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A NOTE ON THE CONVERGENCE OF TRANSFINITE SEQUENCES

PAVEL KOSTYRKO

The aim of the present paper is to complete some of the results of paper [1]. We shall use the terminology and definitions according to [1].

Let X be a topological space and (Y, σ) be a metric space which has two elements at least. Let $\{f_\xi\}_{\xi<\Omega}$ be a functional transfinite sequence of the type Ω (t.s t. Ω), $f_\xi\colon X \to Y$, Ω — the first uncountable ordinal. In [1] the notions of pointwise, uniform, locally uniform and quasi-uniform convergences of a functional t.s.t. Ω are introduced and the following problem is solved: Let (A) and (B) be two of the introduced kinds of convergences. It is said that the convergences (A) and (B) are equivalent (on X) if a functional t.s.t. $\Omega\{f_\xi\}_{\xi<\Omega}$ converges to f (on X) in the sense (A) if and only if it converges to f in the sense (B). In [1] only sufficient conditions are given (Theorem 1 (vi), (vii)) for the equivalence (on X) of uniform and locally uniform convergences, and also for the equivalence (on X) of locally uniform and quasi-uniform convergences. Further we shall give the complete answer to the equivalence (on X) of the above mentioned convergences.

Let $C(X)$ be the family of all compact subsets of X.

Definition 1. *We shall say that a t.s.t. $\Omega\{C_\xi\}_{\xi<\Omega}$, $C_\xi \in C(X)$, is an a-sequence if $C_1 \neq \emptyset$ and $C_\xi - \bigcup_{\eta<\xi} C_\eta \neq \emptyset$ holds for each ξ, $1<\xi<\Omega$. $C(a)$ will stand for the family of all a-sequences.*

Definition 2. *We shall say that a t.s.t. $\Omega\{C_\xi\}_{\xi<\Omega}$, $C_\xi \in C(X)$, $C_\xi \neq \emptyset$, is a b-sequence if for each $C \in C(X)$ there exists $\mu_C < \Omega$ such that $C \cap C_\xi = \emptyset$ holds for each ξ, $\mu_C \leq \xi < \Omega$. $C(b)$ will stand for the family of all b-sequences.*

Recall the definitions of uniform, locally uniform and quasi-uniform convergences.

Definition ([1], p. 233). *We shall say that a functional t.s.t. $\Omega\{f_\xi\}_{\xi<\Omega}$ converges (on X) uniformly to a function f if for each $\varepsilon>0$ there exists $\mu(\mu<\Omega)$ such that for every $\xi \geq \mu$ and each $x \in X$ $\sigma(f_\xi(x), f(x)) < \varepsilon$ holds.*

Let X be a topological space. A functional t.s.t. Ω is said to converge (on X) *locally uniformly to f if for each compa t C $(C\subset X)$ the fun ti nal t s. $\Omega\{f_\xi|C\}_{\xi\ \Omega}$ converges uniformly to $f|C$.*

We shall say that a functional t.s.t. $\Omega\{f_\xi\}_{\xi\ \Omega}$ converges (on X) quasi-uniformly to f if for each $x\in X$ $\lim_{\xi\to\Omega} f_\xi(x) = f(x)$ and to every $\varepsilon>0$ and e y η_1 Ω there exists η, $\eta_1<\eta<\Omega$ such that $\inf_{\eta_1<\xi\ \eta} \sigma(f_\xi(x), f(x))<\varepsilon$ for each .

Theorem. *Let the sets X and Y have the introduced meaning Th*

(i) *uniform and locally uniform convergences are aquivalent (on X) if and nly if $C(a)\cap C(b)=\emptyset$;*

(ii) *locally uniform and quasi-uniform convergences are equivalent (c n X) f and only if X is a countable topological space.*

Proof. (i): Indirectly we shall show that the condition $C(a)\cap C(b)-\emptyset$ is sufficient to the equivalence (on X) of uniform and locally unifoim convergences. Recall that $f_\xi\to f$ uniformly if and only if there is $\mu(\mu<\Omega)$ such that $f\ =f$ holds for each $\xi\geq\mu$ ([1], Lemma). Let $f_\xi\to f$ locally uniformly and not uniformly. Obviously it is possible to choose an increasing t.s.t. $\Omega\{\xi_\mu\}_{\mu<\Omega}$ and a one to one t.s.t. $\Omega\{x_\mu\}_{\mu<\Omega}$ such that $f_{\xi_\mu}(x_\mu)\neq g(x_\mu)$. Let us put $f_{\xi_\mu}=g_\mu$. Obviousl $g_\mu\to f$)callv uniformly and not uniformly. Let us consider an a-sequence $\{C_\mu\}_{\mu<\Omega}$ $C_\mu=\{x_\mu\}\in C(X)$. According to the assumption $\{C_\mu\}_{\mu<\Omega}\notin C(b)$, hence there is a compact C such that the set $\Omega_0 = \{\mu\colon C\cap C_\mu\neq\emptyset\}$ is uncountable. Consequently Ω_0 is a cofinal subset of $\{\mu\colon \mu<\Omega\}$ and so the convergence $g_\mu|C\to f|C$ is not uniform, and also the convergence $g_\mu\to f$ is not locally uniform. A contradiction.

