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ON AN APPLICATION OF A NEWTON–LIKE METHOD TO THE APPROXIMATION OF IMPLICIT FUNCTIONS

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ABSTRACT. We use a Newton-like method to approximate implicit functions in Banach spaces. The nonlinear equations involved contain a nondifferentiable term. Our hypotheses are more general than Chen-Yamamoto's [3], in this case.

I. Introduction

Let E, Λ be Banach spaces and denote by $U(x^0, R)$ the closed ball with center $x^0 \in E$ and of radius R in E. We will use the same symbol for the norm $\| \ \|$ in both spaces. Suppose that the nonlinear operators $F(x, \lambda)$, $G(x, \lambda)$ and the linear operator $A(x, \lambda)(\cdot)$ with values in E defined for $x \in U(x^0, R)$ and $\lambda \in U(\lambda_0, S)$ are such that F is Fréchet differentiable there, $A(x^0, \lambda_0)^{-1}$ exists and

$$\|A(x^0, \lambda_0)^{-1}(A(x, \lambda) - A(x^0, \lambda))\| \le v_s(\|x - x^0\|) + b, \tag{1}$$

$$\|A(x^0, \lambda_0)^{-1}(A(x^0, \lambda) - A(x^0, \lambda_0))\| \le k_1(s)\|\lambda - \lambda_0\|, \tag{2}$$

$$\begin{aligned}&\|A(x^0, \lambda_0)^{-1}[F'(x + t(y - x), \lambda) - A(x, \lambda)]\| \\ &\qquad \le w_s(\|x - x^0\| + t\|y - x\|) - v_s(\|x - x^0\|) + c\end{aligned} \tag{3}$$

for all $t \in [0,1]$ and

$$\|A(x^0, \lambda_0)^{-1}(G(x, \lambda) - G(y, \lambda))\| \le e_s(r)\|x - y\|, \tag{4}$$

for all $x, y \in U(x^0, r) \subset U(x^0, R)$ and $\lambda \in U(\lambda_0, s) \subset U(\lambda_0, S)$, where $w_s(r + t) - v_s(r)$, $t \ge 0$, $k_1(s)$ and $e_s(r)$ are nondecreasing non-negative

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functions with $w_0(0) = v_0(0) = e_0(0) = k_1(0) = 0$, $v_s(r)$ is differentiable, $v_s'(r) > 0$ at every point of $[0, R]$, and the constants b, c satisfy $b \geq 0$, $c \geq 0$ and $b + c < 1$. We need to define the functions

$$a_s = k(s)\left\|A(x^0, \lambda_0)^{-1}\big(F(x^0, \lambda_0) + G(x^0, \lambda_0)\big)\right\|, \qquad (s = 0 \quad \text{if} \quad \lambda = \lambda_0),$$

$$k_2(s) = \int_0^s k_1(t)\,\mathrm{d}t, \quad k(s) = \big(1 - k_2(s)\big)^{-1} \qquad \text{provided that} \quad k_2(S) < 1,$$

$$\varphi_s(r) = a_s - r + k(s)\int_0^r w_s(t)\,\mathrm{d}t,$$

$$\psi_s(r) = k(s)\int_0^r e_s(t)\,\mathrm{d}t,$$

$$\chi_s(r) = \varphi_s(r) + \psi_s(r) + k(s)(b + c)r$$

and the iterations

$$x_0 = x^0,$$
$$x_{n+1}(\lambda) = x_n(\lambda) - A\big(x_n(\lambda), \lambda\big)^{-1}\big(F\big(x_n(\lambda), \lambda\big) + G\big(x_n(\lambda), \lambda\big)\big), \qquad n \geq 0 \tag{5}$$

and

$$y_0 \in U(x^0, R),$$
$$y_{n+1}(\lambda) = y_n(\lambda) - A\big(y_n(\lambda), \lambda\big)^{-1}\big(F\big(y_n(\lambda), \lambda\big) + G\big(y_n(\lambda), \lambda\big)\big), \qquad n \geq 0. \tag{6}$$

By x_0, y_0, x^0, we mean $x_0(\lambda), y_0(\lambda), x^0(\lambda)$, that is, e.g., x_0 depends on the λ used in (5). We use the iterations given by (5) and (6) to approximate a solution $x^*(\lambda)$ of the equation

$$F(x, \lambda) + G(x, \lambda) = 0 \qquad \text{in} \quad U(x^0, R), \tag{7}$$

under the hypotheses (1)–(4).

