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In memoriam Štefan Znám

ON THE MINIMUM NUMBER OF COMPONENTS IN A COTREE OF A GRAPH

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ABSTRACT. The decay number $\zeta(G)$ of a graph G is the smallest number of components a cotree of G can have. Recently, Nebeský [NEBESKÝ, L.: *Characterization of the decay number of a connected graph*, Math. Slovaca **45** (1995), 349–352] showed that $\zeta(G)$ can be expressed as the maximum of $c(G-A)-|A|-1$, where $A \subseteq E(G)$ and $c(G-A)$ denotes the number of components of $G-A$. In this paper we establish a different but related characterization of the decay number and present an application to graphs of diameter 2.

1. Introduction

Let G be a connected graph which is allowed to have both multiple edges and loops. Define the *decay number* of G to be

$$\zeta(G) = \min\{c(G-E(T))\,;\ T \text{ a spanning tree of } G\},$$

where $c(H)$ denotes the number of components of a graph H.

This invariant was introduced by the author in [7] in connection with the problem of determining the maximum genus of a graph with loops. To be more specific, the maximum genus of any graph is greater than or equal to $\lceil(\beta(G)-\zeta(G))/2\rceil$, where $\beta(G)$ is the cycle rank (Betti number) of G. In the same paper, the decay number of a loopless cubic graph of order n was shown to be $n/2-1$ and the decay number of a 2-connected graph of diameter two was bounded from above by 4, the bound being best possible.

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Let G be a graph with p vertices and q edges. Take a spanning tree T of G, and let $Z = G - E(T)$ be the corresponding cotree. Since the cycle rank of Z is

$$\beta(Z) = |E(Z)| - |V(Z)| + c\big(G - E(T)\big) = q - 2p + c\big(G - E(T)\big) + 1\,,$$

we get $c\big(G - E(T)\big) = 2p - q - 1 + \beta(Z)$, and hence

$$\zeta(G) = 2p - q - 1 + \min \beta(Z)\,,$$

where the minimum is taken over all cotrees Z of G. Thus we have

THEOREM 1. *Let G be a connected graph with p vertices and q edges. Then*

$$\zeta(G) \geq 2p - q - 1$$

with equality occurring if and only if G has an acyclic cotree.

Nebeský [4] has found a so called "good characterization" of the decay number by expressing it as a *maximum* of a certain combinatorial function. The function is strongly related to the bound in Theorem 1.

THEOREM 2. (Nebeský [4]) *Let G be a connected graph. Then*

$$\zeta(G) = \max\big\{2c(G - Q) - |Q| - 1\,;\;\; Q \subseteq E(G)\big\}\,.$$

Note that the bound in Theorem 1 (but not the structure of extremal graphs) immediately follows from Theorem 2 by taking $Q = E(G)$.

The proof of this theorem is not easy. Our aim in this paper is to present a new proof which seems to give more insight to the structure of the extremal sets Q and their relationship to the spanning trees where the decay number is attained. In fact, we prove a somewhat different characterization from which Theorem 2 can be derived. At the end we also give a simple but surprising application of the bound given in Theorem 1.

2. Main result and proof

We prove a characterization of the decay number of a graph based on counting leaves rather than components. The proof technique is similar to that of Theorem 24 in [6]. Recall that a *leaf* of a graph H is any 2-edge-connected subgraph of H, trivial or not, maximal with respect to inclusion. Thus every vertex belongs to a unique leaf of H. Let $l(H)$ denote the number of leaves of H. Our main result is:

THEOREM 3. *Let G be a connected graph. Then*

$$\zeta(G) = \max\{l(G-A) - |A|;\ A \subseteq E(G)\}.$$

It is easy to see that the maximum in Theorem 3 is attained by a set A such that $G - A$ is connected. Therefore, in the sequel we shall study pairs (F, A) where $F = G - A$ is a connected spanning subgraph of G (a *frame*) and $A = E(G) - E(F)$. We call (F, A) a *frame decomposition* of G. A *leaf* of a frame decomposition is any leaf of its frame.

LEMMA 4. *Let (F, A) be a frame decomposition of a connected graph G. Then*

$$\zeta(G) \geq l(F) - |A|.$$

Proof. It is obvious that $\zeta(F) \geq l(F)$. On the other hand, for every edge e of G that is not a bridge we have $\zeta(G) \leq \zeta(G-e) \leq \zeta(G) + 1$. By a trivial induction we get $\zeta(F) = \zeta(G - A) \leq \zeta(G) + |A|$, whence $\zeta(G) \geq \zeta(F) - |A| \geq l(F) - |A|$. □

Important features of every frame decomposition (F, A) of G are reflected by its *coframe* $G - B(F) = \big(F - B(F)\big) \cup A$; here $B(H)$ denotes the set of all bridges of a graph H. We say that a frame decomposition (F, A) of a graph G is *smooth* if

(F1) every leaf of F has a connected cotree, and
(F2) the set of bridges of the coframe is A
(that is to say, $B\big(G - B(G-A)\big) = A$).

