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Metin Başarir; Ekrem Savaş

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ON MATRIX TRANSFORMATIONS OF SOME GENERALIZED SEQUENCE SPACE

METIN BAŞARIR — EKREM SAVAŞ

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ABSTRACT. P. Schaefer [9] defined the concepts of σ-conservative, σ-regular, and σ-coercive matrices and characterized these classes of matrices, i.e. (c, V_σ), $(c, V_\sigma)_{\mathrm{reg}}$, and (ℓ_∞, V_σ). Recently Mursaleen [5] determined the classes $(\ell(p), V_\sigma)$ and $(M_0(p), V_\sigma)$. The object of this paper is to obtain necessary and sufficient conditions to characterize the matrices of the classes $(c_0(\boldsymbol{p}), V_{0\sigma}(\boldsymbol{q}))$ and $(c_0(\boldsymbol{p}), V_\sigma(\boldsymbol{q}))$.

1. Preliminaries

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional φ on ℓ_∞, the space of bounded sequences, is said to be an *invariant mean*, or a *σ-mean*, if and only if

(i) $\varphi(\boldsymbol{x}) \ge 0$ when the sequence $\boldsymbol{x} = (x_n)$ has $x_n \ge 0$ for all n,
(ii) $\varphi(\boldsymbol{e}) = 1$, where $\boldsymbol{e} = (1, 1, \dots)$,
(iii) $\varphi(x_{\sigma(n)}) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \ell_\infty$.

In case, σ is the translation mapping $n \mapsto n+1$, a σ-mean is often called a *Banach limit* ([2]), and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set f of *almost convergent sequences* ([3]).

Let f_0 denote the space of almost convergent null sequences.

If $\boldsymbol{x} = (x_n)$, set $T\boldsymbol{x} = (Tx_n) = (x_{\sigma(n)})$. It is known that

$$V_\sigma = \left\{ \boldsymbol{x} \in \ell_\infty : \lim_{m\to\infty} d_{mn}(\boldsymbol{x}) = L\boldsymbol{e}, \text{ uniformly in } n, \text{ and } L = \sigma\text{-}\lim \boldsymbol{x} \right\}, \tag{1.1}$$

where

$$d_{mn}(\boldsymbol{x}) = \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n .$$

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The special case of (1.1) in which $\sigma(n) = n+1$ was given by Lorentz [3; Theorem 1]; the general result can be proved in a similar way.

It is familiar that a Banach limit extends the limit functional on c, the space of convergent sequences. It is known ([5]) that a σ-mean extends the limit functional on c in the sense that $\varphi(\mathbf{x}) = \lim \mathbf{x}$ for all $\mathbf{x} \in c$ if and only if σ has no finite orbits, that is to say, if and only if for all $n \geq 0$, $j \geq 1$, $\sigma^j(n) \neq n$.

P. Schaefer [9] defined the concepts of *σ-concervative*, *σ-regular*, and *σ-coercive matrices* and obtained conditions to characterize these classes of matrices.

Let $V_{0\sigma}$ denote the set of all bounded sequences which are σ-convergent to zero.

Recently, in [5] and [7] the spaces V_σ, $V_{0\sigma}$, f, and f_0 were extended to $V_\sigma(\mathbf{p})$, $V_{0\sigma}(\mathbf{p})$, $f(\mathbf{p})$, and $f_0(\mathbf{p})$ in the following manner.

If $\mathbf{p} = (p_m)$ is a sequence of real numbers such that $p_m > 0$ and $\sup\limits_m p_m < \infty$, we define

$$V_{0\sigma}(\mathbf{p}) = \left\{ \mathbf{x} : \lim_{m\to\infty} |d_{mn}(\mathbf{x})|^{p_m} = 0\,, \text{ uniformly in } n \right\},$$

$$V_\sigma(\mathbf{p}) = \left\{ \mathbf{x} : \lim_{m\to\infty} |d_{mn}(\mathbf{x} - L\mathbf{e})|^{p_m} = 0\,, \text{ uniformly in } n\,, \ \sigma\text{-}\lim \mathbf{x} = L \right\},$$

$$f_0(\mathbf{p}) = \left\{ \mathbf{x} : \lim_{m\to\infty} \left| \frac{1}{m+1} \sum_{i=0}^{m} x_{i+n} \right|^{p_m} = 0\,, \text{ uniformly in } n \right\},$$

$$f(\mathbf{p}) = \left\{ \mathbf{x} : \lim_{m\to\infty} \left| \frac{1}{m+1} \sum_{i=0}^{m} (x_{i+n} - L) \right|^{p_m} = 0 \text{ for some } L\,, \text{ uniformly in } n \right\}.$$

In particular, if $p_m = p > 0$ for all m, we have $V_{0\sigma}(\mathbf{p}) = V_{0\sigma}$ and $V_\sigma(\mathbf{p}) = V_\sigma$. If $\sigma(n) = n+1$, we get $V_{0\sigma}(\mathbf{p}) = f_0(\mathbf{p})$ and $V_\sigma(\mathbf{p}) = f(\mathbf{p})$.

