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SOME RESULTS ON THE HADWIGER NUMBERS OF GRAPHS

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1. Introduction

In the paper presented we prove some theorems generalizing the results in [4]. We also disprove two of Zelinka's conjectures.

We consider only finite undirected graphs without loops and multiple edges. The concept of the Hadwiger number of a graph was defined in [2] (see also [4]), however, we use the equivalent definition introduced in [4].

Let G be a connected graph. Then a decomposition of its vertex set $V(G)$ into nonempty subsets $V_1, \ldots, V_m$ with the following properties

(i) $\bigcup_{i=1}^{m} V_i = V(G)$,

(ii) $V_i \cap V_j = \emptyset$ for any two integers i, j such that $1 \leqq i \leqq m$, $1 \leqq j \leqq m$, $i \neq j$,

(iii) V_i induces a connected subgraph of G for all $i = 1, \ldots, m$,

(iv) $V_i \cup V_j$ induces a connected subgraph of G for all $i = 1, \ldots, m$ and all $j = 1, \ldots, m$,

is called an H-decomposition of G.

The Hadwiger number $\eta(G)$ of a connected graph is the maximal positive integer m such that there exists an H-decomposition of G into m subsets.

Other definitions not given here will be found in [1].

2. Bounds on $\eta(G)$

In this section we establish strict bounds of $\eta(G)$ depending on $\omega(G)$ (i.e. the maximal number of vertices in a clique of G) and $\alpha_0(G)$ (i.e. the vertex covering number of G).

Lemma 1. (B. Zelinka [4]) *Let G' be a subgraph of the graph G. Then*

$$\eta(G') \leqq \eta(G).$$

Theorem 1. *Let G be a connected graph. Then*

$$\omega(G) \leqq \eta(G) \leqq \min(1 + \alpha_0(G), [\tfrac{1}{2}(\omega(G) + |V(G)|)]).$$

Proof. As the Hadwiger number of the complete graph with n vertices is equal to n, the inequality $\omega(G) \leqq \eta(G)$ follows evidently from Lemma 1.

Now let us assume that $V_1, \ldots, V_{\eta(G)}$ is an H-decomposition of G.

Let C be a minimal vertex covering set of G, i.e. $|C| = \alpha_0(G)$. Then the set $V(G) - C$ is an independent set of G. If $\eta(G) > 1 + \alpha_0(G) = 1 + |C|$, then there exist at least two sets V_i, V_j $(i \neq j)$ of the H-decomposition of G such that $V_i \cap C = V_j \cap C = \emptyset$, i.e. $V_i \cup V_j \subseteq V(G) - C$. So $V_i \cup V_j$ is independent and it does not induce a connected subgraph of G, which is a contradiction. Thus

$$\eta(G) \leqq 1 + \alpha_0(G). \tag{1}$$

Let I be a maximal subset of $\{1, \ldots, \eta(G)\}$ such that $|V_i| = 1$ for $i \in I$. Then $\bigcup_{i \in I} V_i$ induces a complete subgraph of G with $|I|$ vertices and hence $|I| \leqq \omega(G)$. Since $|V_j| \geqq 2$ for $j \in \{1, \ldots, \eta(G)\} - I$, we get

$$|V(G)| = |V_1| + \ldots + |V_{\eta(G)}| \geqq |I| + 2(\eta(G) - |I|) \geqq 2\eta(G) - \omega(G).$$

This implies

$$\eta(G) \leqq \tfrac{1}{2}(\omega(G) + |V(G)|).$$

As $\eta(G)$ is an integer, we may write

$$\eta(G) \leqq [\tfrac{1}{2}(\omega(G) + |V(G)|)]. \tag{2}$$

Thus (1) and (2) give the desired upper bound on $\eta(G)$.

3. The Hadwiger number of a complete multipartite graph

The complete k-partite graph is a graph whose vertices can be partitioned into k classes $U_1, \ldots, U_k$ such that two vertices are adjacent if and only if they belong to distinct classes. If $|U_i| = n_i$ for all $i = 1, \ldots, k$, then the complete k-partite graph is denoted by $K(n_1, \ldots, n_k)$.

B. Zelinka [4] determined the Hadwiger number of a complete bipartite graph. Now we determine the Hadwiger number of a complete k-partite graph for all $k \geqq 2$.

