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Introduction

Let I^n be the n-dimensional cube. In [2] some special factor spaces of the cube I^n called s-cubes were introduced and a necessary and sufficient condition (called the property "M") was given for an s-cube X to be a manifold.

In the present paper the full topological classification of those s-cubes of dimension $n \leqq 3$, which are manifolds, is given.

1. Notation and basic definitions

Let $n \geqq 1$ be an integer. According to [2] we shall use the following notation:
$N_n = \{1, 2, \dots, n\}$
$I^n = \{x \in R^n \colon |x_i| \leqq 1,\ i \in N_n\}$ is the n-dimensional cube
∂I^n is the boundary of I^n
$B^n = \{x \in R^n ; \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \leqq 1\}$ is the n-dimensional ball
$S^n = \partial B^{n+1}$ is the n-dimensional sphere, $n \geqq 0$
$J_i^n = \{x \in \partial I^n ; |x_i| = 1\}$ is the i-th double face of I^n
$s_i \colon I^n \to I^n,\ x \mapsto (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n)$ is the symmetry of I^n with respect to the hyperplane $x_i = 0$, $i \in N_n$.

Let G be a subgroup of the group of all tranformations of I^n generated by the set $\{s_i ; i \in N_n\}$. Since $s_i \circ s_j = s_j \circ s_i$ for every $i, j \in N_n$, the group G is abelian and $G \cong Z_2^n$. Each $s \in G$, $s \neq \mathrm{id}$, is a product of mutually different transformations $s_{i_1}, \dots, s_{i_k}$ and it can be uniquely written in the form

$$s_{i_1 i_2 \dots i_k} = s_{i_1} \circ s_{i_2} \circ \dots \circ s_{i_k}, \text{ where } i_1 < i_2 < \dots < i_k.$$

Further, the map $\tau_n \colon G \to 2^{N_n}$, $\tau_n(s_{i_1 i_2 \dots i_k}) = \{i_1, i_2, \dots, i_k\}$, $\tau_n(\mathrm{id}) = \emptyset$, is a bijection.

Definition 1.1. *Let $u^1, \dots, u^n \in G$. An s-cube $X = I^n/(u^1, \dots, u^n)$ is a factor space I^n/T, where T is the equivalence relation on I^n defined as follows:*

$x\,T\,y \Leftrightarrow x = y$ or there are integers $i_1, \dots, i_k \in N_n$ such that $x, y \in \bigcap_{j=1}^{k} J_{i_j}^n$ and $x = u^{i_1} \circ u^{i_2} \circ \dots \circ u^{i_k}(y)$.

The integer n is called the *dimension* of the s-cube X. The s-cube X can be alternately written in the form $X = I^n/(U_1, \ldots, U_n)$, where $U_i = \tau_n(u^i)$, $i \in N_n$.

Definition 1.2. *An s-cube $X = I^n/(u^1, \ldots, u^n)$ is called regular if for every $i, j \in N_n u^i = s_j$ implies $u^j = s_j$. Regular cubes are called briefly r-cubes.*

Definition 1.3. *An r-cube $X = I^n/(u^1, \ldots, u^n)$ has the property "M" if for each nonempty subset $P \subset N_n$ such that*

i) $\forall i, j \in P$: $i \neq j \Rightarrow u^i \neq u^j$,

ii) $\forall i \in P$: card $\tau_n(u^i) \neq 1$

we have

$$P \cap \tau_n \left(\prod_{j \in P} u^j \right) \neq \emptyset$$

According to [2], Proposition 2.10, every s-cube is homeomorphic to some r-cube. Further, an r-cube is a manifold if and only if it has the property "M" ([2], Theorem 3.18).

Definition 1.4. *An r-cube $I^n/(U_1, \ldots, U_n)$ is called cube-fibrable (briefly c-fibrable) if there is a set Q, $\emptyset \subsetneqq Q \subsetneqq N_n$, such that*

i) $Q \cap \left(\bigcup_{j \in N_n - Q} U_j \right) = \emptyset$,

ii) if $U_i = U_j$ for some $i, j \in N_n$, then $i, j \in Q$ or $i, j \in N_n - Q$. An r-cube which is not c-fibrable is called c-nonfibrable.

