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MINIMUM MAXIMAL GRAPHS WITH FORBIDDEN SUBGRAPHS

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0. Abstract

Extremal graph theory was begun by Paul Turán when he determined the maximum size of an extremal graph G not containing a prescribed complete graph K_n. Paul Erdős and colleagues later found the minimum size of a maximal G not containing K_n. The corresponding questions are studied here when the forbidden subgraph F is a path or a star or a matching.

1. Introduction

Given graphs F and G, we say that G is *F-maximal* if $F \not\subset G$ but $F \subset G + e$ for each line e of $\bar{G}$. The original extremal result of Turán [11] gave for each $n \geq 3$ an exact formula for the maximum *size* (number of lines) $t(K_n, p)$ in a K_n-maximal graph G of given *order* (number of points) p. Erdős, Hajnal and Moon [3] developed a corresponding formula for the size $b(K_n, p)$ of a minimum K_n-maximal graph G. The most celebrated problem area of extremal graph theory has been to extend Turán's formula to other forbidden subgraphs F, by determining the value of $t(F, p)$. The author of the definitive treatise [1] on extremal graph theory, Béla Bolobás (verbal communication) asserted to one of us that even for the simple (in appearance only) case when $F = C_4$, the quadrilateral, the problem is utterly intractable.

Since the general problem is so difficult, we consider for the forbidden subgraph F three simple families of forests: the stars $K_{1,n}$, the matchings nK_2 and the paths P_n. For each of these families we investigate not only the Turán problem of determining the maximum size $t(F, p)$ of a maximal graph not containing F, but also the corresponding minimum size $b(F, p)$ as introduced in [5]. Values of $b(K_{1,n}, p)$, $b(nK_2, p)$ and $b(P_n, p)$ are obtained when p is sufficiently large with respect to n. In general, determining $b(F, p)$ seems to be a much more tractable problem than finding $t(F, p)$.

The notation of [4] is followed, in particular $\Delta(G)$ is the maximum degree and $\langle U\rangle$ is the subgraph induced by $U \subset V(G)$. Also $p(G)$ is the order and $q(G)$ the size of the graph.

2. Extremal numbers for forbidden stars

Determining $t(K_{1,n}, p)$ and $b(K_{1,n}, p)$ are the two easiest extremal problems of this type. Note that if G is a $K_{1,n}$-maximal graph, then $\Delta(G) \leq n-1$ but $\Delta(G+e)=n$ for any line e in $\bar{G}$. For the remainder of this section, we assume that $p \geq n+1$; if not, then obviously $t(K_{1,n}, p) = b(K_{1,n}, p) = p(p-1)/2$ and the unique extremal graph is K_p.

Theorem 1. *Let ζ be 1 if both p and $n-1$ are odd, and 0 otherwise, so $\zeta = p(n-1)$ mod 2). Then*

$$t(K_{1,n}, p) = ((n-1)p-\zeta)/2.$$

Proof. Let G be a graph of order p and size $q > ((n-1)p-\zeta)/2$. Then the average degree of the points of G is at least $((n-1)p-\zeta+2)/p > n-1$, so that $\Delta(G) \geq n$, i.e., $K_{1,n} \subset G$. Therefore, $t(K_{1,n}, p) \leq ((n-1)p-\zeta)/2$.

To see that $t(K_{1,n}, p) \geq ((n-1)p-\zeta)/2$, we merely note [4, p. 89] that K_p is the sum of $(p-1)/2$ spanning cycles for p odd and the sum of a 1-factor and $(p-2)/2$ spanning cycles for p even. Using these facts, it is trivial to construct a graph G of order p and size $((n-1)p-\zeta)/2$ such that $\Delta(G)=n-1$.

Given n and p, let $\mathcal{M}(n, p)$ be the family of $K_{1,n}$-maximal graphs of order p and size $t(K_{1,n}, p)$. Obviously all points of any graph in $\mathcal{M}(n, p)$ have degree $n-1$ unless both p and $n-1$ are odd, in which case there is a unique point w with degree $n-2$. Somewhat surprisingly, we shall use these graphs to obtain the minimum $K_{1,n}$-maximal graphs.

