beta_1<\cdots<i<\beta_r\le n\\ 1\le\alpha_1<\cdots<j<\alpha_r\le m}} \lambda_{\alpha,\beta}\cdot A\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ \beta_1 & \dots & \beta_r\end{pmatrix}\right) = a_{ij} \qquad \text{for all} \quad i,j\,.$$

A partial solution of this problem that is appropriate for the class of $\{1,2\}$ inverses is given by (2.4).

3. Several additional $\{1,2\}$ inverses

Selecting the matrices $W_1 \in \mathbb{C}^{n\times r}$ and $W_2 \in \mathbb{C}^{r\times m}$ satisfying conditions (2.2), by means of (2.1) we may generate an infinite set of various definitions of determinants of a rectangular matrix and $\{1,2\}$ generalized inverses. In the second section, the determinantal representations of $\{1,2,3\}$, $\{1,2,4\}$ inverses, the Moore-Penrose, weighted Moore-Penrose inverse and the group inverse were emphasized. Consequently, the determinantal representation of generalized inverses derived so far are only special cases of the representation (2.1).

In this section we select several additional characteristic examples.

EXAMPLE 3.1. Suppose that W_1, W_2 satisfy

$$W_1W_2 = \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{I}_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \in \mathbb{C}^{n\times m}.$$

The denominator in (2.1) (i.e. the corresponding determinant of a matrix $A \in \mathbb{C}_r^{m\times n}$) is equal to $A\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix}$, representing the first principal $r\times r$ minor of A. The corresponding $\{1,2\}$ inverse of A is

$$A^{-1}_{(R,r)} = \frac{1}{A\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix}} \begin{pmatrix} A_{11}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & \dots & A_{r1}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{1r}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & \dots & A_{rr}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

EXAMPLE 3.2. For a given matrix $A \in \mathbb{C}_r^{m\times n}$, let k be the first integer such that $A\begin{pmatrix} k & \dots & k+r \\ k & \dots & k+r \end{pmatrix} \neq 0$. If the rectangular determinant of A, i.e. the denominator in (2.1) is equal to $A\begin{pmatrix} k & \dots & k+r \\ k & \dots & k+r \end{pmatrix}$, then the corresponding $\{1,2\}$ inverse of A is

$$A^{-1}_{(R,r)} = \frac{1}{A\begin{pmatrix} k & \dots & k+r \\ k & \dots & k+r \end{pmatrix}} \begin{pmatrix} 0 \dots 0 & 0 & \dots & 0 & 0 \dots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \dots 0 & 0 & \dots & 0 & 0 \dots 0 \\ 0 \dots 0 & A_{kk}\begin{pmatrix} k & \dots & k+r \\ k & \dots & k+r \end{pmatrix} & \dots & A_{k+r,k}\begin{pmatrix} k & \dots & k+r \\ k & \dots & k+r \end{pmatrix} & 0 \dots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \dots 0 & A_{k,k+r}\begin{pmatrix} k & \dots & k+r \\ k & \dots & k+r \end{pmatrix} & \dots & A_{k+r,k+r}\begin{pmatrix} k & \dots & k+r \\ k & \dots & k+r \end{pmatrix} & 0 \dots 0 \\ 0 \dots 0 & 0 & \dots & 0 & 0 \dots 0 \\ \vdots\ \vdots\ \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots\ \vdots \\ 0 \dots 0 & 0 & \dots & 0 & 0 \dots 0 \end{pmatrix}.$$

4. Solution of a system of linear equations

In this section, the determinantal representation of the solution of a system of linear equations is developed in terms of the determinantal representation of $\{1,2\}$ inverses described.