The necessity of the condition $C(a)\cap C(b)=\emptyset$ for the equivalence (on X) of uniform and locally uniform convergences will be shown also indirectly. Let $\{C_\xi\}_{\xi<\Omega}\in C(a)\cap C(b)$. Let us define a functional t.s.t. $\Omega\{f_\xi\}_{\xi<\Omega}(f_\xi\colon X\to Y)$ in the following way: $f_1\equiv d$. If $1<\xi<\Omega$, then $f_\xi(x)=d$ for $x\in C_\xi - \bigcup_{\eta<\xi} C_\eta$, $f_\xi(x) = c\neq d$ for $x\notin C_\xi - \bigcup_{\eta<\xi} C_\eta$. For each $C\in C(X)$ there is $\mu_C<\Omega$ such that $C\cap C_\xi=\emptyset$ whenever $\xi\geq\mu_C$. Hence $f_\xi|C\to f|C\equiv c$ uniformly, and $f_\xi\to f\equiv c$ locally uniformly. It follows from the definition of $\{f_\xi\}_{\xi<\Omega}$ that the convergence $f_\xi\to f$ is not uniform

(ii): The fact that the countability of the topological space X is sufficient for the equivalence (on X) of locally uniform and quasi-uniform convergences is proved in [1], Theorem 1 (vii). The necessity of the above mentioned condition will be proved by contradiction. Let X be an uncountable topological space. In X there are possible the following two cases: each set $C\in C(X)$ is countable; there is an uncountable set $C\in C(X)$.

Let us suppose that each set $C\in C(X)$ is countable. Let $\{x_\xi\}_{\xi<\Omega}$, $x_\xi\in X$, be a one to one t.s.t. Ω. We define $f_\xi(x_\eta)=b$ for $\eta\geq\xi$ and $f_\xi(x) = a\neq b$ for $x\in X$

$-\{x_\eta\colon \eta \geq \xi\}$. Obviously the convergence $f_\xi|C \to f|C \equiv a$ is uniform for each $C \in C(X)$ and hence $f_\xi \to f \equiv a$ locally uniformly. On the other hand there is $\eta_0(=1)$ such that $\inf\limits_{\eta_0 < \xi \leq \eta} \sigma(f_\xi(x_\eta), f(x_\eta)) = \sigma(b, a) > 0$ holds for each η, $\eta_0 < \eta < \Omega$, therefore the convergence $f_\xi \to f$ is not quasi-uniform.

Let us suppose that there exists an uncountable compact $C \in C(X)$ and let $\{x_\xi\}_{\xi<\Omega}$, $x_\xi \in C$ be a one to one t.s.t. Ω. We put $f_\xi(x_\xi) = b$ for each $\xi < \Omega$ and $f_\xi(x) = a \neq b$ for $x \neq x_\xi$. The functional t.s.t. $\Omega\{f_\xi\}_{\xi<\Omega}$ obviously converges pointwise to $f \equiv a$. For each η_0 there exists $\eta = \eta_0 + 2$ such that $\inf\limits_{\eta_0 < \xi \leq \eta} \sigma(f_\xi(x), f(x)) = \min\{\sigma(f_{\eta_0+1}(x), f(x)), \sigma(f_{\eta_0+2}(x), f(x))\} = 0$ holds for each $x \in X$. Consequently $f_\xi \to f$ quasi-uniformly. Since the convergence $f_\xi|C \to f|C$ is not uniform, the convergence $f_\xi \to f$ is not locally uniform.

Our Theorem is therefore completely proved.

Corollary. *Let the sets X and Y have the introduced meaning. Then uniform and locally uniform convergences are equivalent (on X) if and only if $C(b) = \emptyset$.*

Proof. It is sufficient to prove, according to Theorem (i), that $C(a) \cap C(b) = \emptyset$ if and only if $C(b) = \emptyset$.

Let X be a σ-compact, i.e. $X = \bigcup\limits_{n<\omega} C_n$, $C_n \in C(X)$ (ω — the first non-finite ordinal). If there is a b-sequence $\{C_\xi\}_{\xi<\Omega}$, then $C_\mu = \emptyset$ holds for each μ, $\mu > \mu_{C_n}$, $n = 1, 2, \ldots$ (μ_{C_n} from Definition 2). A contradiction. Hence $C(b) = \emptyset = C(a) \cap C(b)$.

If X is not σ-compact and $C(b) = \emptyset$, then obviously $C(a) \cap C(b) = \emptyset$. Let $C(b) \neq \emptyset$, $\{C_\xi\}_{\xi<\Omega} \in C(b)$. We can choose by induction a subsequence $\{C_{\xi_\eta}\}_{\eta<\Omega}$ of $\{C_\xi\}_{\xi<\Omega}$ such that $\{C_{\xi_\eta}\}_{\eta<\Omega} \in C(a) \cap C(b)$. Let $C_{\xi_1} = C_1$. Let us suppose that for τ, $1 < \tau < \Omega$ the sets $\{C_{\xi_\eta}\}_{\eta<\tau}$ are chosen. Let us define $\xi_\tau = \min\{\xi\colon \xi > \xi_\eta$ and $\xi > \mu_{C_{\xi_\eta}}$ for $\eta < \tau\}$. Obviously $C_{\xi_\eta} \cap C_{\xi_\tau} = \emptyset$ for $\eta < \tau$, consequently $\{C_{\xi_\eta}\}_{\eta<\Omega} \in C(a) \cap C(b)$.

Remark. The condition $C(a) \cap C(b) = \emptyset$ ($C(b) = \emptyset$), which is according to Theorem (i) (Corollary) equivalent to the equivalence of uniform and locally uniform convergences, is in general unknown with respect to the usual topological notions. It would be interesting to find out connections between the above mentioned conditions and notions which are familiar in topology, e.g. σ-compactness.

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ПРИМЕЧАНИЕ К СХОДИМОСТИ ТРАНСФИНИТНЫХ ПОСЛЕДОВАТЕЛЬНОСТЕЙ

Павел Костырко

Резюме

В работе даны необходимые и достаточные условия для эквивалентности равномерной и почти равномерной сходимости, а также для эквивалентности почти равномерной и квазиравномерной сходимости функциональных трансфинитных последовательностей.