2. Homeomorphism Theorem

Let X be a topological space, f: $X \to X$ a homeomorphism. The symbol $X \times I/E_f$ will always denote a quotient space of $X \times I$ which arises by the identification of the pairs $(x, -1)$, $(f(x), 1)$, $x \in X$, in the space $X \times I$.

Lemma 2.1. *Let X be a topological space and let f, g: $X \to X$ be isotopic homeomorphisms. Then $X \times I/E_f \approx X \times I/E_g$.*

Proof. Let H: $\langle 0, 1 \rangle \times X \to X$ be an isotopy such that $H_0 = f$ and $H_1 = g$. Denote

$$F\colon\ X \times I \to X \times I,\ (x, t) \mapsto (g \circ H^{-1}_{\frac{1-t}{2}}(x), t)$$

$$G\colon\ X \times I \to X \times I,\ (x, t) \mapsto (H_{\frac{1-t}{2}} \circ g^{-1}(x), t).$$

One easily verifies that F, G are homeomorphisms inverse to each other and compatible with the equivalences E_f and E_g. This clearly implies that F and G induce a homeomorphism between the spaces $X \times I/E_f$ and $X \times I/E_g$.

Lemma 2.2. *Let $X = I^n/(u^1, \ldots, u^n)$ be an r-cube, $u \in G$. Then the homeomorphism u: $I^n \to I^n$ induces a map $\tilde{u}$: $X \to X$, $[x] \mapsto [u(x)]$ which is a homeomorphism.*

Proof. Let $X = I^n/T$. Making use of Definition 1.1 it is not difficult to prove that

$$x\ T\ y \Leftrightarrow u(x)\ Tu(y)$$

for every $x, y \in I^n$. Since the map u is a homeomorphism, the map $\tilde{u}$ is a homeomorphism.

Now we are going to prove that in some special cases an n-dimensional r-cube X can be represented as a space $Y \times I/E_f$, where Y is an $(n-1)$-dimensional r-cube and $f\colon\ Y \to Y$ is a homeomorphism.

Lemma 2.3. *Let $X = I^n/(U_1, \ldots, U_n)$ be an r-cube such that $n \in U_n$, $n \notin U_i$ for $i \in N_{n-1}$. Denote $Y = I^{n-1}/(U_1, \ldots, U_{n-1})$, $f = \tau_{n-1}^{-1}(U_n - \{n\})$, $f\colon I^{n-1} \to I^{n-1}$. Let $\tilde{f}\colon\ Y \to Y$ be the map induced by f. Then $X \approx Y \times I/E_{\tilde{f}}$.*

Proof: Let $X = I^n/T$. We prove that for every $x, y \in I^n$ we have

$$x\ T\ y \Leftrightarrow ([(x_1, \ldots, x_{n-1})], x_n)\, E_{\tilde{f}}([(y_1, \ldots, y_{n-1})], y_n) \tag{2}$$

We shall discus two cases:

a) $x, y \notin J_n^n$, b) $x, y \in J_n^n$.

In the case a) and in the case b) for $x_n = y_n$ the condition (2) is satisfied. Now we prove (2) in the case b) for $x_n = y_n$. Denote $\bar{x} = (x_1, \ldots, x_{n-1})$, $\bar{y} = (y_1, \ldots, y_{n-1})$.

Let $x\ T\ y$. Then there are integers $i_1, \ldots, i_k \in N_n$, $i_1 < i_2 < \ldots < i_k = n$ such that

$$x, y \in \bigcap_{j=1}^{k} J_{i_j}^n,\ x = u^{i_1} \circ u^{i_2} \circ \ldots \circ u^{i_k}(y).$$

Since $\tau_n(u^n \circ s_n) \cap \{n\} = \emptyset$, we have

$$\bar{x} = u^{i_1} \circ \ldots \circ u^{i_{k-1}} \circ (u^n \circ s_n)(\bar{y}) = u^{i_1} \circ \ldots \circ u^{i_{k-1}} \circ f(\bar{y}).$$

Hence $f(\bar{x}) = u^{i_1} \circ \ldots \circ u^{i_{k-1}}(\bar{y})$, because G is commutative and $s^2 = \mathrm{id}$ for every $s \in G$. Then we have $[f(\bar{x})] = [\bar{y}]$, $\tilde{f}[\bar{x}] = [\bar{y}]$ and finally $([\bar{x}], x_n)\, E_{\tilde{f}}([\bar{y}], y_n)$.