THEOREM 4.1. *The ith component of the solution $A^{(1,2)}z$ of the linear system $Ax = z$, $A \in \mathbb{C}_r^{m\times n}$, $x \in \mathbb{C}^n$, $z \in \mathbb{C}^m$ has the following determinantal representation:*

$$x_i^{1,2} = \frac{\displaystyle\sum_{\substack{1\le\beta_1<\dots<\beta_r\le n\\ 1\le\alpha_1<\dots<\alpha_r\le m}} (W_1W_2)^T\begin{pmatrix} \alpha_1 & \dots & \dots & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix} A\begin{pmatrix} \alpha_1 & \dots & \dots & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix}(i \to {}_\alpha z)}{\displaystyle\sum_{\substack{1\le\delta_1<\dots<\delta_r\le n\\ 1\le\gamma_1<\dots<\gamma_r\le m}} (W_1W_2)^T\begin{pmatrix} \gamma_1 & \dots & \gamma_r \\ \delta_1 & \dots & \delta_r \end{pmatrix} A\begin{pmatrix} \gamma_1 & \dots & \gamma_r \\ \delta_1 & \dots & \delta_r \end{pmatrix}}, \tag{3.1}$$

where ${}_\alpha z$ denotes the vector $\{z_{\alpha_1}, \dots, z_{\alpha_r}\}$.

Proof. Starting from $x_i^{(1,2)} = \left(A^{(1,2)}z\right)_i$ and from the determinantal representation of all the reflexive g-inverses, given by (2.1), we get

$$x_i^{(1,2)} = \sum_{k=1}^{r} \frac{\displaystyle\sum_{\substack{1\le\beta_1<\dots<\beta_r\le n\\ 1\le\alpha_1<\dots<\alpha_r\le m}} (W_1W_2)^T\begin{pmatrix} \alpha_1 & \dots & k & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix} A_{ki}\begin{pmatrix} \alpha_1 & \dots & k & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix}}{\displaystyle\sum_{\substack{1\le\delta_1<\dots<\delta_r\le n\\ 1\le\gamma_1<\dots<\gamma_r\le m}} (W_1W_2)^T\begin{pmatrix} \gamma_1 & \dots & \gamma_r \\ \delta_1 & \dots & \delta_r \end{pmatrix} A\begin{pmatrix} \gamma_1 & \dots & \gamma_r \\ \delta_1 & \dots & \delta_r \end{pmatrix}} z_k$$

$$= \frac{\displaystyle\sum_{\substack{1\le\beta_1<\cdots<\beta_r\le n\\ 1\le\alpha_1<\cdots<\alpha_r\le m}} (W_1W_2)^T \binom{\alpha_1 \ \dots\ k\ \dots\ \alpha_r}{\beta_1\ \dots\ i\ \dots\ \beta_r} \sum_{k=1}^{r} A_{ki} \binom{\alpha_1 \ \dots\ k\ \dots\ \alpha_r}{\beta_1\ \dots\ i\ \dots\ \beta_r} z_k}{\displaystyle\sum_{\substack{1\le\delta_1<\cdots<\delta_r\le n\\ 1\le\gamma_1<\cdots<\gamma_r\le m}} (W_1W_2)^T \binom{\gamma_1\ \cdots\ \gamma_r}{\delta_1\ \dots\ \delta_r} A \binom{\gamma_1\ \cdots\ \gamma_r}{\delta_1\ \dots\ \delta_r}}$$

$$= \frac{\displaystyle\sum_{\substack{1\le\beta_1<\cdots<\beta_r\le n\\ 1\le\alpha_1<\cdots<\alpha_r\le m}} (W_1W_2)^T \binom{\alpha_1\ \dots\ \dots\ \dots\ \alpha_r}{\beta_1\ \dots\ i\ \dots\ \beta_r} A \binom{\alpha_1\ \dots\ \dots\ \dots\ \alpha_r}{\beta_1\ \dots\ i\ \dots\ \beta_r} (i \to {}_{\alpha}z)}{\displaystyle\sum_{\substack{1\le\delta_1<\cdots<\delta_r\le n\\ 1\le\gamma_1<\cdots<\gamma_r\le m}} (W_1W_2)^T \binom{\gamma_1\ \cdots\ \gamma_r}{\delta_1\ \dots\ \delta_r} A \binom{\gamma_1\ \cdots\ \gamma_r}{\delta_1\ \dots\ \delta_r}}.$$

□

Remark 4.1. In the case $W_1 = Q^*$, $W_2 = P^*$, we get the determinantal representation for the Moore-Penrose solution of the system of linear equations, introduced in [5]. In [20] we obtain the same results, using simpler proof. From Theorem 3.1, we can obtain an elegant derivation of the determinantal representation of the best approximate solution, and improve the proof from [20].