Theorem 2. *Let $k \geqq 2$, $1 \leqq n_1 \leqq \ldots \leqq n_k$ be integers. For the complete k-partite graph $K(n_1, \ldots, n_k)$ there holds*

$$\eta(K(n_1, \ldots, n_k)) = \min(1 + n_1 + \ldots + n_{k-1}, [\tfrac{1}{2}(k + n_1 + \ldots + n_k)]). \tag{3}$$

Proof. For $n_{k-1} = 1$ we have $\alpha_0(K(1, \ldots, 1, n_k)) = k - 1$ and $\omega(K(1, \ldots, 1, n_k)) = k$ and thus by Theorem 1 we get

$$k \leqq \eta(K(1, \ldots, 1, n_k)) \leqq \min(1 + (k - 1), [\tfrac{1}{2}(k + (k - 1) + n_k)]) = k.$$

Hence

$$\eta(K(1, \ldots, 1, n_k)) = \min(1 + (k - 1), [\tfrac{1}{2}(k + (k - 1) + n_k)]).$$

Now, let us assume that $K(m_1, \ldots, m_k)$ $(1 \leqq m_1 \leqq \ldots \leqq m_k)$ is a complete k-partite graph with the minimal possible number of vertices such that (3) does not hold. Then $m_{k-1} \geqq 2$, hence $K(m_1, \ldots, m_k) - \{u, v\}$ (for vertices u, v belonging to classes of cardinalities m_{k-1} and m_k respectively) is again a complete k-partite graph, i.e. $K(m_1, \ldots, m_{k-1} - 1, m_k - 1)$. This graph contains less vertices than $K(m_1, \ldots, m_k)$, therefore we may determine its Hadwiger number by (3). Two cases must be distinguished.

Case 1. If $k = 2$ or $k > 2$ and $m_k > m_{k-2}$, then

$$\eta(K(m_1, \ldots, m_{k-1} - 1, m_k - 1)) = \min(1 + m_1 + \ldots + (m_{k-1} - 1), [\tfrac{1}{2}(k + m_1 + \ldots + (m_{k-1} - 1) + (m_k - 1))]) = \min(1 + m_1 + \ldots + m_{k-1}, [\tfrac{1}{2}(k + m_1 + \ldots + m_k)]) - 1. \quad (4)$$

Case 2. If $k > 2$ and $m_k = m_{k-2}$, then

$$\eta(K(m_1, \ldots, m_{k-1} - 1, m_k - 1)) = \min(1 + m_1 + \ldots + m_{k-3} + (m_{k-1} - 1) + (m_k - 1), [\tfrac{1}{2}(k + m_1 + \ldots + m_{k-2} + (m_{k-1} - 1) + (m_k - 1))]) = \min(m_1 + \ldots + m_{k-1}, [\tfrac{1}{2}(k + m_1 + \ldots + m_k)]) - 1. \quad (5)$$

However, in this case $m_{k-2} = m_k = m_{k-1} \geqq 2$ and hence

$$m_1 + \ldots + m_{k-2} + 2 + (m_{k-1} + m_k - 2m_k) > k.$$

This implies

$$2 + 2(m_1 + \ldots + m_k) - 2m_k > k + m_1 + \ldots + m_k.$$

Hence

$$1 + m_1 + \ldots + m_{k-1} > \tfrac{1}{2}(k + m_1 + \ldots + m_k) \geqq [\tfrac{1}{2}(k + m_1 + \ldots + m_k)].$$

As all numbers appearing in the relation are integers, we have

$$m_1 + \ldots + m_{k-1} \geqq [\tfrac{1}{2}(k + m_1 + \ldots + m_k)]. \quad (6)$$

By (5) and (6) we get

$$\eta(K(m_1, \ldots, m_{k-1} - 1, m_k - 1)) = \min(1 + m_1 + \ldots + m_{k-1}, [\tfrac{1}{2}(k + m_1 + \ldots + m_k)]) - 1. \quad (7)$$

If $V_1, \ldots, V_\eta$ is an H-decomposition of $K(m_1, \ldots, m_k) - \{u, v\}$, then $V_1, \ldots, V_\eta$, $\{u, v\}$ is evidently an H-decomposition of $K(m_1, \ldots, m_k)$. Thus

$$\eta(K(m_1, \ldots, m_k)) \geqq 1 + \eta(K(m_1, \ldots, m_k) - \{u, v\}) = \\ = 1 + \eta(K(m_1, \ldots, m_{k-1} - 1, m_k - 1)).$$

So by (4) and (7) we get

$$\eta(K(m_1, \ldots, m_k)) \geqq \min(1 + m_1 + \ldots + m_{k-1}, [\tfrac{1}{2}(k + m_1 + \ldots + m_k)]). \quad (8)$$

Since $\omega(K(m_1, \ldots, m_k)) = k$ and $\alpha_0(K(m_1, \ldots, m_k)) = m_1 + \ldots + m_{k-1}$, by Theorem 1 we have

$$\eta(K(m_1, \ldots, m_k)) \leqq \min(1 + m_1 + \ldots + m_{k-1}, [\tfrac{1}{2}(k + m_1 + \ldots + m_k)]). \quad (9)$$

(8) and (9) result in a contradiction to our assumption, which completes the proof.