Let now $([\bar{x}], x_n)\, E_{\tilde{f}}([\bar{y}], y_n)$. Since $x_n \neq y_n$, we can suppose $\tilde{f}[\bar{x}] = \bar{y}$. Then $[f(\bar{x})] = [\bar{y}]$ and there are integers $i_1, \ldots, i_k \in N_{n-1}$ such that $f(\bar{x}), \bar{y} \in \bigcap_{j=1}^{k} J_{i_j}^{n-1}$ and $f(\bar{x}) = u^{i_1} \circ \ldots \circ u^{i_k}(\bar{y})$. Then

$$\bar{x} = u^{i_1} \circ \ldots \circ u^{i_k} \circ f(\bar{y}) = u^{i_1} \circ \ldots \circ u^{i_k} \circ (u^n \circ s_n)(\bar{y}).$$

Since $|x_n| = |y_n| = 1$, $x_n \neq y_n$, we have $x, y \in J_n^n \cap \left(\bigcap_{j=1}^{k} J_{i_j}^n\right)$ and $x = u^{i_1} \circ \ldots \circ u^{i_k} \circ u^n(y)$. Hence $x\ T\ y$.

Homeomorphism Theorem. *Let $X_U = I^n/(U_1, \ldots, U_n)$, $X_V = I^n/(V_1, \ldots, V_n)$ be such r-cubes that $n \in U_n \cap V_n$ and for every $i \in N_{n-1}$ there is $U_i = V_i$ and $n \notin U_i$. Let $f_U, f_V\colon\ I^{n-1} \to I^{n-1}$, $f_U = \tau_{n-1}^{-1}(U_n - \{n\})$, $f_V = \tau_{n-1}^{-1}(V_n - \{n\})$ and let $\tilde{f}_U, \tilde{f}_V\colon\ I^{n-1}/$*

$/(U_1, \ldots, U_{n-1}) \to I^{n-1}/(U_1, \ldots, U_{n-1})$ *be the isotopic homeomorphisms induced by* f_U, f_V. *Then* $X_U \approx X_V$.

Proof. Let $Y = I^{n-1}/(U_1, \ldots, U_{n-1})$. With regard to Lemma 2.3 we get $X_U \approx Y \times I/E_{f_U}$, $X_V \approx Y \times I/E_{f_V}$. Then by Lemma 2.1 we have $X_U \approx X_V$.

3. Classification in dimension 1 and 2

In the classification we can limit ourselves only to r-cubes because every s-cube is homeomorphic to some r-cube.

There is only one r-cube with the property "M" in dimension 1, it is the r-cube $I/(s_1) \approx S^1$.

Let $I^2/(u^1, u^2)$ be the 2-dimensional r-cube with the property "M". There are only three possibilities for u^1, u^2; namely s_1, s_2, s_{12}, and only 6 possibilities for X: $I^2/(s_1, s_1)$, $I^2/(s_2, s_2)$, $I^2/(s_1, s_2)$, $I^2/(s_1, s_{12})$, $I^2/(s_{12}, s_2)$, $I^2/(s_{12}, s_{12})$. Making use of [2], Proposition 1.3, we obtain

$$I^2/(s_1, s_1) \approx I^2/(s_2, s_2), \quad I^2/(s_1, s_{12}) \approx I^2/(s_{12}, s_2).$$

Let

$$X_1 = I^2/(s_1, s_1), \quad X_2 = I^2/(s_1, s_2), \quad X_3 = I^2/(s_1, s_{12}),$$

$$X_4 = I^2/(s_{12}, s_{12}). \tag{3}$$

It is not difficult to see that

$$X_1 \approx S^2, \quad X_2 \approx S^1 \times S^1, \quad X_3 \approx Kb, \quad X_4 \approx RP^2 \tag{4}$$

where Kb is the Klein bottle and RP^2 is the real projective plane.

