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# A NOTE ON THE EXTENSIBILITY OF STATES

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In the paper the extensibility of states from a Boolean subalgebra of a logic to the logic is treated.

## 1. Notation and known results

Let $(L, \leq)$ be a partially ordered set *(poset)* with the least element 0 and the greatest element 1. An orthocomplementation on $L$ is a mapping $a \mapsto a^\perp$ on $L$ such that (i) $(a^\perp)^\perp = a$, (iii) $a \vee a^\perp$ exists and is equal to 1, and (iii) $a \leq b$ if and only if $b^\perp \leq a^\perp$. A poset admitting an orthocomplementation is called orthocomplemented. A pair $a, b \in L$ is said to be orthogonal, denoted $a \perp b$, if $a \leq b^\perp$. An orthocomplemented poset is called an *orthomodular poset* if (i) $a \perp b$ implies that $a \vee b$ exists, and (ii) $a \leq b$ implies that there is a $d \in L$ such that $d \perp a$ and $b = a \vee d$. An orthomodular poset is called a logic if $\vee \{a_i, i = 1, 2, \dots\}$ exists provided $a_i \perp a_j$, $i \neq j$, $i, j = 1, 2, \dots$. A logic which is a lattice, will be called a *lattice-logic.*

Let $L$ be a logic, a mapping $m$: $L \rightarrow [0, 1]$ satisfying (i) $m(1) = 1$, (ii) if $\{a_i, i = 1, 2, \dots\}$ are pairwise orthogonal, then $m(\vee_i a_i) = \Sigma_i m(a_i)$ is called a *state.* The set of all states is strongly convex, i.e. if $m_i$, $i = 1, 2, \dots$ are states, then $m(a) = \Sigma_i t_i m_i(a)$ $(a \in L)$, where $0 \leq t_i \leq 1$, $\Sigma_i t_i = 1$ is also a state. A set of states is said to be *quite full for* $L$ if $\{m \in M\colon m(a) = 1\} \subset \{m \in M\colon m(b) = 1\}$ implies $a \leq b$ [1]. A set $M$ of states is said to be *unital for* $L$ if for every $a \in L$, $a \neq 0$, there exists a state $m \in M$ such that $m(a) = 1$ [2]. If $M$ is quite full for $L$, then it is also unital for $L$ [1].

A subset $L_0 \subset L$ containing 1 is called a *sublogic* of $L$ if it is a logic with the same ordering $\leq$, orthocomplementation $\perp$ and the operation $\vee$ as $L$. If the sublogic $L_0$ of $L$ is a Boolean $\sigma$-algebra, it is called a *Boolean sub-$\sigma$-algebra* of $L$. Two elements $a, b \in L$ are said to be *compatible,* written $a \leftrightarrow b$ if there are elements $a_1$, $b_1$, $c \in L$ mutually orthogonal and such that $a = a_1 \vee c$, $b = b_1 \vee c$. A logic $L$ is a Boolean $\sigma$-algebra if and only if $a \leftrightarrow b$ for any $a, b \in L$. If $L$ is a lattice-logic, then a collection of elements of $L$ are mutually compatible if and only if the collection is contained in a Boolean sub-$\sigma$-algebra of $L$ [3, 4]. A Boolean sub-$\sigma$-algebra is called *maximal* if it is not contained in any other Boolean sub-$\sigma$-algebra.

An element $q \in L$ is called an *atom* if $b \leq q$, $b \in L$ implies $b = 0$ or $b = q$. A Boolean $\sigma$-algebra is *discrete* if it is generated by an at most countable set of atoms.

An observable $x$ on the logic $L$ is a $\sigma$-homomorphism from the Borel subsets $B(R)$ of the real line $R$ to $L$. We denote the range of $x$ by $R(x)$. $R(x)$ is a Boolean sub-$\sigma$-algebra of $L$. If $x$ is an observable and $u$ is a Borel function on $R$, we define the observable $u(x)$ by $u(x)(E) = x(u^{-1}(E))$ for all $E \in B(R)$. If $x$ and $y$ are observables, then $R(x) \subset R(y)$ if and only if there is a Borel function $u$ such that $x = u(y)$ [5]. The *spectrum* $\sigma(x)$ of the observable $x$ is the smallest closed set $C \subset R$ such that $x(C) = 1$. An observable $x$ is *bounded* if its spectrum $\sigma(x)$ is bounded. Observables $x$, $y$ on $L$ are *compatible*, written $x \leftrightarrow y$ if $x(E) \leftrightarrow y(F)$ for any $E$, $F \in B(R)$. Let $X$ be a set of observables and $y$ be any observable; we shall write $y \leftrightarrow X$ if $y \leftrightarrow x$ for every $x \in X$. If $x$ is an observable and $m$ is a state, then the *expectation* of $x$ in the state $m$ is $m(x) = \int \lambda m(x(d\lambda))$ if the integral exists.

