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Mathematica Slovaca, Vol. 28 (1978), No. 3, 321--328

Persistent URL: http://dml.cz/dmlcz/132374

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ALMOST UNIFORM CONVERGENCE FOR CONTINUOUS PARAMETERS

TIBOR NEUBRUNN

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Various papers deal with Jegoroff's theorem of almost uniform convergence. The classical version proved in [3] (see e.g. [2] p. 88) concerns sequences of functions. It is well known that it fails if instead of sequences a system $\{f^t\}$ $(t \in T)$ of functions is considered. More precisely, a function $f(x, t)$ on $X \times T$ is given and by means of this function the collection $\{f^t\}$ $(t \in T)$ where f^t are t — sections of t is considered. The counterexamples were given in [11], [12] (see also [10]). If the notion of the almost uniform convergence is weakened, then a weaker analogy of Jegoroff's theorem may be obtained also for continuous parameters. Such results were proved in [13] and [14].

On the other hand, there is a possibility to obtain also for a continuous parameter the classical version of Jegoroff's theorem, if certain assumption on $f(x, t)$ as a function of two variables are given. Thus, e.g., the Borel measurability of f as a function of two real variables is sufficient, as was proved by Tolstoff [11], using the properties of the analytic sets in the plane. In the present paper we give two theorems of this kind covering the cases when X and T are sufficiently general spaces. Of course the proofs will differ from those of Tolstoff.

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Definition 1. *Let $(X, \mathcal{S}, \mu)$ be a measure space (in the sense of [2]) and T a topological space. Let $F: X \times T \to R$ be a function and $t_0 \in T$ a point. A system $\{f^t\}$ $(t \in T)$, where $f^t(x) = f(x, t)$ for $x \in X$ is said to be almost uniformly convergent to a function φ defined on X, if to any $\varepsilon > 0$ there exists a set $E \in \mathcal{S}$ such that $\mu(E) < \varepsilon$ and the system $\{f^t\}$ converges uniformly to φ on $X - E$ as t tends to t_0.*

Our first result concerning the almost uniform convergence uses the notion of the quasicontinuity.

Definition 2. *If X, Y are topological spaces and $g: X \to Y$ a mapping, then g is said to be quasicontinuous $x_0 \in X$ if for every two open sets U, V such that $x_0 \in U$,*

$g(x_0) \in V$, *there exists* G *open*, $G \neq \emptyset$, $G \subset U$ *and* $g(G) \subset V$. *It is said to be quasicontinuous on* X *if it is quasicontinuous at any* $x_0 \in X$..

Remark 1. The notion of quasicontinuity was used already in [4]. The relations between quasicontinuity continuity and other types of continuities was discussed in [6], [7], [8] and elsewhere. The characterization of discontinuity points of a quasicontinuous function was given in [5].

It seems to be worth mentioning that there exists a quasicontinuous function which is not Lebesgue measurable (see [6]).

Lemma 1. *If* $g : X \to Y$ *is a quasicontinuous function on* X, *then the following is true: If* $Z \subset X$ *is any open set and* $D \subset Z$ *is dense in* Z, *then* $f(D)$ *is dense in* $f(Z)$.

Proof. Let $y \in f(Z)$ and $x \in Z$ such that $f(x) = y$. Let V be any open set containing y. We have from the quasicontinuity at x that a nonempty open $G \subset Z$ exists with $f(G) \subset V$. Since $G \cap D \neq \emptyset$ and $f(G \cap D) \subset V$, a point $v \in V$ belonging to $f(D)$ exists. The proof is finished.

Remark 2. It is known that the converse of Lemma 1 holds too. However, we do not use this fact.