In a smooth frame decomposition the leaves of the coframe coincide with the leaves of the frame, and the *reduced coframe graph* obtained from the coframe by contracting each leaf to a vertex is acyclic. Each of these two properties is equivalent to condition (F2).

Condition (F1) implies that the frame F of every smooth frame decomposition (F, A) of a graph G contains a spanning tree T such that $c\big(F - E(T)\big) = l(F)$. We call T a *smooth* spanning tree of G associated with (F, A).

LEMMA 5. *Let (F, A) be a smooth frame decomposition of a connected graph G and let T be a smooth spanning tree of G associated with (F, A). Then*

$$l(F) - |A| = \zeta(G) = c\big(G - E(T)\big),$$

which in turn equals the number of components of the coframe of (F, A).

Proof. Equality $c\big(G - E(T)\big) = c\big(G - B(F)\big)$ follows immediately from condition (F1). By condition (F2), the reduced coframe graph G' is acyclic, whence $c\big(G - E(T)\big) = c(G') = |V(G')| - |E(G')| = l(F) - |A|$. The result follows. □

The proof of Theorem 3 will be completed by establishing:

THEOREM 6. *Every connected graph admits a smooth frame decomposition.*

Before proving this we sketch the main idea. We take a spanning tree T of the graph in question; trivially, $(T, \emptyset)$ is a smooth frame decomposition of T. Now, we start adding edges to T. In each step we extend or transform the actual decomposition to a smooth frame decomposition of the larger graph until we reach a smooth frame decomposition of the whole graph.

We proceed to describing the transformation process – recombination. Consider a connected graph G with a smooth frame decomposition (F, A) and a connected supergraph $G+e$ where $e \notin E(G)$. If the end-vertices of e belong to different components of the coframe, then $(F, A \cup e)$ is a smooth frame decomposition of $G+e$. If they belong to the same component, say S, and also to the same leaf of S, then $(F \cup e, A)$ is a smooth frame decomposition of $G+e$. So we are left with the case where e joins vertices from different leaves of S. Let T be a smooth spanning tree associated with (F, A). Define a *recombination chain* from e to be a sequence of edges $\Sigma = (e_0, f_0, e_1, f_1, \dots, e_{n-1}, f_{n-1}, e_n)$ which starts with $e_0 = e$ and alternates edges f_i of $S \cap A$, $i \geq 0$, with edges e_i of $T \cap B$, $i \geq 1$, in such a way that for $i = 0, 1, \dots, n-1$, f_i is a cyclic edge of $S + e_i$ and e_{i+1} is a cyclic edge of $T + f_i$. An edge in $A \cup B$ which is contained in some recombination chain from e will be said to be *accessible* from e. A leaf of F is *accessible* from e if one of its vertices is an end-vertex of an accessible edge.

By means of Σ we can construct a sequence of frame decompositions (F_i, A_i) of $G+e$ if we set $(F_0, A_0) = (F, A \cup e)$ and $(F_{i+1}, A_{i+1}) = (F_i \cup f_i - e_{i+1}, A_i \cup e_{i+1} - f_i)$ for $i = 1, 2, \dots, n-1$. We say that the frame decomposition (F_n, A_n) has been obtained from (F_0, A_0) by *recombination*.

For $i < n$ the reduced coframe graph of (F_i, A_i) contains a unique cycle, the one that contains e_i. The same is true for (F_n, A_n) unless the end-vertices of e_n belong to different components of the coframe of (F, A). Then (F_n, A_n) is a smooth frame decomposition of $G+e$.

Now we are ready to prove Theorem 6.

P r o o f o f T h e o r e m 6. We proceed by induction on the number of edges. Let G be a connected graph. If G has no more than one edge, then $(G, \emptyset)$ is a smooth frame decomposition. So we may assume that G has at least two edges and that the statement of our theorem holds for all graphs with less than $|E(G)|$ edges.