Our assumptions (1)–(4) generalize the ones made by Chen-Yamamoto (for $F(x, \lambda) = F(x)$ and $G(x, \lambda) = G(x)$) [3], Zabrejko-Nguen [12], Yamamoto [11] and Potra-Pták (in [7] for $G = 0$). Moreover, several authors have treated the case when $G = 0$ provided that k_s and k_1 are constants (or not) [1], [2], [5], [6], [8], [9], [10]. Note that conditions of the form (3) are variants of the usual Lipschitz condition with the constant depending on the radius of the ball. Such conditions have also been considered in [12] and [3], as well as in [1], [4], [6], [7] and [9]. Several applications and justifications for the use of conditions like (3) can also be found in the above references. One can also claim that for certain choices of the functions w_s, v_s and the parameter c the right-hand side of (3) does not depend on the radius of the ball. Choose for example w_s such that for all $x, y \in U(x^0, r)$ and $t \in [0,1]$, $w_s(\|x - x^0\| + t\|y - x\|) = v_s(\|x - x^0\|) + tq\|x - y\|$ for some constant q (independent of r) with $q \in (0,1)$, and $c = 0$ in (3). Then the right-hand side of (3) is independent of the radius of the ball and constitutes the usual Lipschitz condition for the operator $F'(x, \lambda)$.

We provide sufficient conditions for the convergence of iterations (5) and (6) to a locally unique solution $x^*(\lambda)$ of equation (7) as well as several error bounds on the distances $\|x_{n+1}(\lambda) - x_n(\lambda)\|$, $\|x_n(\lambda) - x^*(\lambda)\|$, $\|y_{n+1}(\lambda) - y_n(\lambda)\|$ and $\|y_n(\lambda) - x^*(\lambda)\|$.

II. Convergence results

Let us define the numerical sequence

$$r_0 \in [0, R], \quad r_{n+1,s} = r_{n+1} = r_n + \frac{u_s(r_n)}{\overline{w}_s(r_n)}, \qquad n \geq 0, \tag{8}$$

where $u_s(r) = x_s(r) - \alpha^*$ and $w_s(r) = 1 - k(s)(v_s(r) + b)$, $\overline{w}_s(r) = k(s)w_s(r)$. Here $\alpha_s^* = \alpha^*$ denotes the minimal value of $\chi_s(r)$ in $[0, R]$. Let us denote the minimal point by $r_s^* = r^*$. If $\chi_s(R) \leq 0$, then $\psi_s(r)$ has a unique zero $t_s^* = t^*$ in $(0, r^*]$, since $\chi_s(r)$ is strictly convex. Moreover, it is a simple calculus to show as in Lemma 1 in [3, p. 40] that the sequence $\{r_n\}$, $k \geq 0$ is strictly monotonically increasing and converges to the unique zero of $u_s(r)$ in $[0, r^*]$, r^* for any $r_0 \in [0, r^*)$.

Furthermore, let us define the sets

$$\tilde{U}_s = \tilde{U} = \begin{cases} U(x^0, R), & \text{if} \quad \chi_s(R) < 0 \quad \text{or} \quad \chi_s(R) = 0 \quad \text{and} \quad t^* = R \\ U^0(x^0, R), & \text{if} \quad \chi_s(R) = 0 \quad \text{and} \quad t^* < R, \end{cases}$$

$$\Omega_s = \Omega = \bigcup_{r \in [0, r^*)} \left\{ y \in U(x^0, r) / \|A(y, \lambda)^{-1}(F(y, \lambda) + G(y, \lambda))\| \leq \frac{u_s(r)}{\overline{w}_s(r)} \right\}$$

for $y \in \Omega$,

$$R_{y,s} = R_y = \left\{ r \in [0, r^*) / \left\| A(y,\lambda)^{-1} \big(F(y,\lambda) + G(y,\lambda)\big) \right\| \le \frac{u_s(r)}{\overline{w}_s(r)}, \right.$$
$$\left. \|y - x^0\| \le r \right\}$$

and the numerical sequence $\{\rho_n\}$, $n \ge 0$, by

$$\rho_0 = 0, \quad \rho_{n+1,s} = \rho_{n+1} = \rho_n + \frac{u_s(\rho_n)}{\overline{w}_s(\rho_0)}, \qquad n \ge 0. \tag{9}$$

We can now formulate the main result:

THEOREM 1. *Suppose that* $\chi_s(R) \le 0$.