In what follows, we suppose that the functions f which are dealt with assume real values. A generalization for metric spaces or some uniform spaces is possible. But we are of the opinion that in that direction a sufficiently general version is given in [14]. By the same method as in [14] our results may be transferred to suitable uniform spaces. As to the spaces X, T on the product $X \times T$ of which the function f will be defined, we shall suppose that $(X, \mathcal{S}, \mu)$ is a totally finite measure space and T a separable topological space satisfying the first countability axiom. These conditions will not be repeated in the formulations of the theorems. Only additional conditions if necessary, will be explicitely stated. It can be easily verified that the first countability axiom may be weakened in some of the results. The same is true for the total finiteness of the space $(X, \mathcal{S}, \mu)$ if we restrict ourselves to functions defined on $E \times T$, where $E \in \mathcal{S}$, and $\mu(E) < \infty$.

Lemma 2. *Let* $f: X \times T \to R$. *Let* $t_0 \in T$. *If for ever* $t \in T$ *the functions* f^t *are measurable and* $\lim_{t \to t_0} f^t(x) = \varphi(x)$ *for every* $x \in X$, *then* φ *is a measurable function.*

Proof. From the assumption of the first countability of T it follows that $\varphi(x) = \lim_{n \to \infty} f(x, t_n)$. Hence φ is measurable as a limit of a sequence of measurable functions.

Remark 3. Obviously the assumption $\lim_{t \to t_0} f^t(x) = \varphi(x)$ for every $x \in X$ may be substituded by the assumption that $\lim_{t \to t_0} f^t(x) = \varphi(x)$ almost everywhere.

Lemma 3. *Let $f: X \times T \to R$ be such that f^t are measurable for every $t \in T$ and f_x quasicontinuous for every $x \in X$. Suppose that $\lim_{t \to t_0} f^t(x) = \varphi(x)$ for every $x \in X$. Then for every open set $G \subset T$ and any $\varepsilon > 0$ the set $F = \{x: |f(x, t) - \varphi(x)| \leqq \varepsilon$, for every $t \in G\}$ is measurable.*

Proof. Let D be a coutable dense subset of G. Suppose that D is the set of values of a sequence $\{t_k\}_{k=1}^{\infty}$. Put

$$F_1 = \{x: |f(x, t_k) - \varphi(x)| \leqq \varepsilon, \quad \text{for } k = 1, 2, \ldots\}.$$

Since f^{t_k} are measurable functions for $k = 1, 2, \ldots$ and φ is measurable according to Lemma 2, we have that F_1, as a countable intersection of measurable sets, is measurable. It is sufficient to show that $F = F_1$. The inclusion $F \subset F_1$ is obvious. Let $x \in F_1$ and t any point in G. According to Lemma 1, we have that $f_x(D)$ is dense in $f_x(G)$. Hence a sequence $\{t_{k_n}\}_{n=1}^{\infty}$ exists such that $f_x(t_{k_n})$ tends to $f(x, t)$. Since $x \in F_1$, it follows $|f(x, t_{k_n}) - \varphi(x)| \leqq \varepsilon$ for $n = 1, 2, \ldots$ Hence $|f(x, t) - \varphi(x)| \leqq \varepsilon$. Since t was arbitrary, we have $x \in F$. The inclusion $F_1 \subset F$ is proved.

Theorem 1. *Let $f: X \times Y \to R$. Let $t_0 \in T$ and $\lim_{t \to t_0} f(x, t) = \varphi(x)$ almost everywhere on X. Let f_x be quasicontinuous for every $x \in X$. Then for $t \to t_0$, f^t tends to φ almost uniformly.*