S. M. Zaidi [10] has determined necessary and sufficient conditions for some matrix $\mathbf{A} = (a_{nk})$, $n, k = 1, 2, \dots$, such that the $\mathbf{A}$-transform of $\mathbf{x} = (x_k)$ belongs to the set $V_\sigma(\mathbf{q})$, where in particular $\mathbf{x} \in \ell_\infty(\mathbf{p})$.

Just as boundedness is related to convergence, it was quite natural to expect that the sequence space ℓ_∞^σ of σ-boundedness is related to σ-convergence.

We write

$$\ell_\infty^\sigma = \left\{ \mathbf{x} : \sup_{m,n} |d_{mn}(\mathbf{x})| < \infty \right\}.$$

But in [8], Savaş has observed that this concept coincides with ℓ_∞, viz., $\ell_\infty^\sigma = \ell_\infty$.

2. Notation

If $\boldsymbol{p} = (p_k)$ is a sequence of real numbers such that $p_k > 0$ and $\sup_k p_k < \infty$, we write

$$\ell_\infty(\boldsymbol{p}) = \Big\{\boldsymbol{x} : \sup_k |x_k|^{p_k} < \infty\Big\},$$

$$c(\boldsymbol{p}) = \Big\{\boldsymbol{x} : \lim_{k\to\infty} |x_k - L|^{p_k} = 0\,,\ \text{for some } L\Big\},$$

$$c_0(\boldsymbol{p}) = \Big\{\boldsymbol{x} : \lim_{k\to\infty} |x_k|^{p_k} = 0\Big\}.$$

As special cases of the above, with $p_k = 1$ for all k, we get ℓ_∞, c, and c_0. By the Köthe-Toeplitz dual of a set $E \subset s$, the set of complex sequences, $E \neq \emptyset$, we mean the linear space

$$E^+ = \Big\{\boldsymbol{a} : \sum_k a_k x_k \ \text{convergent for all } \boldsymbol{x} \in E\Big\}.$$

E^* denotes the dual space of the continuous linear functionals of E.

We want to add that $\boldsymbol{p} = (p_k)$ and $\boldsymbol{q} = (q_k)$ in the sequel will denote sequences with $p_k > 0$ and $q_k > 0$.

We use the fact that $c_0(\boldsymbol{p})$ is a compete paranormed space with paranorm

$$g(\boldsymbol{x}) = \Big(\sup_k |x_k|^{p_k}\Big)^{\frac{1}{M}}, \qquad M = \max\Big(1, \sup_k p_k\Big).$$

The purpose of this paper is to obtain necessary and sufficient conditions to characterize the matrices of classes $\big(c_0(\boldsymbol{p}), V_{0\sigma}(\boldsymbol{q})\big)$ and $\big(c_0(\boldsymbol{p}), V_\sigma(\boldsymbol{q})\big)$.

3. Main results

If X and Y are two sequence spaces, let (X, Y) denote the set of all matrices $\mathbf{A} = (a_{nk})$, $n, k = 1, 2, \dots$, that transform $\boldsymbol{x} = (x_k) \in X$ into $\boldsymbol{y} = (y_n) = \mathbf{A}\boldsymbol{x} = \big(A_n(\boldsymbol{x})\big) \in Y$, defined by $y_n = \sum_k a_{nk} x_k$ $(n = 1, 2, \dots)$. Let us write for all integers $n, m \geq 1$,

$$t_{mn} = t_{mn}(\mathbf{A}\boldsymbol{x}) = \sum_k a(n, k, m) x_k\,,$$

where $a(n, k, m) = \dfrac{1}{m+1} \sum_{j=0}^{m} a\big(\sigma^j(n), k\big)$.

In the sequel, we can assume $p_k \leq 1$ for all k without loss of generality because $c_0(\boldsymbol{p}) = c_0\big(\boldsymbol{p}/M\big)$ for $M = \max\Big(1, \sup_k p_k\Big)$.