A logic is *countably generated* if every Boolean sub-$\sigma$-algebra of it is countably generated. If $L$ is a lattice-logic which is countably generated, then the following theorems hold true [3], [6].

**Theorem 1.** *A subset of $L$ is the range of an observable if and only if it is a Boolean sub-$\sigma$-algebra.*

**Theorem 2.** *$\{x_\alpha: \alpha \in A\}$ are compatible (i.e. $x_\alpha \leftrightarrow x_\beta$, $\alpha$, $\beta \in A$) if and only if there exist an observable $x$ and Borel functions $u_\alpha$ such that $u_\alpha(x) = x_\alpha$, $\alpha \in A$.*

Let $L(H)$ be the logic consisting of all closed subspaces of a complex, separable Hilbert space $H$ with $\dim H \geq 3$. It is known that $L(H)$ is a countably generated lattice logic. By the Gleason theorem [7], [4], each state on $L(H)$ is of the form $m(a) = \Sigma_i t_i(\varphi_i, a\varphi_i)$ $(a \in L(H))$, where $0 \leq t_i \leq 1$, $\Sigma_i t_i = 1$ and $\varphi_i \in H$, $\|\varphi_i\| = 1$. The set of all states is quite full for $L(H)$. The (bounded) observables on $L(H)$ are in a one-to-one correspondence with the (bounded) self-adjoint linear operators on $L(H)$. Let us denote by $\mathcal{O}$ the set of all bounded observables on $L(H)$. Then $\mathcal{O}$ is the self-adjoint part of the von Neumann algebra $B(H)$ of all bounded operators on $H$. Any operator $x \in B(H)$ can be written in the form $x = x_1 + ix_2$, where $x_1$, $x_2 \in \mathcal{O}$. For any $x$, $y \in \mathcal{O}$, $x \leftrightarrow y$ is equivalent with $xy = yx$. If $A \subset B(H)$ is a von Neumann algebra, then the set of all projection operators in $A$ is a sublogic of $L(H)$. In the sequel we shall need the following theorems [8, p. 68, Ex. 9 and 10]. We recall that the ultraweak topology on $B(H)$ is defined by the system of seminorms: $y \in B(H)$, $y \mapsto \left|\sum_{i=1}^{\infty}(y\varphi_i, \psi_i)\right|$, where $\{\varphi_i\}$ and $\{\psi_i\}$ are sequences of vectors from $H$ such that $\sum_{i=1}^{\infty}\|\varphi_i\|^2 < \infty$ and $\sum_{i=1}^{\infty}\|\psi_i\|^2 < \infty$.

**Theorem 3.** *Let $A \subset B(H)$ be a von Neumann algebra and $g$ a positive linear functional on $A$. The restriction of $g$ to the logic $L(A)$ consisting of all projection* *operators in $A$ is a $\sigma$-additive state on $L(A)$ if and only if $g$ is ultraweakly continuous on $A$ and $g(1)=1$.*

**Theorem 4.** *To ewery positive, ultraweakly continuous linear functional $g$ on a von Neumann algebra $A\subset B(H)$ there exists a positive, ultraweakly continuous linear functional $g^{\sim}$ on the algebra $B(H)$ such that $g^{\sim}/A=g$ and $g^{\sim}(1)=g(1)$.*

## 2. Extensibility of states

**Theorem 5.** *Let $L$ be a logic such that the set of states $M$ is unital for $L$. Let $B$ be a discrete Boolean sub-$\sigma$-algebra of $L$. Then any state on $B$ can be extended to a state on $L$.*

Proof. Let $\{a_1, a_2, \ldots\}$ be the set of all atoms in $B$. If $m$ is a state on $B$, let us set $m^*(b) = \Sigma_i m(a_i)m_i(b)$, $b\in L$, where $m_i$ are states on $L$ such that $m_i(a_i)=1$ for $i=1, 2, \ldots$. Then $m^*(1) = m(\vee_i a_i) = \Sigma_i m(a_i)=1$, because the atoms of $B$ are mutally orthogonal and $\vee_i a_i=1$. From this it follows that $m^*$ is a state on $L$. Clearly, $m^*(a_i)=m(a_i)$, $i=1, 2, \ldots$, which implies that $m^*(b)=m(b)$ for any $b\in B$. Q.E.D.