Also, the substitutions $W_1 = (QN)^*$, $W_2 = (MP)^*$ lead to the well-known determinantal representation of the weighted Moore-Penrose solution of a system of linear equations over the field of complex numbers ([19]), or over an integral domain ([14]).

COROLLARY 4.1. *Let $A = PQ$ be a full-rank factorization of A. Then $x_i^{(1,2)} = \left(A^{(1,2)}z\right)_i$ can be represented as the linear combination of the solutions $x_i^{(\alpha,\beta)}$ of all uniquely solvable $r\times r$ subsystems $A\binom{\alpha_1\ \cdots\ \alpha_r}{\beta_1\ \dots\ \beta_r} x = {}_{\alpha}z$ of the starting system $Ax = z$:*

$$x_i^{(1,2)} = p_\alpha q_\beta x_i^{(\alpha,\beta)}, \qquad p_\alpha, q_\beta \in \mathbb{C},$$

where

$$\sum_{1\le\alpha_1<\cdots<\alpha_r\le m} p_\alpha = 1, \qquad \sum_{1\le\beta_1<\cdots<\beta_r\le n} q_\beta = 1.$$

Proof. In the case when $A\binom{\alpha_1\ \dots\ \alpha_r}{\beta_1\ \dots\ \beta_r} \neq 0$, the canonical embedding of the solution of the system $A\binom{\alpha_1\ \cdots\ \alpha_r}{\beta_1\ \dots\ \beta_r} x = {}_{\alpha}z$ into the m-dimensional space, denoted by $x_i^{(\alpha,\beta)}$, is equal to

$$x_i^{(\alpha,\beta)} = \begin{cases} \dfrac{A\binom{\alpha_1\ \dots\ \dots\ \dots\ \alpha_r}{\beta_1\ \dots\ i\ \dots\ \beta_r}(i \to {}_{\alpha}z)}{A\binom{\alpha_1\ \dots\ \dots\ \dots\ \alpha_r}{\beta_1\ \dots\ i\ \dots\ \beta_r}}, & i \in \{\beta_1,\dots,\beta_r\}, \\ 0, & i \notin \{\beta_1,\dots,\beta_r\}. \end{cases}$$

In the singular case we define $x_i^{(\alpha,\beta)}$ to be the zero vector. Now, we easily verify the following:

$$
\begin{aligned}
x_i^{(1,2)} &= \frac{\displaystyle\sum_{\substack{11\le\alpha_1<\dots<\alpha_r\le m\\ 1\le\beta_1<\dots<\beta_r\le n}} W_2^T\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ 1 & \dots & r\end{pmatrix} W_1^T\begin{pmatrix}1 & \dots & \dots & \dots & r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} A\begin{pmatrix}\alpha_1 & \dots & \dots & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}(i \to {}_\alpha z)}{|QW_1||W_2P|}\\
&= \sum_{\substack{1\le\alpha_1<\dots<\alpha_r\le m\\ 1\le\beta_1<\dots<i<\dots<\beta_r\le n}} p_\alpha q_\beta x_i^{(\alpha,\beta)},
\end{aligned}
$$

where

$$
p_\alpha = \frac{W_2^T\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ 1 & \dots & r\end{pmatrix} P\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ 1 & \dots & r\end{pmatrix}}{|W_2P|}, \qquad q_\beta = \frac{W_1^T\begin{pmatrix}1 & \dots & r\\ \beta_1 & \dots & \beta_r\end{pmatrix} Q\begin{pmatrix}1 & \dots & r\\ \beta_1 & \dots & \beta_r\end{pmatrix}}{|QW_1|}.
$$

The equations $\sum_{1\le\alpha_1<\dots<\alpha_r} p_\alpha = 1$ and $\sum_{1\le\beta_1<\dots<\beta_r} q_\beta = 1$ can be easily verified. $\square$

Remark 4.2. In the case $W_1 = Q^*$, $W_2 = P^*$ we conclude $p_\alpha \ge 0$, $q_\beta \ge 0$, which implies that the Moore-Penrose solution of a system of linear equations is the convex combination of the solutions of all uniquely solvable $r\times r$ subsystems. This is the well-known result of [5]. In [4] a similar result is obtained for the least-squares solution of an overdetermined system.