Classification Theorem A. *Let* X *be the n-dimensional s-cube which is a manifold.*

1) *If* $n = 1$, *then* $X \approx I/(s_1)$.

2) *If* $n = 2$, *then* X *is homeomorphic to one of the r-cubes* $X_1, \ldots, X_4$ (*see* (3), (4)). *The r-cubes* $X_1, \ldots, X_4$ *are mutually nonhomeomorphic.*

4. Classification in dimension 3

Let $X = I^3/(U_1, U_2, U_3)$ be an r-cube with the property "M". In the case when X is c-nonfibrable, there is $X \approx X_1 = I^3/(s_1, s_1, s_1)$ or $X \approx X_2 = I^3/(s_{123}, s_{123}, s_{123})$, see [2], Proposition 3.13.

Now let us suppose that X is c-fibrable and consider the following two cases:

I. If X is c-fibrable with regard to a subset $Q \subset N_3$, then card $Q = 2$.

II. X is c-fibrable with regard to a subset $Q \subset N_3$ with card $Q = 1$.

First we shall discuss the case I, supposing without loss of generality

$$\text{card } U_1 \leqq \text{card } U_2 \leqq \text{card } U_3. \tag{5}$$

Our assumptions imply that X can be c-fibrable only with regard to $Q=\{1, 2\}$, $\{1, 3\}$ or $\{2, 3\}$. If there were $Q=\{1, 2\}$, there would be, by Definition 1.4, $U_3=\{3\}$, and therefore, by (5), card $U_1=$card U_2. This and the definitions 1.2—1.4 would, however, imply $3 \notin U_1 \cup U_2$, and thus X would be c-fibrable with regard to $Q'=\{3\}$, which would be contrary to our assumption. Similarly for $Q=\{1, 3\}$ we would obtain a contrary by showing X to be c-fibrable with regard to $Q'=\{3\}$ or $\{2\}$. Hence X is c-fibrable with regard to $Q=\{2, 3\}$ and, clearly, $U_1=\{1\}$.

Lemma 4.1. *Under the assumptions* I *and* (5) *we have card* $U_2>1$ *and card* $U_3>1$.

Proof. Supposing card $U_2=$card $U_3=1$ we obtain that X is c-fibrable with regard to $Q'=\{1\}$. Similarly the assumption card $U_2=1$ and card $U_3>1$ yields that X is c-fibrable with regard to $Q'=\{3\}$.

Lemma 4.2. *Under the assumptions* I *and* (5) *we have* $X=I^3/(s_1, s_{123}, s_{123})$.

Proof. By virtue of Lemma 4.1 we only need to show that there can be neither card $U_2=$card $U_3=2$ nor $2=$card $U_2<$card $U_3=3$. This is, however easily done by considering all the possibilities and showing that each of them leads to a contrary either to the assumption I or to the property "M".

Now we shall continue with the case II. With regard to [2], Proposition 1.3, we can take $Q=\{3\}$. Since X has the property "M", $Y=I^2/(U_1, U_2)$ is the 2-dimensional r-cube with the property "M" ([2], Lemma 3.16). Hence there are only four possibilities for the r-cube Y, namely $I^2/(s_1, s_1)$, $I^2/(s_1, s_2)$, $I^2/(s_1, s_{12})$, $I^2/(s_{12}, s_{12})$.

Proposition 4.3. *In the case* II *the r-cube X is homeomorphic to one of the following r-cubes:* $X_4=I^3/(s_1, s_1, s_3)$, $X_5=I^3/(s_1, s_1, s_{13})$, $X_6=I^3/(s_1, s_2, s_3)$, $X_7=I^3/(s_1, s_2, s_{13})$, $X_8=I^3/(s_1, s_2, s_{123})$, $X_9=I^3/(s_1, s_{12}, s_{23})$, $X_{10}=I^3/(s_{12}, s_{12}, s_3)$.