Proof. Without loss of generality we may suppose that $f^t(x)$ converges to $\varphi(x)$ for every $x \in X$. Let $V_1 \supset V_2 \supset \ldots$ be a base of open neighbourhoods at the point t_0. Let $\varepsilon > 0$. Put for $k = 1, 2, \ldots, n = 1, 2, \ldots$ $E_k^n = \{x: x \in X, |f(x, t) - \varphi(x)| \leqq \frac{1}{n}$, for every $t \in V_k\}$. According to Lemma 3 the sets E_k^n are measurable. Since $\lim_{t \to t_0} f^t(x) = \varphi(x)$ for every $x \in X$, we have $\bigcup_{k=1}^{\infty} E_k^n = X$ for $n = 1, 2, \ldots$ Moreover $E_k^n \subset E_{k+1}^n$ for $k = 1, 2, \ldots$ Thus for every n a number $k(n)$ exists such that $\mu(X - E_{k(n)}^n) < \frac{\varepsilon}{2^n}$. Put $E = \bigcap_{n=1}^{\infty} E_{k(n)}^n$. The fact that $\mu(X - E) < \varepsilon$ follows now in the same way as in the proof of the classical Jegoroff's theorem, as well as the fact that the convergence is uniform on E is now quite similar to the one in the classical case. If $\eta > 0$ is arbitrary, then we can choose n such that $\frac{1}{n} < \eta$. Now, since

$$E \subset E_{k(n)}^n, \text{ we have } |f(x, t) - \varphi(x)| \leqq \frac{1}{n}$$

for every $t \in V_{k(n)}$ and every $x \in E$.

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In this section another sufficient condition will be given for an almost uniform convergence in the case of a continuous parameter. The proof will be based on a method which was used for proving the measurability of functions of two variables by the author in his thesis (1963) and then applied in [1]. At first we shall give some notes concerning this method. The functions will be again real valued. The assumption concerning the domain will be specified in each of the assertions which follow.

Definition 3. *If X is a nonempty set, then a collection $\mathcal{P} = \{P_n^k\}$ of nonempty sets $k = 1, 2, \ldots\ n \in N(k)$, where $N(k)$ is either a set $\{1, 2, \ldots, n_k\}$ or the set of all positive integers, and $\bigcup_n P_n^k = X$ for $k = 1, 2, \ldots$, is said to be a $\mathcal{P}$-system on X.*

Remark 5. Note that in [1] we used the notion of a $\mathcal{P}$-system only in the case when a measurable space $(X, \mathcal{S})$ was considered and we supposed $\mathcal{P}_n^k \in \mathcal{S}$. This will not be the case here in general. If $\mathcal{P}_n^k \in \mathcal{S}$ for $n \in N(k)$ and $k = 1, 2, \ldots$, we say that $\mathcal{P}$ is a measurable $\mathcal{P}$-system.

Definition 4. *If $\mathcal{P}$ is a $\mathcal{P}$-system on X, then a function f defined in X is said to be regular at x_0 with respect to $\mathcal{P}$, provided that for any open G containing $f(x_0)$ there exists k_0 such that if $k > k_0$, then $x, x_0 \in P_n^k$ for some n implies $f(x) \in G$. It is said to be regular on X with respect to $\mathcal{P}$ if it is regular at any $x_0 \in X$, with respect to $\mathcal{P}$.*

Definition 5. *If X is a topological space with the topology $\mathcal{T}$ and $\mathcal{P}$ a $\mathcal{P}$-system on X, then $\mathcal{P}$ is said to be regular with respect to $\mathcal{T}$ provided that for any $U \in \mathcal{T}$ and any $x \in U$ there exists k_0 such that if $k > k_0$, then $x \in \mathcal{P}_n^k$ implies $\mathcal{P}_n^k \subset U$.*

Lemma 4. *if $\mathcal{P}$ is a $\mathcal{P}$-system on a set X, then a function f on X which is regular on X relative to $\mathcal{P}$, is measurable with respect to the σ-algebra generated by $\mathcal{P}$.*

Proof. Let G be an open set. If $x_0 \in f^{-1}(G)$, then there exists k_0 such that if $k > k_0$ and x, $x_0 \in \mathcal{P}_n^k$, then $f(x) \in G$. Since for x_0 there is n such that $x_0 \in \mathcal{P}_n^k$, we have for this n $f(\mathcal{P}_n^k)G$ if $k > k_0$. Thus to any $x_0 \in f^{-1}(G)$ a set $\mathcal{P}_n^k$ may be associated such that $f(\mathcal{P}_n^k) \subset G$. This means that $f^{-1}(G)$ is a union of some sets belonging to $\mathcal{P}$.