Theorem 2. *Let σ equal one if both $n-1$ and $p \dot{-} n/2$ are odd integers, and zero otherwise, so that $\sigma = (n-1)(p-n/2)$ (mod 2). Then if $p \geq \lfloor 3n/2 \rfloor$,*

$$b(K_{1,n}, p) = (p(n-1) - \lfloor n/2 \rfloor + \lfloor n/2 \rfloor^2 + \sigma)/2.$$

Proof. Let G be a minimum $K_{1,n}$-maximal graph of order p and as usual let $V(G) = \{v_1, \ldots, v_p\}$. Let r be the number of points of G with degree less than $n-1$, and assume without loss of generality that these points are $v_1, \ldots, v_r$ (we may regard $r > 0$, for otherwise G has size $t(K_{1,n}, p)$, the maximum size among all maximal graphs). Because G is maximal, its induced subgraph $\langle v_1, \ldots, v_r\rangle$ is K_r. Also, deg $(v_i) = n-1$ for $r+1 \leq i \leq p$, and since G is a minimum maximal graph, this implies that $\langle v_{r+1}, \ldots, v_p\rangle \in \mathcal{M}(n, p-r)$. If both $n-1$ and $p-r$ are odd, then $\langle v_{r+1}, \ldots, v_p\rangle$ contains just one point, say v_{r+1}, with degree $n-2$, so that G has a line joining v_{r+1} and v_i for some $i \leq r$. If they are not both odd, then G has no lines between point sets $\{v_1, \ldots, v_r\}$ and $\{v_{r+1}, \ldots, v_p\}$.

Consequently, G has size

$$q = (r(r-1) + (p-r)(n-1) + \sigma)/2. \tag{1}$$

If the σ term is disregarded, equation (1) is a quadratic in the variable r, and then it follows that its minimum among integers r is attained at both $r = \lfloor n/2 \rfloor$ and $\lfloor n/2 \rfloor$. However, $\sigma = 1$ only if n is even, in which case $r = n/2$ still gives a minimum for (1), since it previously was the unique minimum. Therefore, $r = \lfloor n/2 \rfloor$ always yields a minimum for q (in (1), giving the value in the statement of the theorem.

3. Extremal values for forbidden matchings

The evaluation of $t(nK_2, p)$ is not very difficult. In fact, Simonovits [10] determined the maximum nK_s-maximal graphs for any positive integer s, after Erdős [2] and Moon [8] obtained partial solutions. We state the result only for the case $s = 2$.

Theorem A [10]. *If $2n \leq p$, then the unique maximum nK_2-maximal graph of order p is $K_{n-1} + \bar{K}_{p-n+1}$.*

The value of $t(nK_2, p)$ now follows immediately.

Corollary. If $2n \leq p$, then $t(nK_2, p) = (n-1)(2p-n)/2$.

The value of $b(nK_2, p)$ is also easy to find when p is large enough relative to n.

Theorem 3. *If $p \geq 3(n-1)$ then $b(nK_2, p) = 3(n-1)$.*

Proof. Obviously $(n-1)K_3 \cup (p-3n+3)K_1$ is nK_2-maximal, so $b(nK_2, p) \leq 3(n-1)$.

Now let G be an nK_2-maximal graph of order p, and let $S = \{u_1v_1, \ldots, u_{n-1}v_{n-1}\}$ be a set of $n-1$ independent lines in G. Let $G_i = G - [V(S) - u_i - v_i]$ for $i = 1, \ldots, n-1$. Because G is nK_2-maximal, the lines of G_i must form either a triangle or a star, so that $q(G_i) = 3$ or $p - 2n + 3$ for each i. Since $E(G_j) \cap E(G_i) = \emptyset$ for $j \neq i$, it follows that $q(G) \geq 3(n-1)$.

In fact, the nK_2-maximal graphs have been characterized by Mader.

Theorem B [7]. *The nK_2-maximal graphs are the join graphs $K_{k_0} + (K_{k_1} \cup K_{k_2} \cup \ldots \cup K_{k_t})$ where $k_0 \geq 0$, $t \geq k_0 + 2$, every k_j is odd for $j \geqq 1$, and $n - 1 = k_0 + \sum_{i=1}^{t} (k_i - 1)/2$.*

Note that $k_0 = 0$ gives the "null graph" K_0 which was intensively investigated in [6].

If $p < 3(n-1)$, then the formula given in Theorem 3 for $b(nK_{2,p})$ is not valid. However, the method of the proof of Theorem 3 can be used to show that the unique minimum nK_2-maximal graph G has the form of Theorem B with $k_0 = 0$ and $|k_i - k_j| \leq 2$ for $1 \leq i, j \leq t$, so that G is a union of complete graphs.