We shall say that the sublogic $L_0$ of $L$ has the *extension property* if any state on $L_0$ can be extended to a state on $L$.

**Theorem 6.** *Let $L$ be a lattice -logic and let the set of all states be unital for $L$. Moreover, let any state on $L$ have the following property: $m(a)=1$, $m(b)=1$ $(a, b\in L)$ imply $m(a\wedge b)=1$. Then a finite sublogic $L_0$ of $L$, which is indeed a finite orthomodular sublattice of $L$, has the extension property if and only if it is a Boolean subalgebra of $L$.*

Proof. If $L_0$ is a Boolean subalgebra, it has the extension property by Theorem 5. Now let $L_0$ have the extension property. Then to any state $m$ on $L_0$ there is a state $m^*$ on $L$ such that $m(b)=m^*(b)$ for any $b\in L_0$. From this it follows that $m(a)=1$, $m(b)=1$, $a, b\in L_0$ imply $m(a\wedge b)=1$ for any state $m$ on $L_0$. On the other hand, the restriction of a state $m$ on $L$ to $L_0$ is a state on $L_0$. From this it follows that the set of states on $L_0$ is unital for $L_0$. By [2, Theorem 4.3], $L_0$ is a Boolean algebra. Q.E.D.

Theorem 6 for the special case $L=L(H)$ is proved in [2, Theorem 5.3].

**Theorem 7.** *Any Boolean sub-$\sigma$-algebra of the logic $L(H)$ has the extension property.*

Proof. Let $B\subset L(H)$ be a Boolean sub-$\sigma$-algebra. Let $B''$ be the bicommutant of $B$ in $B(H)$. A theorem of Bade [9], [10, XVII, P. 286] proves, that for a complete Boolean sublattice $C$ of $L(H)$ the following holds:

$$C=\{P\in C'':\ P \text{ is projection operator}\}.$$

As $H$ is separable, any Boolean sub-$\sigma$-algebra of $L(H)$ is a complete lattice [11]. From this it follows that $B$ is the logic of all projection operators in the von Neumann algebra $B''$. For any $x \in B''$ we can set $x = x_1 + i x_2$, where $x_1, x_2 \in B''$ are self-adjoint operators such that $R(x_1), R(x_2) \subset B$ [8]. Let $m$ be a state on $B$. We define a functional $f$ on $B''$ by setting

$$f(x) = \int tm(x_1(dt)) + i \int tm(x_2(dt))$$

for $x \in B''$, $x = x_1 + i x_2$. We shall show that $f$ is a positive linear functional on $B''$. It is enough to show the linearity of $f$ on the set of all self-adjoint operators in $B''$. Let $x_1, x_2 \in \mathcal{O} \cap B''$. As $R(x_1)$ and $R(x_2)$ are contained in the Boolean sub-$\sigma$-algebra $B \subset L(H)$, $x_1$ and $x_2$ are compatible. Let $R(x_1) \vee R(x_2)$ be the minimal Boolean sub-$\sigma$-algebra of $L(H)$ containing $R(x_1)$ and $R(x_2)$. Then $R(x_1) \vee R(x_2) \subset B$. Let $x_0$ be an observable with the range $R(x_0) = R(x_1) \vee R(x_2)$. There are real Borel functions $u_1$ and $u_2$ such that $x_1 = u_1(x_0)$ and $x_2 = u_2(x_0)$. For any $\alpha, \beta \in R$ then

$$\begin{aligned} f(\alpha x_1 + \beta x_2) &= \int tm((\alpha x_1 + \beta x_2)(dt)) = \int tm(\alpha u_1(x_0) + \beta u_2(x_0)(dt)) = \\ &= \int (\alpha u_1(t) + \beta u_2(t)) m(x_0(dt)) = \alpha \int u_1(t) m(x_0(dt)) + \beta \int u_2(t) m(x_0(dt)) = \\ &= \alpha \int tm(x_1(dt)) + \beta \int tm(x_2(dt)) = \alpha f(x_1) + \beta f(x_2). \end{aligned}$$

By Theorem 3, $f$ is ultraweakly continuous and by Theorem 4 there is a positive, ultraweakly continuous extension $\tilde{f}$ of $f$ to $B(H)$. Then $\tilde{f}/L(H)$ is a $\sigma$-additive state on $L(H)$ and $\tilde{f}/B = f/B = m$.

Q.E.D.

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## ЗАМЕЧАНИЕ О ПРОДОЛЖЕНИИ СОСТОЯНИЙ

Сылвия Пулманнова

Резюме

В данной статье исследуется возможность продолжения состояний из булевой подалгебры данной логики на всю эту логику.