Corollary. *If X is a topological space and $\mathcal{P}$ a $\mathcal{P}$-system on X which is Borel measurable (i.e. every element of $\mathcal{P}$ belongs to the σ-algebra generated by all open sets), then any function regular on X with respect to $\mathcal{P}$ is Borel measurable.*

Lemma 5. *If X is a topological space with the topology $\mathcal{T}$ and $\mathcal{P}$ a regular $\mathcal{P}$-system on X, then the σ-algebra generated by $\mathcal{P}$ contains the collection of all Borel sets.*

Proof. It is sufficient to prove that any open set U belongs to the σ-algebra generated by $\mathscr{P}$. Let $x \in U$. From the regularity of $\mathscr{P}$ it follows that there exists k_0 such that if $k > k_0$ and $x \in \mathscr{P}_n^k$, then $\mathscr{P}_n^k \subset U$. Since to any x and any k there exists n such that $x \in \mathscr{P}_n^k$. we have that U is a (countable) union of sets belonging to $\mathscr{P}$. The lemma is proved.

Corollary. *If X is a topological space and $\mathscr{P}$ a regular, Borel measurable $\mathscr{P}$-system on X, then the σ-algebra generated by $\mathscr{P}$ concides with the σ-algebra of all Borel sets.*

Lemma 6. *If X is a topological space and $\mathscr{P}$ a regular $\mathscr{P}$-system on X, then any continous function on X is regular on X with respect to $\mathscr{P}$.*

We omit the simple proof. It is contained in [1].

Lemma 7. *Let X be a set and $\mathscr{P}$ a $\mathscr{P}$-system on X. Let f be a function regular on X with respect to $\mathscr{P}$. Then if $\emptyset \neq Y \subset X$, there exists a $\mathscr{P}$-system $\mathscr{P}^*$ on Y such that the restriction $f \mid Y$ is regular on Y with respect to $\mathscr{P}^*$.*

Proof. If $\mathscr{P} = \{\mathscr{P}_n^k\}$, $k = 1, 2, \ldots$; $n \in N(k)$, then put $Q_n^k = \mathscr{P}_n^k \cap Y$ for $k = 1, 2, \ldots$; $n \in N(k)$. We may suppose that the sets Q_n^k are nonempty. If this is not the case, the only thing which will be different is that $N(k)$ will be substituted by another finite set which is a subset of $N(k)$. Evidently $\mathscr{P}^* = \{Q_n^k\}$ $k = 1, 2, \ldots$; $n \in N(k)$ is a $\mathscr{P}$-system on Y. Now if $x_0 \in Y$ and G is an open set containing $(f \mid Y)$ (x_0), we have $f(x_0) \in G$. Since f is regular with respect to $\mathscr{P}$, there is k_0 such that if $k > k_0$ x, $x_0 \in \mathscr{P}_n^k$, then $f(x) \in G$. Since $Q_n^k \subset \mathscr{P}_n^k$, we obtain inmediately that if $k > k_0$ and x, $x_0 \in Q_n^k$, then $(f \mid Y)$ $(x) \in G$.

Remark 6. It is evident from the proof of Lemma 7 that if a class $\mathscr{F}$ of functions and a $\mathscr{P}$-system $\mathscr{P}$ are given such that each $f \in \mathscr{F}$ is regular on X with respect to $\mathscr{P}$, then the class $\mathscr{F}^*$ of all $f \mid Y$, $f \in \mathscr{F}$ is regular on Y with respect to the $\mathscr{P}$-system $\mathscr{P}^*$ constructed in the proof of Lemma 7.

In what follows let $(X, \mathscr{S}, \mu)$ be again a totally finite measure space, T a topological space and $t_0 \in T$ a point possessing a countable base of neighbourhoods.