4. Extremal numbers for forbidden paths

We turn now to the problem of computing extremal values with a path as the forbidden subgraph. By way of comparison, $b(C_4, p)$ has been determined by Ollman [9], but $b(C_n, p)$ is unknown for $n \geq 5$. Also, as mentioned in the introduction, determining $t(C_n, p)$ is an extremely difficult problem even for $n = 4$.

As might be expected, evaluating $t(P_n, p)$ also appears to be quite hard and no major results are known. However, it is possible to compute $b(P_n, p)$ for values of p that are 'large enough' with respect to n. First we require some preliminary results on maximal trees, which are developed in four lemmas.

Lemma 4.1. *If T is a P_n-maximal tree of order $p > 3$, then T has no points of degree 2 (is homeomorphically irreducible).*

Proof. Suppose v is a point of T of degree 2 with neighbors u, w. Then the line uw is not in T, so $T + uw$ contains a path P^* of order n and uw is in P^*. Now P^* must also contain either vu or vw, for otherwise removing uw from P^* and adding vu and vw to it will yield a path on $n+1$ points in T. So, suppose without loss of generality that line uv is in P^*. Since v has degree 2 in T, it is an endpoint of P^*, which must now have the form $v, u, w, x_4, \ldots, x_n$. But then $P' - u, v, w, x_4, \ldots, x_n$ is an n-point path in T, a contradiction.

Lemma 4.2. *If T is a P_n-maximal tree of order $p \geqq 3$, then $T \supset O_{n\ 1}$.*

Proof. Let uv, vw be adjacent lines in T. Then $T + uw$ contains P_n by the maximality of T. But it is easily verified that the maximum length of a path in $T + uw$ is at most one more than its length in T, so that $T \supset P_{n\ 1}$.

For $n \geqq 4$, let $f(n)$ be the minimum order of a P_n-maximal tree T other than the trivial cases K_1 and K_2. Also, we define $f(3) = 2$ and $f(2) = 1$. It will soon be shown that $f(n)$ is finite.

Lemma 4.3. *If $n \geq 4$, then $f(n) \geq 2f(n-2) + 2$.*

Proof. Let T be a P_n-maximal tree of order $f(n)$, $n \geq 4$. Let T' be the graph obtained from T by deleting all its endpoints. Since T has at least three points, diam $(T') =$ diam $(T) - 2$, so that $P_{n-2} \not\subset T'$. Obviously any line e in $\bar{T}'$ is also in $\bar{T}$. Since the unicyclic graph $T + e$ contains P_n by the maximality of T, it follows that $P_{n-2} \subset T' + e$ because by Lemma 4.1, a longest path in $T + e$ begins and ends at endpoints of T. Hence T' is a P_{n-2}-maximal tree.

By Lemma 4.2, T' can be K_1 or K_2 only if n is 4 or 5, so

$$p(T') \geq f(n-2). \tag{4.1}$$

On the other hand, recall that $p(T) = f(n)$. By Lemma 4.1, no points of T have degree 2 and since $q(T)=f(n)-1$, it follows that T has at least $1+f(n)/2$ endpoints. Therefore by the construction of T',

$$p(T') \leq f(n) - (1 + f(n)/2). \tag{4.2}$$

Combining inequalities (4.1) and (4.2) then yields the desired result. □

Now define recursively a family of trees T_n by setting $T_2 = K_1$, $T_3 = K_2$, $T_4 = K_{1,3}$ and for $n > 4$, let T_n be the tree obtained from T_{n-2} by adding to each endpoint v of T_{n-2}, two new endpoints adjacent only to v, as illustrated in Figure 1.