To prove Proposition 4.3, we shall need some lemmas.

Lemma 4.4. *a)* $I^3/(s_1, s_1, s_3)\approx I^3/(s_1, s_1, s_{123})$, *b)* $I^3/(s_1, s_1, s_{13})\approx I^3/(s_1, s_1, s_{23})$.

Proof. Let $X_U=I^3/(s_1, s_1, s_3)$, $X_V=I^3/(s_1, s_1, s_{123})$. Making use of the Homeomorphism Theorem it is sufficient to prove that the maps $\tilde{f}_U$, $\tilde{f}_V$, induced by the maps $f_U=\mathrm{id}$, $f_V=s_{12}$, are isotopic. It is easy to see that identifying $I^2/(s_1, s_1)$ with S^2 in a suitable way, we can view $\tilde{f}_U$, $\tilde{f}_V$ as the homeomorphisms $\bar{f}_U$, $\bar{f}_V$: $S^2 \to S^2$ defined by $\bar{f}_U(x)=x$, $\bar{f}_V(x)=(-x_1, -x_2, x_3)$. These homeomorphisms are, however, wellknown to be isotopic. The assertion b) is proved in a similar way.

Lemma 4.5. *a)* $I^3/(s_1, s_2, s_{13})\approx I^3/(s_1, s_2, s_{23})$,
b) $I^3/(s_1, s_2, s_{13})\approx I^3/(s_1, s_{12}, s_3)$.

Proof. In [2], Proposition 1.3, it is sufficient to take f: $N_3 \to N_3$, $f(1)=2$, $f(2)=1$, $f(3)=3$ in the case a) and $f(1)=1$, $f(2)=3$, $f(3)=2$ in the case b).

Lemma 4.6. *a)* $I^3/(s_1, s_{12}, s_3)\approx I^3/(s_1, s_{12}, s_{13})$,
b) $I^3/(s_1, s_{12}, s_{23})\approx I^3/(s_1, s_{12}, s_{123})$.

Proof. a) We shall use the Homeomorphism Theorem. Let $X_U = I^3/(s_1, s_{12}, s_3)$, $X_V = I^3/(s_1, s_{12}, s_{13})$. We prove that the maps $\tilde{f}_U, \tilde{f}_V$: $I^2/(s_1, s_{12}) \to I^2/(s_1, s_{12})$ are isotopic. Let

$$H: \langle 0, 1\rangle \times I^2/(s_1, s_{12}) \to I^2/(s_1, s_{12})$$

$$H_t[(x_1, x_2)] = \begin{cases} [(x_1, x_2+2t)] & \text{if} \quad 1-x_2 \geqq 2t \\ [-(x_1, 2t-2+x_2)] & \text{if} \quad 1-x_2 \leqq 2t \end{cases}$$

We see that for every $t \in \langle 0, 1\rangle$ H_t is a homeomorphism and $H_0 = \tilde{f}_U = \widetilde{\mathrm{id}}$, $H_1 = \tilde{f}_V = \tilde{s}_1$.

b) It is sufficient to apply [2], Proposition 3.7, for $k=2$.

Lemma 4.7. *a)* $I^3/(s_{12}, s_{12}, s_3) \approx I^3/(s_{12}, s_{12}, s_{123})$,

b) $I^3/(s_{12}, s_{12}, s_{23}) \approx I^3/(s_{12}, s_{12}, s_{123})$,

c) $I^3/(s_{12}, s_{12}, s_{13}) \approx I^3/(s_{12}, s_{12}, s_{123})$.

