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SOLUTION OF A PROBLEM OF M. KAZT CONCERNING THE OPTIMIZATION OF A FUNCTIONAL

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I. The Theorem

In the paper presented we give a solution to a problem of M. Katz formulated in [1].

Denote by $\mathcal{F}$ the class of functions f: $\langle 0, 1\rangle \to \langle 0, 1\rangle$ generalized in the following way: There is a subset $A(f) \subset \langle 0, 1\rangle$ of the Lebesgue measure zero such that f is singlevalued function on $\langle 0, 1\rangle$ except for $A(f)$ and f is a multi-function on the set $A(f)$. Also we use convention

$$\int_{A(f)} f(x)\,\mathrm{d}x = 0.$$

We say that $f \in \mathcal{F}$ is symmetric when the graph of f is symmetric with respect to the axis $y = x$.

For $m \in (0, 1)$ let $\mathcal{F}(m)$ be the class of all measurable functions of $\mathcal{F}$ with

$$\int_0^1 f(x)\,\mathrm{d}x = m.$$

For every $f \in \mathcal{F}(m)$ let

$$g(x) = \operatorname{meas}\{y, f(y) \geqq x\} \tag{1}$$

and

$$I_1(f) = \int_0^1 f(x) g(x)\,\mathrm{d}x$$

$$I_2(f) = \int_0^1 (f^2(x) + g^2(x))\,\mathrm{d}x$$

The problem consists in finding the supremum over $\mathcal{F}(m)$ on the functional

$$I(\alpha, f) = \alpha I_1(f) + I_2(f) \tag{2}$$

where $\alpha > 0$ [1, problem (a)]. In the present paper we construct explicitly the function solving this problem.

Theorem. *For every* $f(x) \in \mathcal{F}(m)$ *and* $\alpha > 0$ *we have*

$$I(\alpha, f) \leqq h(m, \alpha) \cdot \begin{cases} 2m - 1 + (1-m)^{3/2} & \text{if } m \leqq 1/2 \\ m^{3/2} & \text{otherwise,} \end{cases} \tag{3}$$

where

$$h(m, \alpha) = \begin{cases} 4 \dfrac{m(\alpha+1)+1}{1+4m-m^2} & \text{if } 0 < \alpha \leqq \dfrac{1-m}{1+m} \\ \alpha + 2 & \text{otherwise} \end{cases} \tag{4}$$

and the bounds in (3) are attained by the functions (5), (6):

$$u(x) = \begin{cases} 1 & 0 \leqq x < 1 - (1-m)^{1/2} \\ 1-(1-m)^{1/2} & \text{otherwise} \end{cases} \tag{5}$$

if $m \leqq 1/2$

$$v(x) = \begin{cases} m^{1/2} & 0 \leqq x < m^{1/2} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

if $m \geqq 1/2$.

As a consequence of this theorem we obtain the recent result of M. Katz for $\alpha = 2$ (see [1]). The proof of this theorem is given in a few steps.

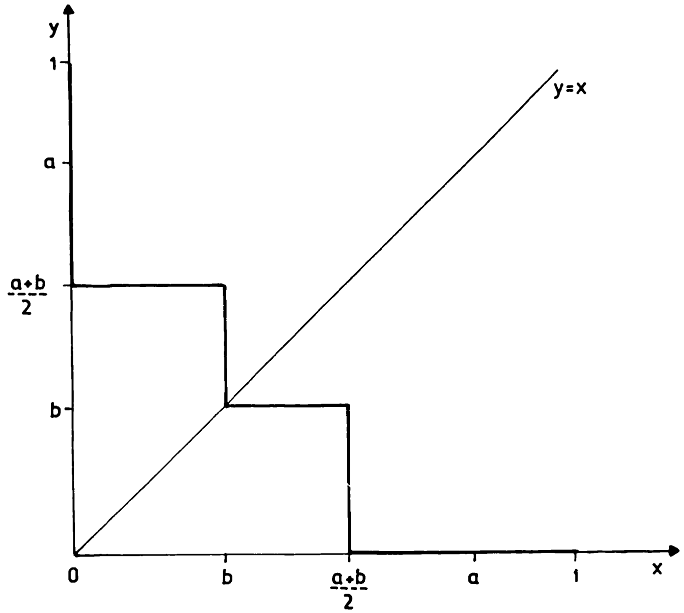

Fig. 1

Remark. Let us note that our notation differs somewhat from that of [1]. It can be easily verified that Katz's results from [1], [2] apply also for functions generalized in the above sense. So we can use them in sequel without any further remarks.