Then

(a) *the equation* (7) *has a solution* $x^*(\lambda)$ *in* $U(x^0, t^*)$, *which is unique in* $\tilde{U}$;

(b) *for any* $y_0 \in \Omega$, *the iterations* (5) *and* (6) *are well defined for all* $n \ge 0$, *remain in* $U^0(x^0, r^*)$ *and satisfy the estimates*

$$\|y_{n+1}(\lambda) - y_n(\lambda)\| \le r_{n+1} - r_n, \qquad n \ge 0, \tag{10}$$
$$\|y_n(\lambda) - x^*(\lambda)\| \le r^* - r_n, \qquad n \ge 0, \tag{11}$$
$$\|x_{n+1}(\lambda) - x_n(\lambda)\| \le \rho_{n+1} - \rho_n, \qquad n \ge 0 \tag{12}$$

and

$$\|x_n(\lambda) - x^*(\lambda)\| \le r^* - \rho_n, \qquad n \ge 0, \tag{13}$$

provided that $r_0 \in R_{y_0}$.

P r o o f. We will only show that the estimate (10) is true for all $n \ge 0$. The estimates (11)-(13) will then follow immediately (note that $x^0 \in \Omega$).

For $n = 0$ we must show $\|y_1(\lambda) - y_0\| \le r_1 - r_0$. Let $r_0 \in R_{y_0}$ for $y_0 \in \Omega$, then

$$\|y_0 - x^0\| \le r_0 < r^*$$

and

$$\|y_1(\lambda) - y_0\| = \left\| A(y_0, \lambda)^{-1} \big(F(y_0,\lambda) + G(y_0,\lambda)\big) \right\| \le \frac{u_s(r_0)}{\overline{w}_s(r_0)} = r_1 - r_0.$$

Hence

$$\|y_1(\lambda) - x^0\| \le r_1.$$

We will show that $\|y_n(\lambda)-x^0\| \le r_n$, $n \ge 0$, which we showed to be true above for $n=0$. Let

$$\|y_n(\lambda) - y_{n-1}(\lambda)\| \le r_n - r_{n-1} \qquad \text{and} \qquad \|y_n - x^0\| \le r_n$$

hold for all $n \le k$.

The inverse of the linear operator $A(x,\lambda)$, $x \in U(x^0, R)$, $\lambda \in U(\lambda_0, S)$ exists if the right-hand side of

$$\begin{aligned} &A(x,\lambda)^{-1}A(x^0,\lambda_0) \\ &\qquad = \left[I + A(x^0,\lambda)^{-1}\left(A(x,\lambda) - A(x^0,\lambda)\right)\right]^{-1} A(x^0,\lambda)^{-1}A(x^0,\lambda_0) \end{aligned}$$

exists. But, $A(x^0,\lambda)^{-1}$ exists, since

$$A(x^0,\lambda)^{-1}A(x^0,\lambda_0) = \left[I + A(x^0,\lambda_0)^{-1}\left(A(x^0,\lambda) - A(x^0,\lambda_0)\right)\right]^{-1}$$

exists. Moreover, by (1) - (8) we get that $A(x,\lambda)^{-1}$ exists,

$$\left\|A\big(y_k(\lambda),\lambda\big)^{-1}A(x^0,\lambda_0)\right\| \le k(s)w_s(r_k)^{-1}$$