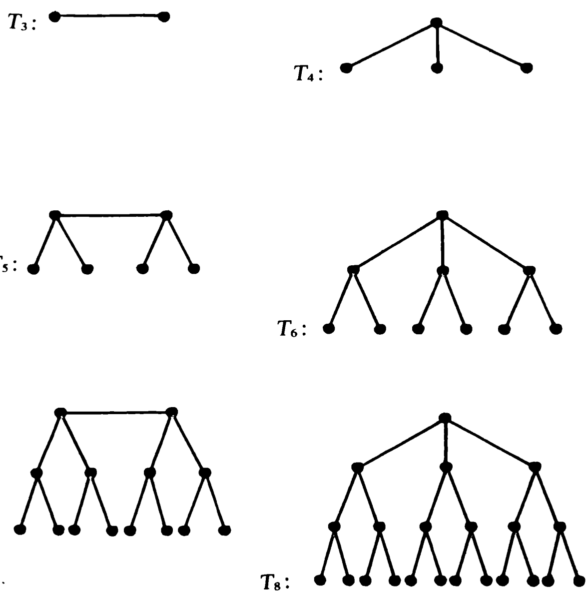

Fig. 1. The maximal graphs $T_3, \ldots, T_8$

It is straightforward to verify that T_n is P_n-maximal. Moreover, $p(T_n) = 2p(T_{n-2}) + 2$ for $n \geq 4$, so that by Lemma 4.3, T_n is a minimum order P_n-maximal graph and $f(n) = p(T_n)$ for $n \geqq 2$. A simple counting argument then shows that

$$f(n) = \begin{cases} 3 \cdot 2^{(n-2)/2} - 2 & \text{for } n \text{ even} \\ 2^{(n+1)/2} - 2 & \text{for } n \text{ odd.} \end{cases} \tag{4.3}$$

It can also be shown that T_n is the unique P_n-maximal tree of order $f(n)$ although the tedious proof will not be included here.

Lemma 4.4. *Let u, w be endpoints of T_n, $n \geq 4$, which are both adjacent to the point v. Let T_n^r be the graph obtained from T_n by adding $r \geq 1$ new endpoints, each adjacent to v. Then T_n^r is also P_n-maximal.*

Proof. Since $p(T_n) \geq 3$, diam $(T_n^r) =$ diam (T_n) and so $P_n \not\subset T_n^r$. Given a line e in $\overline{T_n^r}$, let $\{x_1, ..., x_n\}$ be r endpoints in $T_n^r + e$, which are adjacent to v but not incident with e. Then $T_n^r - \{x_1, ..., x_n\} = T_n$ and it follows that $P_n \subset T_n^r + e$.

We are finally ready to return to the problem of determining $b(P_n, p)$. Obviously, $b(P_2, p) = 0$ and $b(P_3, p) = [p/2]$. Also $b(P_4, p) = p/2$ for p even and $(p+3)/2$ for p odd, $p \geq 3$. When p is 'large enough', we have the following result.

Theorem 4. *If $p \geq f(n)$, then*

$$b(P_n, p) = \begin{cases} p - 1 - \lfloor (p-2)/f(n) \rfloor & \text{for } n = 5 \\ p - \lfloor p/f(n) \rfloor & \text{for } n \geq 6, \end{cases}$$

where $f(n)$ is as in (4.3).

Proof. Obviously $b(P_n, p) \leq p - 1$ since by Lemma 4.4 the tree $T_n^{p, f(n)}$ is P_n-maximal. In general, if $b(P_n, p) = p - k$, then at least k of the components of any minimum size P_n-maximal graph G must be trees. If $T \neq K_1$ is any tree such that $P_n \not\subset T$, then obviously $T \cup K_1$ is not P_n-maximal, so K_1 cannot be a component of G when $k \geq 2$.

If $n = 5$, it is easy to see that $K_2 \cup T_n$ is P_n-maximal but $2K_2$ is not. It follows that k is equal to $1 + [(p-2)/f(n)]$.

If $n \geq 6$ and T is any P_n-maximal tree, then $K_2 \cup T$ is not P_n-maximal since adding a line e between a point of K_2 and a central point of T would still leave diam $(K_2 \cup T + e) = n - 2$. Thus in this case $k = [p/f(n)]$, that is, k is the maximum integer r such that $r \cdot f(n) \leq p$.

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МАКСИМАЛЬНЫЕ ГРАФЫ С ЗАПРЕЩЕННЫМИ ПОДГРАФАМИ ОБЛАДАЮЩИЕ МИНИМАЛЬНЫМ ЧИСЛОМ РЕБЕР

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Резюме

П. Туран установил максимальное число ребер графа несодержающего заданный полный граф K_n. П. Эрдёш и его сотрудники нашли минимальное число ребер максимального графа несодержающего K_n. В работе изучаются соответсвующие вопросы для случаев, когда запрещенным графом является путь, или звезда, или паросочетание.