Now, let us quote some known results as the following.

We remark that $\ell_\infty^\sigma(\boldsymbol{q}) = \ell_\infty(\boldsymbol{q})$ in the lemmas below.

LEMMA A. ([4]) *Let X be a complete paranormed space with Schauder basis (b_k), and (A_n) a sequence of elements of X^* with $A_n(\mathbf{x}) = \sum_k a_{nk}x_k$ for all $\mathbf{x} \in X$ and $n \in \mathbb{N}$. Furthermore, let $\mathbf{q} = (q_k)$ be a bounded sequence. Then*

$$\mathbf{A} \in \big(X, V_{0\sigma}(\mathbf{q})\big) \iff \text{i)} \quad \big(t_{mn}(b_k)\big) \in V_{0\sigma}(\mathbf{q}) \ \textit{for all } k\,,$$
$$\text{ii)} \quad \lim_{M\to\infty} \limsup_{m} \big(\|t_{mn}\|_M\big)^{q_m} = 0\,.$$

LEMMA B. ([4]) *Let X be a complete paranormed space with Schauder basis (b_k), and (A_n) a sequence of elements of X^* with $A_n(\mathbf{x}) = \sum_k a_{nk}x_k$ for all $\mathbf{x} \in X$ and $n \in \mathbb{N}$. Furthermore, let $\mathbf{q} = (q_k)$ be a bounded sequence. Then*

$$\mathbf{A} \in \big(X, V_{\sigma}(\mathbf{q})\big) \iff \text{i)} \quad \textit{there exists an } L \in X^* \textit{ with}$$
$$\big(t_{mn}(b_k) - L(b_k)\big) \in V_{0\sigma}(\mathbf{q}) \ \textit{for all } k\,,$$
$$\text{ii)} \quad \lim_{M\to\infty} \limsup_{m} \big(\|t_{mn}\|_M\big)^{q_m} = 0\,.$$

LEMMA C. ([4]) *Let $\mathbf{p}, \mathbf{q} \in \ell_\infty$. Then*

$$\mathbf{A} \in \big(c_0(\mathbf{p}), \ell_\infty^\sigma(\mathbf{q})\big) \iff \sup_{m,n}\Big(\sum_k |a(n,k,m)|M^{\frac{-1}{p_k}}\Big)^{q_m} < \infty$$
$$\textit{for some} \quad M > 1\,.$$

Additionally, we use the characterization of the Köthe-Toeplitz dual of $c_0(\mathbf{p})$:

$$c_0^+(\mathbf{p}) = \bigcup_{N>1}\Big\{\mathbf{a}:\ \sum_k |a_k|N^{\frac{-1}{p_k}} < \infty\Big\}$$

and the fact that $c_0^+(\mathbf{p}) \cong c_0^*(\mathbf{p})$ (isometrically isomorphic for bounded sequences $\mathbf{p}$).

We now establish the following theorems.

THEOREM 1. *Let $\mathbf{p}, \mathbf{q} \in \ell_\infty$. Then*

$$\mathbf{A} \in \big(c_0(\mathbf{p}), V_{0\sigma}(\mathbf{q})\big) \iff \text{i)} \quad \lim_{m\to\infty} |a(n,k,m)|^{q_m} = 0\,, \ \textit{uniformly in } n\,,$$
$$\text{ii)} \quad \lim_{M\to\infty} \limsup_{m}\Big(\sum_k |a(n,k,m)|M^{\frac{-1}{p_k}}\Big)^{q_m} = 0\,.$$

P r o o f. Let $\mathbf{A} \in \big(c_0(\mathbf{p}), V_{0\sigma}(\mathbf{q})\big)$. Since $V_{0\sigma}(\mathbf{q}) \subset \ell_\infty^\sigma(\mathbf{q})$, we have $\mathbf{A} \in \big(c_0(\mathbf{p}), \ell_\infty^\sigma(\mathbf{q})\big)$. Then $t_{mn}(\mathbf{Ax}) = \sum_k a(n,k,m)x_k$ is defined for all $\mathbf{x} \in c_0(\mathbf{p})$, m

and n. That is $a(n,k,m) \in c_0^+(\boldsymbol{p})$ and $t_{mn} \in c_0^*(\boldsymbol{p})$ for all m,n, and $\|t_{mn}\|_M = \sum_{k=1}^{\infty} |a(n,k,m)| M^{\frac{-1}{p_k}}$ if $\|t_{mn}\|$ is defined. $c_0(\boldsymbol{p})$ being complete, we obtain (ii) by Lemma A and (i) by using $\boldsymbol{e}^{(k)} \in c_0(\boldsymbol{p})$.