and

$$\begin{aligned} &\left\|y_{k+1}(\lambda) - y_k(\lambda)\right\| = \left\|A\big(y_k(\lambda),\lambda\big)^{-1}\left(F\big(y_k(\lambda),\lambda\big) + G\big(y_k(\lambda),\lambda\big)\right)\right\| \\ &\le \left\|A\big(y_k(\lambda),\lambda\big)^{-1}A(x^0,\lambda_0)\right\| \left\|A(x^0,\lambda_0)^{-1}\left\{F\big(y_k(\lambda),\lambda\big) + G\big(y_k(\lambda),\lambda\big)\right.\right. \\ &\quad \left.\left. - A\big(y_{k-1}(\lambda),\lambda\big)\big(y_k(\lambda) - y_{k-1}(\lambda)\big) - F\big(y_{k-1}(\lambda),\lambda\big) - G\big(y_{k-1}(\lambda),\lambda\big)\right\}\right\| \\ &\le k(s)w_s(r_k)^{-1}\left\{\int_0^1 \left\|A(x^0,\lambda_0)^{-1}\left(F'\big(y_{k-1}(\lambda) + t\big(y_k(\lambda) - y_{k-1}(\lambda)\big)\big)\right.\right.\right. \\ &\quad \left.\left. - A\big(y_{n-1}(\lambda),\lambda\big)\right)\right\| \left\|y_k(\lambda) - y_{k-1}(\lambda)\right\| \mathrm{d}t \\ &\quad \left. + \left\|A(x^0,\lambda_0)^{-1}\left(G\big(y_k(\lambda),\lambda\big) - G\big(y_{k-1}(\lambda),\lambda\big)\right)\right\|\right\} \le \end{aligned}$$

(As in [3, p. 42 relation (13)], and [12, Proposition 1], we get)

$$\leq k(s)w_s(r_k)^{-1}\Bigg\{\int_0^1 \big(w_s(\|y_{k-1}(\lambda)-x^0\|+t\|y_k(\lambda)-y_{k-1}(\lambda)\|)$$

$$-\,v_s(\|y_{k-1}(\lambda)-x^0\|)\big)\|y_k(\lambda)-y_{k-1}(\lambda)\|\,\mathrm{d}t$$

$$+\,c\|y_{k-1}(\lambda)-y_k(\lambda)\|+\int_{r_{k-1}}^{r_k} e_s(t)\,\mathrm{d}t\Bigg\}$$

$$\leq \overline{w}_s(r_k)^{-1}\big(u_s(r_k)-u_s(r_{k-1})+\overline{w}_s(r_{k-1})(r_k-r_{k-1})\big)$$

$$=\overline{w}_s(r_k)^{-1}u_s(r_k)=r_{k+1}-r_k\,.$$

Moreover, we have

$$\|y_{k+1}(\lambda)-x^0\|\leq\|y_{k+1}(\lambda)-y_k(\lambda)\|+\|y_k(\lambda)-x^0\|\leq r_{k+1}\,.$$

That is, (10) is true for all $n\geq 0$ and $y_n(\lambda)\in U^0(x^0,\,r_n)\subset U^0(x^0,\,r^*)$. Hence, the sequence $\{y_n(\lambda)\}$ is a Cauchy sequence in $U(x^0,\,r^*)$ and converges to a solution $x^*(\lambda)\in U^0(x^0,\,r^*)$ of equation (7).

The proof of the theorem will be complete if we show that $x^*(\lambda)$ is the unique solution of equation (7) in $\tilde{U}$. Let $y^*(\lambda)$ be any solution in $\tilde{U}$. Then we have

$$\|y^*(\lambda)-x^0\|-a_s\leq\|y^*(\lambda)-x_1(\lambda)\|$$

$$\leq\int_0^1 w_s\big(t\|y^*(\lambda)-x^0\|+c\big)\|y^*(\lambda)-x^0\|\,\mathrm{d}t+\int_0^{\|y^*(\lambda)-x^0\|} e_s(t)\,\mathrm{d}t\,.$$

The above inequality shows $\chi_s(\|y^*(\lambda)-x^0\|)\geq 0$. Hence, $\|y^*(\lambda)-x^0\|\leq t^*$.