II. The Proof

Let $\mathcal{F}_s(m)$ be the set of all nonincreasing step functions in $\mathcal{F}(m)$.

For $f \in \mathcal{F}_s(m)$ we define a symmetric function $\varphi(x)$ (similarly as in [1]) as follows:

Let x_0 be the least upper bound of those $x>0$ for which $f(x)+g(x) \geqq 2x$ (clearly $x_0>0$) and put $\psi(x)=(f(x)+g(x))/2$ for $x \in \langle 0, x_0 \rangle$. Now for $x \in (0, x_0)$ define $\tilde{\varphi}(x)=\psi(x)$ if ψ is continuous at x and let

$$\begin{aligned} &\tilde{\varphi}(x)=\langle \psi(x_+), \psi(x_-) \rangle, \\ &\tilde{\varphi}(0)=\langle \psi(0_+), 1 \rangle, \\ &\tilde{\varphi}(x_0)=\langle x_0, \psi(x_{0-}) \rangle, \text{ otherwise,} \end{aligned}$$

where $\psi(x_+)$ denotes $\lim_{t \to x_+} \psi(t)$ and similarly for $\psi(x_-)$. Finally let φ be the unique symmetric function with $\varphi(x)=\tilde{\varphi}(x)$ for

$$x \in \langle 0, x_0 \rangle. \tag{7}$$

We prove now this

Lemma. *For every $f(x) \in \mathcal{F}_s(m)$ we have*

$$I(\alpha, f) < h(m, \alpha) \cdot \int_0^1 \varphi^2(x)\, \mathrm{d}x \tag{8}$$

where $\varphi(x)$, $h(m, \alpha)$ are defined by (7), (4), respectively.

Proof. We use induction relative to the number of nonzero values of $f(x) \in \mathcal{F}_s(m)$. Let $f(x)$ have exactly one nonzero value. Then

$$f(x)=\begin{cases} b & \text{if } 0 \leqq x < a \\ 0 & \text{otherwise,} \end{cases} \qquad g(x)=\begin{cases} a & \text{if } 0 \leqq x \leqq b \\ 0 & \text{otherwise.} \end{cases}$$

Let $b \leqq a$. (We shall omit the detailed analysis of the case $b>a$, since it leads to the same formulas up to (9) with b replaced by a.) According to (7) we have (see Figure 1)

$$\varphi(x)=\begin{cases} \langle (a+b)/2, 1 \rangle & x=0 \\ (a+b)/2 & 0<x<b \\ \langle b, (a+b)/2 \rangle & x=b \\ b & b<x<(a+b)/2 \\ \langle 0, b \rangle & x=(a+b)/2 \\ 0 & (a+b)/2<x \leqq 1. \end{cases}$$

Then

$$I(\alpha, f)=ab^2+a^2b+\alpha ab^2$$

$$\int_0^1 \varphi^2(x)\, \mathrm{d}x=b(a+b)^2/4+b^2(a-b)/2.$$

Denote $F(b, \alpha, m) = I(\alpha, f) \Big/ \int_0^1 \varphi^2(x)\,\mathrm{d}x$. Then we have

$$F(b, \alpha, m) = 4m \frac{(\alpha+1)b^2 + m}{m^2 + 4mb^2 - b^4} \tag{9}$$

where

$$m \leqq b \leqq m^{1/2}. \tag{10}$$

Now we find the supremum of $F(b, \alpha, m)$ over b for fixed $\alpha > 0$, $0 < m < 1$. We use the substitution $b^2 = x$. Then

$$G(x, \alpha, m) = 4m \frac{(\alpha+1)x + m}{m^2 + 4mx - x^2}, \quad m^2 \leqq x \leqq m. \tag{11}$$

By differentiation of (11) we obtain

$$G'(x, \alpha, m) = \frac{(\alpha+1)x^2 + 2mx + (\alpha-3)m^2}{(m^2 + 4mx - x^2)^2}.$$