Conversely, suppose that the conditions (i) and (ii) hold and $\boldsymbol{x} \in c_0(\boldsymbol{p})$. By (ii) it follows that for some $M > 1$,

$$\sup_{m,n} \Big(\sum_k |a(n,k,m)| M^{\frac{-1}{p_k}} \Big)^{q_m} < \infty .$$

Due to the convergence of $\sum_{k=1}^{\infty} |a(n,k,m)| M^{\frac{-1}{p_k}}$, we have $a(n,k,m) \in c_0^+(\boldsymbol{p})$, and therefore $t_{mn} \in c_0^*(\boldsymbol{p})$ and $\|t_{mn}\|_M = \sum_{k=1}^{\infty} |a(n,k,m)| M^{\frac{-1}{p_k}}$ if $\|t_{mn}\|$ is defined. Trivially $(\boldsymbol{e}^{(k)})$ is a Schauder basis of $c_0(\boldsymbol{p})$. By Lemma A, $\mathbf{A} \in (c_0(\boldsymbol{p}), V_{0\sigma}(\boldsymbol{q}))$. □

We have

THEOREM 2. *Let $\boldsymbol{p}, \boldsymbol{q} \in \ell_\infty$. Then $\mathbf{A} \in (c_0(\boldsymbol{p}), V_\sigma(\boldsymbol{q}))$ if and only if*

(i) $\sup_{n,m} \sum_k |a(n,k,m)| M^{\frac{-1}{p_k}} < \infty$ *for some $M > 1$,*

(ii) *there exist $\alpha_1, \alpha_2, \dots \in C$ with $|a(n,k,m) - \alpha_k|^{q_m} \to 0$, as $m \to \infty$, uniformly in n, for each k,*

(iii) $\lim_{M\to\infty} \limsup_m \Big(\sum_k |a(n,k,m) - \alpha_k| M^{\frac{-1}{p_k}} \Big)^{q_m} = 0 .$

P r o o f. Suppose that $\mathbf{A} \in (c_0(\boldsymbol{p}), V_\sigma(\boldsymbol{q}))$. Because of $V_\sigma(\boldsymbol{q}) \subset \ell_\infty^\sigma(\boldsymbol{q})$, we have $\mathbf{A} \in (c_0(\boldsymbol{p}), \ell_\infty^\sigma(\boldsymbol{q}))$, and so that $t_{mn} \in c_0^*(\boldsymbol{p})$. By Lemma B, there exists an $L \in c_0^*(\boldsymbol{p})$ with

(1) $\big(t_{mn}(\boldsymbol{e}^{(k)}) - L(\boldsymbol{e}^{(k)})\big) \in V_{0\sigma}(\boldsymbol{q})$ for all k,

(2) $\lim_{M\to\infty} \limsup_m \big(\|t_{mn} - L\|_M\big)^{q_m} = 0$.

This $L \in c_0^*(\boldsymbol{p})$ can be written as

$$L(\boldsymbol{x}) = \sum_k \alpha_k x_k$$

for all $\boldsymbol{x} \in c_0(\boldsymbol{p})$ with $(\alpha_k) \in c_0^+(\boldsymbol{p})$. Then (1) reads as $|a(n,k,m) - \alpha_k|^{q_m} \to 0$, as $m \to \infty$, uniformly in n, for each k, which is (ii).

By (2) and since $\|t_{mn} - L\|_M = \sum_k |a(n,k,m) - \alpha_k| M^{\frac{-1}{p_k}}$ for all M, for which $\|t_{mn} - L\|_M$ is defined, (iii) follows.