Proof. Let $X_U = I^3/(s_{12}, s_{12}, s_3)$, $X_V = I^3/(s_{12}, s_{12}, s_{123})$. We shall use the Homeomorphism Theorem. We prove that the maps $\tilde{f}_U = \widetilde{\mathrm{id}}$, $\tilde{f}_V = \tilde{s}_{12}$, $\tilde{f}_U$, $\tilde{f}_V$: $I^2/(s_{12}, s_{12}) \to I^2/(s_{12}, s_{12})$ are isotopic. By suitable identification of the spaces $I^2/(s_{12}, s_{12})$ and B^2/Ω (Ω identifies the antipodal points on ∂B^2) we can view $\tilde{f}_U, \tilde{f}_V$ as the homeomorphisms $\tilde{f}_U, \tilde{f}_V$: $B^2/\Omega \to B^2/\Omega$, $\tilde{f}_U[(x, y)] = [(x, y)]$, $\tilde{f}_V[(x, y)] = [(-x, -y)]$. It is not difficult to see that the homeomorphisms $\tilde{f}_U, \tilde{f}_V$ are isotopic. To prove assertions b), c) it is sufficient to take k=2, 1 in [2], Proposition 3.7.

Proof of Proposition 4.3. Since $3 \in U_3$, we have only four possibilities for U_3, namely $\{3\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$. Then for $U_1 = U_2 = \{1\}$ we have $X \approx X_4$ or $X \approx X_5$ by Lemma 4.4, for $U_1 = \{1\}$, $U_2 = \{2\}$ we have $X \approx X_6$ or $X \approx X_7$ or $X \approx X_8$ by Lemma 4.5, for $U_1 = \{1\}$, $U_2 = \{1, 2\}$ we have $X \approx X_7$ or $X \approx X_9$ by Lemma 4.5 and Lemma 4.6 and finally for $U_1 = U_2 = \{1, 2\}$ we have $X \approx X_{10}$ by Lemma 4.7.

It was proved in [1] that on any given s-cube X it is possible to introduce a structure of a CW space. In the case when the s-cube X is a manifold, one can sometimes define a CW decomposition of X with a smaller number of cells than in the general case (see [3]).

Let X be an r-cube from the set $X_1, \ldots, X_{10}$. By standard computation making use of the CW decomposition of X introduced in [1] or [3] one can compute the following table of homology groups (over Z) of the r-cubes $X_1, \ldots, X_{10}$.

With regard to the classification procedure, Lemma 4.2, Proposition 4.3 and Table 1 we get

Classification Theorem B. *Let X be a 3-dimensional s-cube which is a manifold. Then X is homeomorphic to one of the r-cubes $X_1, \ldots, X_{10}$ listed in Table 2. The r-cubes $X_1, \ldots, X_{10}$ are mutually nonhomeomorphic.*

Table 1

X	$H_n(X)$ $n>3$	$H_3(X)$	$H_2(X)$	$H_1(X)$	$H_0(X)$
$X_1 \approx S^3$	0	Z	0	0	Z
$X_2 \approx RP^3$	0	Z	0	Z_2	Z
X_3	0	Z	0	Z_2^2	Z
$X_4 \approx S^2 \times S^1$	0	Z	Z	Z	Z
X_5	0	0	Z_2	Z	Z
$X_6 \approx S^1 \times S^1 \times S^1$	0	Z	Z^3	Z^3	Z
$X_7 \approx Kb \times S^1$	0	0	$Z + Z_2$	$Z^2 + Z_2$	Z
X_8	0	Z	Z	$Z + Z_2^2$	Z
X_9	0	0	Z_2	$Z + Z_2^2$	Z
$X_{10} \approx RP^2 \times S^1$	0	0	Z_2	$Z + Z_2$	Z

Table 2

$X_1 = I^3/(s_1, s_1, s_1)$	$X_6 = I^3/(s_1, s_2, s_3)$
$X_2 = I^3/(s_{123}, s_{123}, s_{123})$	$X_7 = I^3/(s_1, s_2, s_{13})$
$X_3 = I^3/(s_1, s_{123}, s_{123})$	$X_8 = I^3/(s_1, s_2, s_{123})$
$X_4 = I^3/(s_1, s_1, s_3)$	$X_9 = I^3/(s_1, s_{12}, s_{23})$
$X_5 = I^3/(s_1, s_1, s_{13})$	$X_{10} = I^3/(s_{12}, s_{12}, s_3)$

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КЛАССИФИКАЦИЯ s-КУБОВ РАЗМЕРНОСТИ $n \leqq 3$

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Резюме

В статье дана полная топологическая классификация тех s-кубов размерности $n \leqq 3$, которые являются многообразиями.