Set $m = \max\{l(P) - |D|;\ (P, D)$ a frame decomposition of $G\}$. Take a frame decomposition (F, A) of G with $l(F) - |A| = m$ such that the number of edges in A is maximum. Call such a decomposition *saturated*. Obviously, each edge of A has end-vertices in different leaves of F. We first show that every leaf in a saturated frame decomposition has a connected cotree. This is certainly true for trivial leaves, so we consider a non-trivial leaf L of F. If $\zeta(L) \geq 2$, then there exists an edge e in L such that $\zeta(L-e) \geq 2$. By the induction hypothesis, $L-e$ admits a smooth frame decomposition, say (R, Z). It follows that $l(R) - |Z| \geq 2$ and hence $l(R) \geq |Z| + 2$. For the frame decomposition $\big(F - (Z \cup e),\ A \cup Z \cup e\big)$ of G we now have

$$\begin{aligned} l\big(F - (Z \cup e)\big) - |A \cup Z \cup e| &= l(F) - 1 + l(R) - |A \cup Z \cup e| \\ &\geq l(F) - 1 + |Z| + 2 - |A| - |Z| - 1 \\ &\geq l(F) - |A| = m\,. \end{aligned}$$

But this is impossible since $|A \cup Z \cup e| > |A|$ and (F, A) is a saturated frame decomposition of G. Therefore each leaf of a saturated frame decomposition has a connected cotree.

It remains to prove that among saturated frame decompositions of G there is one that is smooth, that is, one whose reduced coframe graph is acyclic. Suppose not, and choose a saturated frame decomposition of G, still denoted by (F, A), where A contains a subset C of maximum cardinality such that (F, C) is a smooth decomposition of $H = F \cup C$. Clearly, $C \subsetneq A$. Since (F, C) is smooth, by Lemma 5 we have

$$\zeta(H) = l(F) - |C| = c\big(H - B(F)\big)\,. \tag{1}$$

Take an edge $e \in A - C$. By maximality, e joins different leaves in the same component S of the coframe of (F, C). Let D be the set of all edges in A that are accessible from e (with respect to a smooth spanning tree of H). Clearly, $D \subseteq C \cup e$. An easy inductive argument shows that all edges from D belong to the same leaf of $F \cup D$, denoted by M. We claim that there exists an accessible leaf of (F, C) disjoint from S. If not, we consider the decomposition $(F \cup D, A - D)$. Its leaves include M and the leaves of (F, C) disjoint from S. Since (F, C) is a smooth decomposition, the number of the latter leaves equals the number of edges of C not in S plus the number of components of the coframe of (F, C) different from S. From (1) and the fact that $e \notin C$ we now obtain

$$\begin{aligned} l(F \cup D) - |A - D| &\geq |C - D| + c\big(H - B(F)\big) - |A - D| \\ &= |C - D| + l(F) - |C| - |A - D| \\ &= |C - (D - e)| + l(F) - |C| - |A - D| \\ &= l(F) - |D| + 1 - |A - D| = l(F) - |A| + 1 = m + 1\,, \end{aligned}$$

contradicting the choice of (F, A). Thus in (F, C) there is an accessible leaf disjoint from S. By employing recombination we obtain a saturated frame decomposition (F_1, A_1) of G and a subset $C_1 \subseteq A_1$ such that (F_1, C_1) is a smooth decomposition of $F_1 \cup C_1$ and $|C_1| \geq |C| + 1$. This contradiction proves Theorem 6. □

3. Remarks

Remark 1. Essentially the same proof can be used to establish a stronger result than Theorem 6. Namely, in condition (F1) we may require every leaf L of a smooth frame decomposition to be *stable*, that is, $\zeta(L) = 1$ and $\zeta(L - e) = 1$ for each edge e of L.

Remark 2. Theorem 2 and Theorem 3 are related as follows. Take a set Q for which the function $\theta_G(X) = 2c(G - X) - |X| - 1$, $X \subseteq E(G)$, reaches its maximum and $|Q|$ is maximal. Then each component of $G - Q$ is 2-edge-connected. Now, if we choose any subset $W \subseteq Q$ of $c(G - Q) - 1$ elements such that $(G - Q) \cup W$ is connected, then $(F, A) = \big((G - Q) \cup W, Q - W\big)$ is a frame decomposition where the function $l(F) - |A|$ reaches its maximum. Conversely, if (F, A) is a smooth frame decomposition, then $\theta_G\big(A \cup B(F)\big)$ is the maximal value of $\theta_G(X)$.

Remark 3. Let G be a 2-connected graph of diameter 2. By a result of [7], we have $\zeta(G) \leq 4$. Denoting by p and q the number of vertices and the number of edges of G, respectively, and using Theorem 1 we obtain $4 \geq \zeta(G) \geq 2p - q - 1$ whence

$$q \geq 2p - 5. \tag{2}$$

The bound (2) on the number of edges in a 2-connected graph of diameter 2 was earlier proved by several authors: Murty [3], Palumbíny [5], Gliviak [2] and a proof also appears in Bollobás [1]. A short proof was found by the late Professor Š. Znám and presented in his lectures.

Of course, all the extremal graphs have acyclic cotrees and their decay number is 4. As shown by Palumbíny [5], the only extremal graph with minimum valency $\delta \geq 3$ is the Petersen graph.

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