We can see that the function (11) can have at most one stationary point in $\langle m^2, m \rangle$, namely

$$x = m[-1 + (-\alpha^2 + 2\alpha + 4)^{1/2}]/(\alpha+1)$$

and (11) attains its minimum at this point. Therefore the points of supremum can be the boundary points of the interval $\langle m^2, m \rangle$ only. If we compare the two values

$$G(m, \alpha, m) = \alpha + 2$$

$$G(m^2, \alpha, m) = 4 \frac{(\alpha+1)m + 1}{1 + 4m - m^2}$$

we can see that the point of the maximum depends on α and this dependence is expressed exactly by the function $h(m, \alpha)$ from (4). Then it is easy to verify that the lemma is true for $f(x) \in \mathscr{F}_s(m)$ with one nonzero value.

Let (8) be true for every $f \in \mathscr{F}_s(m)$ having k nonzero values. Let $f \in \mathscr{F}_s(m)$ be a function with $k+1$ nonzero values. Denote

$$\overset{b}{\underset{a}{I}}(\alpha, f) = \int_a^b [f^2(x) + \alpha f(x) g(x) + g^2(x)]\,\mathrm{d}x.$$

Then we can write

$$I(\alpha, f) = \overset{a}{\underset{0}{I}}(\alpha, f) + \overset{c}{\underset{a}{I}}(\alpha, f) + \overset{1}{\underset{c}{I}}(\alpha, f) \tag{14}$$

where a, c are the first and the last point of discontinuity of f, respectively. Then

$$\underset{a}{\overset{c}{I}}(\alpha, f) = \int_a^c [(f-a)^2 + (g-a)^2 + \alpha(f-a)(g-a)] +$$

$$+ \int_a^c (\alpha+2)a(f+g-a)$$

where $f-a$, $g-a$ on $\langle a, c\rangle$ are the functions with k — nonzero values. They have the same geometrical interpretation as f, g on $\langle 0, 1\rangle$. By the hypothesis the following inequality holds

$$\underset{a}{\overset{c}{I}}(\alpha, f) < h(m, \alpha) \cdot \int_a^c (\varphi - a)^2 + (\alpha+2)a \cdot \int_a^c (f+g-a) \cdot \tag{15}$$

If $\alpha \geqq 2(1-m)/(1+m)$, then

$$\underset{a}{\overset{c}{I}}(\alpha, f) < (\alpha+2) \int_a^c [\varphi^2 - 2a\varphi + a^2 + a(f+g) - a^2] = (\alpha+2) \int_a^c \varphi^2. \tag{16}$$

Otherwise, we have

$$4\,\frac{m(1+\alpha)+1}{1+4m-m^2} \geqq \alpha + 2$$

and

$$\underset{a}{\overset{c}{I}}(\alpha, f) < 4\,\frac{m(1+\alpha)+1}{1+4m-m^2} \cdot \int_a^c \varphi^2. \tag{17}$$

Relations (14), (16), (17) imply (8) for every $f(x) \in \mathcal{F}_s(m)$.

Corollary. For every $f(x) \in \mathcal{F}_s(m)$ the inequality (3) holds.

P r o o f. The function $\varphi(x) \in \mathcal{F}_s(m)$ is symmetric. According to [2, p. 64] the following inequality holds:

$$\int_0^1 \varphi^2(x)\,\mathrm{d}x \leqq \begin{cases} 2m - 1 + (1-m)^{3/2}, & m \leqq 1/2 \\ m^{3/2}, & m \geqq 1/2 \end{cases} \tag{18}$$

and the only functions which have attained the right- hand bounds are, respectively $u(x)$, $v(x)$ defined by (5) and (6). Then (3) is the consequence of (8) and (18).

R e m a r k. We extend result (3) to the entire set $\mathcal{F}(m)$ similarly as in [1, p. 166].

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РЕШЕНИЕ ПРОБЛЕМЫ М. КАЦА, КАСАЮШЕЙСЯ ОПТИМАЛИЗАЦИИ ФУНКЦИОНАЛА

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Резюме

Пусть $\mathscr{F}(m)$ — множество всех измеримых функций, отображающих отрезок $\langle 0,1 \rangle$ в себя. Каждой функции f ставится в соответствие функция g по формуле (1).

Проблема М. Каца состоит в следующем: найти супремум функционала (2), где α — положительное число. В работе надо решение этой проблемы, следствием которого является и результат Каца для $\alpha = 2$ (смотри [1]).