Theorem 2. *Let $f: X \times T \to R$. Let a $\mathscr{P}$-system $\mathscr{P}$ on T exist such that f_x is regular on T with respect to the $\mathscr{P}$-system for every $x \in X$ and let f^t be measurable for every $t \in T$. Then if $\lim_{t \to t_0} f(x, t) = \varphi(x)$ for almost every x, the convergence is almost uniform as t tends to t_0.*

Proof. We may again suppose that $\lim_{t \to t_0} f(x, t) = \varphi(x)$ for every $x \in X$. Let $\{V_k\}_{k=1}^{\infty}$ be a decreasing sequence of open sets forming a base at t_0. Exactly as in Theorem 1 the main thing is to prove that the sets $E_k^n = \{x : |f(x, t) - \varphi(x)| \leqq \frac{1}{n}$, for every $t \in V_k\}$ are measurable for $n = 1, 2, \ldots, k = 1, 2, \ldots$ Thus it is sufficient to prove that the set $E = \{x : |f(x, t) - \varphi(x)| \leqq \varepsilon$, for every $t \in V\}$ is measurable for any $\varepsilon > 0$, and any open set $V \subset T$. Construct now a $\mathscr{P}$-system $\mathscr{P}^*$ on V such that for the elements of $\mathscr{P}$ we take the sets $Q_n^k = \mathscr{P}_n^k \cap V.\ \mathscr{P}_n^k \in \mathscr{P}$ (see Lemma 7 and Remark 6). Choose in every nonempty Q_n^k a point t_n^k and denote by D the set of all these points. Since D is countable, the set F, where $F = \{x : f(x, t) - \varphi(x)| \leqq \varepsilon$, for every $t \in D\}$, is measurable as a countable intersection of measurable sets. (The measurability of each of the sets $\{x : |f(x, t) - \varphi(x)| \leqq \varepsilon\}$ for a fixed t follows from the assumption and from the measurability of φ, which in its turn follows in the same way as in Lemma 2). Now we prove $E = F$. The inclusion $E \subset F$ is trivial. Let $x \in F$ and $t \in V$ be any point. Let $\{\eta_i\}_{i=1}^{\infty}$ be a decreasing sequence of positive numbers converging to 0. Since f_x is regular (on V) with respect to $\mathscr{P}^*$, there exists for any η_i a positive integer $k(i)$ such that if $k > k(i)$, then for any t_1 such that t_1, $t \in Q_n^k$ we have $|f_x(t_1) - f_x(t)| < \eta_i$. Especially if we choose instead of t_1 $t_n^{k(i)} \in D$, depending on $i (t_n^{k(i)} = t(i)$, to simplify the notation), which belong to the same Q_n^k as t, we have $|f_x(t_n^{k(i)}) - f_x(t)| < \eta_i$. Thus a sequence of points $t_n^{k(i)} = t(i)$ belonging to D exists, such that $\lim_{i \to \infty} f_x(t(i)) = f_x(t)$. Since $t(i) \in D$, $i = 1, 2, \ldots$, we have

$$|f(x, t(i)) - \varphi(x)| \leqq \varepsilon \quad \text{for } i = 1, 2, \ldots .$$

hence $|f(x, t) - \varphi(x)| \leqq \varepsilon$. Since for $t \in V$ we may choose any point from V we have $|f(x, t) - \varphi(x)| \leqq \varepsilon$ for any $t \in V$, and so $x \in E$. Hence $E = F$ and E is measurable. Thus the measurability of E_k^n, $k = 1, 2, ., n \in N(k)$, is proved and the rest of the proof proceeds as in Theorem 1.

Corollary. *Let $(X, \mathscr{S}, \mu)$ be a totally finite measure space and T a second countable topological space. Let $f : X \times T \to R$ be such that f_x are continuous for every $x \in X$ and f^t measurable for every $t \in T$. Then if $\lim_{t \to t_0} f^t(x) = \varphi(x)$ for almost every $x \in X$, the convergence is almost uniform as t tends to t_0.*

Proof. If T is a second countable topological space, then a regular $\mathscr{P}$-system on T exists (See [1]). Now according to Lemma 6 the sections f_x are regular with respect to $\mathscr{P}$ and the result follows.