Remark 1. By Theorem 2 the complete multipartite graphs (graphs $K(1, \ldots, \ldots, 1, n)$) attain the upper (lower, respectively) bound in Theorem 1. Thus these inequalities cannot be improved in general.

Remark 2. V. G. Vizing [2] suggested the study of the function $\lambda_k(n)$ which denotes the maximal possible number of edges of a graph with n vertices and with the Hadwiger number k. A. A. Zykov [5] and B. Zelinka [3] proved that $\lambda_k(n) = (k-1)n - \binom{k}{2}$ for $k \leqq 4$, $n \geqq k$. B. Zelinka [3] conjectured that this equality is true for any two positive integers n, k, where $n \geqq k$. By Theorem 2 it is easy to prove that the complete m-partite graphs $K(2, 2, \ldots, \ldots, 2)$ (for $m \geqq 7$) are counter-examples to Zelinka's conjecture.

4. The Hadwiger number of the Cartesian product of two stars

The Cartesian product $G_1 \times G_2$ of the graphs G_1, G_2 is the graph whose vertex set is the set of all ordered pairs $[u_1, u_2]$, where u_1 is a vertex of G_1 and u_2 is a vertex of G_2 and in which the vertices $[u_1, u_2]$, $[v_1, v_2]$ are adjacent if and only if either $u_1 = v_1$ and the vertices u_2, v_2 are adjacent in G_2, or $u_2 = v_2$ and the vertices u_1, v_1 are adjacent in G_1.

B. Zelinka [4] proved the following theorem: *Let G_1, G_2 be two finite connected graphs, let $G_1 \times G_2$ be their Cartesian product. Then* $\eta(G_1 \times G_2) \geqq \geqq \eta(G_1) + \eta(G_2) - 1$. He also conjectured that the equality always holds in this relation.

In this part we determine the Hadwiger number of the Cartesian product of two stars which will disprove Zelinka's conjecture.

First we prove

Lemma 2. *Let G and G' be connected homeomorphic graphs. Then*

$$\eta(G) = \eta(G').$$

Proof. Evidently, it is sufficient to consider only an elementary subdivision. Hence let G be a connected graph and uv be its edge. Let G' be a graph which we obtain from G by deleting the edge uv and adding the vertex w and the edges uw and vw.

Let $U_1, \ldots, U_{\eta(G')}$ be an H-decomposition of G' and without loss of generality let $w \in U_1$. If $|U_1| \geqq 1$, then $U_1 - \{w\}, U_2, \ldots, U_{\eta(G')}$ is an H-decomposition of G. Thus

$$\eta(G) \geqq \eta(G'). \tag{10}$$

If $|U_1| = 1$, then by (iv) $\eta(G') \leqq 3$. If $\eta(G') \leqq 2$, then

$$\eta(G) \geqq 2 \geqq \eta(G'), \tag{11}$$

because $\{u, v\}$ induces a complete subgraph of G. If $\eta(G') = 3$, then G' contains a cycle, because the Hadwiger number of any tree is equal to 2. Since a subdivision does not give rise to a new cycle, the graph G must contain a cycle C. By Lemma 1

$$\eta(G) \geqq \eta(C) = 3 = \eta(G'), \tag{12}$$

because the Hadwiger number of any cycle is equal to 3.

Let $V_1, \ldots, V_{\eta(G)}$ be an H-decomposition of G and without loss of generality let $v \in V_1$. Then $V_1 \cup \{w\}, V_2, \ldots, V_{\eta(G)}$ is an H-decomposition of G' and so

$$\eta(G') \geqq \eta(G). \tag{13}$$

The assertion follows already from (10), (11), (12) and (13).

Theorem 3. *Let $K_{1,n}$ and $K_{1,m}$ be stars, let $K_{1,n} \times K_{1,m}$ be their Cartesian product. Then*

$$\eta(K_{1,n} \times K_{1,m}) = 2 + \min(n, m).$$

Proof. It can be easily seen that the graphs $K_{1,n} \times K_{1,m}$ and $K(1, n, m)$ are homeomorphic. By Lemma 2 we have

$$\eta(K_{1,n} \times K_{1,m}) = \eta(K(1, n, m)).$$

By Theorem 2 we get the assertion of Theorem 3.

Remark 3. As $\eta(K_{1,r}) = 2$, the Cartesian product $K_{1,n} \times K_{1,m}$ (for $n \geqq 2$, $m \geqq 2$) is a counter-example to Zelinka's conjecture.

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НЕКОТОРЫЕ РЕЗУЛЬТАТЫ О ЧИСЛАХ ХАДВИГЕРА ГРАФОВ

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Резюме

В статье исследовано число Хадвигера графа. Показаны некоторые оценки для числа Хадвигера графа. Также определены числа Хадвигера полного k-дольного графа и декартова произведения двух звезд. На основании этих результатов опровергнуты две гипотезы, которые высказал Зелинка [3, 4].