We can show

$$\|y^*(\lambda)-x_n(\lambda)\|\leq r^*-\rho_n\,,\qquad n\geq 0\,. \tag{14}$$

For $n=0$, the above inequality is trivially true. Suppose that it is true for all

$n \leq k$, then

$$\begin{aligned}
\|y^*(\lambda) - x_{k+1}(\lambda)\| &= \big\|y^*(\lambda) - x_k(\lambda) + A\big(x_k(\lambda),\, \lambda\big)^{-1}\big(F\big(x_k(\lambda),\, \lambda\big) \\
&\quad + G\big(x_k(\lambda),\, \lambda\big)\big) - A\big(x_k(\lambda),\, \lambda\big)^{-1}\big(F\big(y^*(\lambda),\, \lambda\big) + G\big(y^*(\lambda),\, \lambda\big)\big)\big\| \\
&\leq k(s)w_s(\rho_k)^{-1}\Big\{\int_0^1 \big\|A(x^0,\, \lambda_0)^{-1}F'\big(x_k(\lambda) + t(y^*(\lambda) - x_k(\lambda))\big) \\
&\quad - A\big(x_k(\lambda),\, \lambda\big)\big)\big\|\|y^*(\lambda) - x_k(\lambda)\|\,\mathrm{d}t \\
&\quad + \big\|A(x^0,\, \lambda_0)^{-1}\big(G\big(y^*(\lambda),\, \lambda\big) - G\big(x_k(\lambda),\, \lambda\big)\big)\big\|\Big\} \\
&\leq k(s)w_s(\rho_k)^{-1}\Big\{\int_{\rho_k}^{r^*}(w_s(t) + e_s(t))\mathrm{d}t + (b + c - 1)(r^* - \rho_k) + w_s(\rho_k)(r^* - \rho_k)\Big\} \\
&= \overline{w}_s(\rho_k)^{-1}\big(\chi_s(r^*) - \chi_s(\rho_k)\big) + r^* - \rho_k = r^* - \rho_{k+1}\,.
\end{aligned}$$

Hence, (14) is true for all $n \geq 0$. By taking the limit in (14) as $n \to \infty$ we get $x^*(\lambda) = y^*(\lambda)$.

That completes the proof of the theorem.

We will now generalize the function $\chi_s(r)$. Let $y \in \Omega$ and choose $r_y \in R_y$ (fixed) and set

$$a_{s,y} = a_y = k(s)\big\|A\big(y(\lambda),\, \lambda\big)^{-1}\big(F\big(y(\lambda),\, \lambda\big) + G\big(y(\lambda),\, \lambda\big)\big)\big\|,$$

$$d_{s,y} = d_y = \begin{cases} k(s), & \text{if } \ y = x^0 \ \text{ and } \ r_y = 0 \\ \overline{w}_s(r_y)^{-1}, & \text{otherwise} \end{cases}$$

and

$$\chi_{s,y}(r) = \chi_y(r) = a_y + d_y\left[k(y)\left(\int_0^r \big(w_s(r_y + t) + e_s(r_y + t)\big)\,\mathrm{d}t + b + c\right) - r\right].$$

Moreover, let us define the numerical iteration $\{q_n\}$, $n \geq 0$, by

$$q_{n+1,s} = q_{n+1} = q_n + \frac{\chi_y(q_n)}{d_y\overline{w}_s(r_y + q_n)}, \qquad q_0 = 0, \quad y_0 = y, \quad n \geq 0.$$

Then exactly as in Proposition 1 and Theorem 2 in [3, p. 45] we can show

THEOREM 2. *Suppose that the hypotheses of Theorem* 1 *are true. Then*

(a) *the ball* $U\left(x^0, \dfrac{|\alpha^*|}{2-b}\right) \subset \Omega$;

(b) *the equation* $\chi_y(r)$ *has a unique zero* $q^* = q^*_s$ *in* $[0, r^* - r_y]$ *and* $\chi_y(r^* - r_y) \le 0$.

(c) *Moreover, the following estimates are true*

$$\|y_{n+1}(\lambda) - y_n(\lambda)\| \le q_{n+1} - q_n\,, \qquad n \ge 0$$

and

$$\|y_n(\lambda) - x^*(\lambda)\| \le q^* - q_n \le r^* - r_y\,, \qquad n \ge 0\,.$$

Rall in [8] and Rheinboldt in [10] showed convergence of (5) to $x^*(\lambda)$ in a closed ball centered at x^* when $G = 0$, $F(x, \lambda) = F(x)$ and v_s, $\overline{w}_s$ are constants if $F'(x^*)^{-1}$ exists.

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