4

This section will be devoted to some discussion and lpossible generalization of the obtained results.

The aim of Theorem 1 and Theorem 2 was to state some simple sufficient condition for the validity of Jegoroff's theorem. It is not difficult to give an abstract formulation of the mentioned theorems.

Definition 6. *We shall say that a function f defined on $X \times T$ (T a topological space) satisfies the property (S) if the following is true:*

(S) *There exists a countable dense set $D \subset T$ such that for any open $G \subset T$ and any closed interval I*

$$\{x: f(x,t) \in I, \quad \text{for all} \quad t \in G\} = \{x: f(x,t) \in I\,,$$
$$\text{for all } t \in G \cap D\}$$

Remark 7. Note that the property (S) follows from the assumptions of Theorem 1 (See Lemma 3) as well as from the assumptions of Theorem 2.

Lemma 8. *If f is a real function defined on $X \times T$ and statisfying (S) and φ a real function on X, then there exists a countable dense set $D \subset T$ such that for any real $c \geqq 0$ and any open set G (S_1) holds:*

$$(S_1) \quad \{x: |f(x,t) - \varphi(x)| \leqq c\} \quad \text{for all} \quad t \in G = \{x: |f(x,t) - \varphi(x)| \leqq c$$
$$\text{for all} \quad t \in G \cap D\}\,.$$

Using the property (S_1) we can prove a lemma analogous to Lemma 3 and then the proof of the following theorem is strainghtforward.

Theorem 3. *Let $(X, \mathcal{S}, \mu)$ be totally finite measure space. Let T be a separable topological space satisfying the first countability axiom. Let the property (S_1) be satisfied. Then, if $\lim_{t \to t_0} f(x,t) = \varphi(x)$ for almost every $x \in X$, the convergence is almost uniform as t tends to t_0.*

In the case in which T is a subspace of $(-\infty, \infty)$ with the usual topology and $(X, \mathcal{S}, \mu)$ a probability space, the property (S) is the usual definition of a separable stochastic process. Hence from Theorem 3 and Lemma 8 we obtain as a corollary the following result.

Corollary. *If $(X, \mathcal{S}, \mu)$ is a probability space $T \subset (-\infty, \infty)$ and $f: X \times T \to R$ a separable stochastic process, then the almost everywhere convergence of $f(x,t)$ to $\varphi(x)$ for $t \to t_0$ implies the almost uniform convergence.*

Remark 8. In paper [9] Jegoroff's theorem for sequences was proved in the case when the measure space was substituted by a space with the system of collections of „small measure sets“. It is easy to see that our theorems may be proved also for such spaces.

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Received February 15, 1977.

Katedra matematickej analýzy
Prírodovedeckej fakulty UK
Mlynská dolina
816 31 Bratislava

ПОЧТЫ РАВНОМЕРНАЯ СХОДИМОСТЬ В СЛУЧАЕ НЕПРЕРЫВНЫХ ПАРАМЕТРОВ

Тибор Нойбрун

Резюме

Пусть $(X, \mathcal{S}, \mu)$ пространство с вполне конечной мерой, T-сепарабельное топологическое пространство исполняющее первую аксиому счетности. Вещественная функция f определена на $X \times T$ определяет систему функций на X. Именно, для $t \in T$, $f^t(x) = f(x, t)$. Если эти функции измеримы и если

$$\lim_{t \to t_0} f^t(x) = \varphi(x)$$

почти всюду, то в некоторых случаях эта сходимость μ — почти равномерна. Доказывается, что одним условием для μ — почти равномерной сходнимости является квазинепрерывность x^- — сечений f_x, функции f. В работе дается также другое условие подобного типа.