REFERENCES

[1] ARGHIRIADE, E.—DRAGOMIR, A.: *Une nouvelle definition de l'inverse generalisee d'une matrice*, Atti Accad. Naz. Lincei, Rend., Cl. Sci. Fis. Mat. Nat., VIII. Ser. **35** (1963), 158–165.

[2] BAPAT, R. B.—BHASKARA, K. P. S.—MANJUNATHA PRASAD, K.: *Generalized inverses over integral domains*, Linear Algebra Appl. **140** (1990), 181–196.

[3] BEN-ISRAEL, A.—GREVILE, T. N. E.: *Generalized Inverses: Theory and Applications*, Wiley-Interscience, New York, 1974.

[4] BEN-TAL, A.: *A geometric property of the least squares solution of linear equations*, Linear Algebra Appl. **139** (1990), 165–170.

[5] BERG, L.: *Three results in connection with inverse matrices*, Linear Algebra Appl. **84** (1986), 63–77.

[6] CLINE, R. E.: *Inverses of rank invariant powers of a matrix*, SIAM J. Numer. Anal. **5** (1968), 182–197.

[7] ERDELYI, I.: *On the matrix equation $Ax = \lambda Bx$*, J. Math. Anal. Appl. **17** (1967), 119–132.

[8] GABRIEL, R.: *Extinderea complementilor algebrici generalizati la matrici oarecare*, Stud. Cerc. Mat. **17** (1965), 1566–1581.

[9] GABRIEL, R.: *Das verallgemeinerte Inverse einer Matrix über einem beliebigen Körper-analytish betrachtet*, J. Reine Angew. Math. **244** (1970), 83–93.

[10] JOSHI, V. N.: *A determinant for rectangular matrices*, Bull. Australl. Math. Soc. **21** (1980), 137–146.

[11] MIAO, J.: *Reflexive generalized inverses and their minors*, Linear and Multilinear Algebra **35** (1993), 153–163.

[12] PENROSE, R.: *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955), 406–413.

[13] PRASAD, K. M.—BHASKARA RAO, K. P. S.—BAPAT, R. B.: *Generalized inverses over integral domains. II. Group inverses and Drazin inverses*, Linear Algebra Appl. **146** (1991), 31–47.

[14] PRASAD, K. M.—BAPAT, R. B.: *The generalized Moore-Penrose inverse*, Linear Algebra Appl. **165** (1992), 59–69.

[15] PRASAD, K. M.: *Generalized inverses of matrices over commutative ring*, Linear Algebra Appl. **211** (1994), 35–53.

[16] RADIĆ, M.: *Some contributions to the inversions of rectangular matrices*, Glas. Mat. **1(21)** (1966), 23–37.

[17] RADIĆ, M.: *A definition of the determinant of a rectangular matrix*, Glas. Mat. **1(21)** (1966), 17–22.

[18] RAO, C. R.—MITRA, S. K.: *Generalized Inverse of Matrices and its Applications*, John Wiley & Sons, Inc, New York-London-Sydney-Toronto, 1971.

[19] STANIMIROVIĆ, P.—STANKOVIĆ, M.: *Determinantal representation of weighted Moore-Penrose inverse*, Mat. Vesnik **46** (1994), 41–50.

[20] STANIMIROVIĆ, P.—STANKOVIĆ, M.: *Generalized algebraic complement and Moore-Penrose inverse*, Filomat **8** (1994), 57–64.

[21] STANIMIROVIĆ, P.: *General determinantal representation of pseudoinverses and its computation*, Rev. Acad. Cienc. Zaragoza (2) **50** (1995), 41–49.

[22] STOJAKOVIĆ, M.: *Generalized inverse matrices.* In: Mathematical Structures – Computational Mathematics – Mathematical Modelling, Sofia, 1975, pp. 461–470.

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