Noting that $V_\sigma(\boldsymbol{q}) \subset \ell_\infty^\sigma = \ell_\infty$ and that therefore $\mathbf{A} \in \big(c_0(\boldsymbol{p}), \ell_\infty\big)$, we may apply Lemma C to obtain

$$\sup_{m,n}\Big(\sum_k |a(n,k,m)| M^{\frac{-1}{p_k}}\Big) < \infty .$$

For the converse, let (i), (ii), and (iii) hold. From (i), we have $a(n,k,m) \in c_0(\boldsymbol{p})$ for all n,m, and therefore $t_{mn} \in c_0^*(\boldsymbol{p})$ for all n,m. It follows from (i) and (iii) that for n, m and M large enough

$$\sum_k |\alpha_k| M^{\frac{-1}{p_k}} \le \sum_k |a(n,k,m) - \alpha_k| M^{\frac{-1}{p_k}} + \sum_k |a(n,k,m)| M^{\frac{-1}{p_k}} < \infty .$$

Therefore

$$(\alpha_k) \in c_0^+(\boldsymbol{p}),$$

and with $L\boldsymbol{x} = \sum_k \alpha_k x_k$:

$$L \in c_0^*(\boldsymbol{p}) .$$

So we have for t_{mn}, $L \in c_0^*(\boldsymbol{p})$

$$\|t_{mn} - L\|_M = \sum_k |a(n,k,m) - \alpha_k| M^{\frac{-1}{p_k}} .$$

By Lemma B, $\mathbf{A} \in \big(c_0(\boldsymbol{p}), V_\sigma(\boldsymbol{q})\big)$. This completes the proof. □

4. Corollaries

We deduce the following corollaries.

COROLLARY 1. $\mathbf{A} \in \big(c_0(\boldsymbol{p}), V_{0\sigma}\big)$ *if and only if*

(i) $a(n,k,m) \to 0$ *as* $m \to \infty$, *uniformly in* n, *for each* k,

(ii) $\lim\limits_{M\to\infty} \limsup\limits_{m} \sum\limits_k |a(n,k,m)| M^{\frac{-1}{p_k}} = 0$.

P r o o f. Take $q_k = 1$ for all k in Theorem 1. □

COROLLARY 2. $\mathbf{A} \in \big(c_0(\boldsymbol{p}), V_\sigma\big)$ *if and only if*

(i) $\sup\limits_{n,m} \sum\limits_k |a(n,k,m)| M^{\frac{-1}{p_k}} < \infty$ *for some* $M > 1$,

(ii) *there exist* $\alpha_1, \alpha_2, \dots \in C$ *with* $|a(n,k,m) - \alpha_k| \to 0$, *as* $m \to \infty$, *uniformly in* n, *for each* k,

(iii) $\lim\limits_{M\to\infty} \limsup\limits_{m} \sum\limits_k |a(n,k,m) - \alpha_k| M^{\frac{-1}{p_k}} = 0$.

P r o o f. Take $q_k = 1$ for all k in Theorem 2. □

COROLLARY 3. *Let* $\boldsymbol{p}, \boldsymbol{q} \in \ell_\infty$. *Then* $\mathbf{A} \in \big(c_0(\boldsymbol{p}), f_0(\boldsymbol{q})\big)$ *if and only if*

(i) $|b(n,k,m)|^{q_m} \to 0$ *as* $m \to \infty$, *uniformly in* n, *for each* k,

(ii)
$$\lim_{M\to\infty} \limsup_m \Big(\sum_k |b(n,k,m)| M^{\frac{-1}{p_k}} \Big)^{q_m} = 0,$$

where $b(n,k,m) = \frac{1}{m+1} \sum_{j=0}^{m} a(n+j,k)$.

Taking $\sigma(n) = n+1$ in Theorem 1, we close the proof.

COROLLARY 4. *Let* $\boldsymbol{p}, \boldsymbol{q} \in \ell_\infty$. *Then* $\mathbf{A} \in \big(c_0(\boldsymbol{p}), f(\boldsymbol{q})\big)$ *if and only if*

(i) $\sup_{n,m} \sum_k |b(n,k,m)| M^{\frac{-1}{p_k}} < \infty$ *for some* $M > 1$,

(ii) *there exist* $\alpha_1, \alpha_2, \dots \in C$ *with* $|b(n,k,m) - \alpha_k|^{q_m} \to 0$, *as* $m \to \infty$, *uniformly in* n, *for each* k,

(iii)
$$\lim_{M\to\infty} \limsup_m \Big(\sum_k |b(n,k,m) - \alpha_k| M^{\frac{-1}{p_k}} \Big)^{q_m} = 0,$$

where $\alpha_k = L - \lim_n a_{nk}$.

Proof. Choosing the mapping $\sigma(n) = n+1$ instead of mapping σ as the transformation mapping, the space $V_\sigma(\boldsymbol{q})$ of Theorem 2 reduces to $f(\boldsymbol{q})$. Hence it is proved. □

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Department of Mathematics
Fırat University
TR-23169, Elazığ
Turkey