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A REMARK ON THE COMPARISON OF MACKEY AND SEGAL MODELS

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In the paper the necessary conditions for imbedding of a Segal system into a Mackey system are discussed.

1. Mackey and Segal systems

There are several papers dealing with the comparison of two important axiomatic models of the quantum theory — the Segal system and the Mackey system [1—7]. In some of them problems have arisen which so far have not been solved.

In the present note, an exact formulation of the imbedding of a Segal system into a Mackey system is given and some necessary conditions for this imbedding are discussed.

First we shall shortly describe the original Mackey and Segal systems.

Let $(\mathscr{L}, \leqslant)$ be a partially ordered set (abbreviated to poset) with a one-to-one map $a \mapsto a'$ of $\mathscr{L}$ onto $\mathscr{L}$. $(\mathscr{L}, \leqslant, ')$ is said to be a *σ-orthocomplemented poset* (see [8]), provided that

(a) $a'' = a$ for $a \in \mathscr{L}$.

(b) $a \leqslant b$ implies $b' \leqslant a'$.

(c) If $a_1, a_2, \ldots$ is a sequence of the members of $\mathscr{L}$, where $a_i \leqslant a_j'$ for $i \neq j$, then the least upper bound $a_1 \cup a_2 \cup \ldots$ exists in $\mathscr{L}$.

d) $a \cup a' = b \cup b'$ for all $a, b \in \mathscr{L}$. We denote $a \cup a'$ by 1. A σ-orthocomplemented poset is said to be *orthomodular* if

(e) $a \leqslant b$ implies $b = a \cup (b' \cup a)'$.

Let $\mathscr{L}$ be a σ-orthocomplemented poset. A map $m: \mathscr{L} \to [0, 1]$ is said to be a *state* on $\mathscr{L}$ if $m(1) = 1$ and $m(a_1 \cup a_2 \cup \ldots) = m(a_1) + m(a_2) + \ldots$ where $a_i \leqslant a_j'$ for $i \neq j$.

If for some $a, b \in \mathscr{L}$ we have $a \leqslant b'$, then we say that a is *orthogonal* to b and we write $a \perp b$.

A set of states $\mathscr{M}$ on $\mathscr{L}$ is said to be *full* if $m(a) \leqslant m(b)$ for all $m \in \mathscr{M}$ implies $a \leqslant b$. A σ-orthocomplemented poset with a full set of states is orthomodular [8].

The elements $a, b \in \mathscr{L}$ are *compatible* (written $a \leftrightarrow b$) if there exist mutually orthogonal elements $a_1, b_1, c \in \mathscr{L}$ such that $a = a_1 \cup c$, $b = b_1 \cup c$.

An observable x on $\mathscr{L}$ is a map from the Borel sets $\mathscr{B}(R)$ of the real line R into $\mathscr{L}$, which satisfies

(i) $x(R)=1$,
(ii) $x(E)\perp x(F)$ if $E\cap F=0$,
(iii) $x\left(\bigcup_{i=1}^{\infty} E_i\right)=\bigcup_{i=1}^{\infty} x(E_i)$, if $E_i\cap E_j = \neq$ for $i\neq j$.

The observables x, y are said to be compatible if $x(E)\leftrightarrow y(F)$ for all E, $F\in\mathscr{B}(R)$.

If x is an observable and u a real valued Borel function on R, we define the observable $u(x)$ by $u(x)(E)=x(u^{-1}(E))$ for all $E\in\mathscr{B}(R)$.

A set $\mathcal{O}$ of observables on $\mathscr{L}$ is said to be *full* if (i) $x\in\mathcal{O}$ implies $f(x)\in\mathcal{O}$ for all real valued Borel functions f and (ii) if $a\in\mathscr{L}$, then there are an $x\in\mathcal{O}$ and $E\in\mathscr{B}(R)$ such that $a=x(E)$.

The Mackey system (described in [9], axioms I—VI) can be considered as a full set of observables $\mathcal{O}$ on a σ-orthocomplemented poset $\mathscr{L}$ with a convex full set of states $\mathscr{M}$ [9, 10].

The spectrum $\sigma(x)$ of an observable x in the Mackey system is the smallest closed set $E\in\mathscr{B}(R)$ such that $x(E)=1$. An observable x is *bounded* if $\sigma(x)$ is bounded. The *expectation* of x in the state m is $m(x) = \int_R \lambda m(x(d\lambda))$, if the integral exists.

The norm of a bounded observable x is defined by $\|x\|=\sup\{|m(x)| : m\in\mathscr{M}\}$.

We say that the observable z is the *sum* of the bounded observables x and y if $m(z) = m(x) + m(y)$ for all $m\in\mathscr{M}$. The sum of two bounded observables in the Mackey system need not exist and need not be unique.

We say that x is a *proposition observable* if $\sigma(x)\subset\{0, 1\}$. The following statements are equivalent [10].

(i) x is a proposition observable.
(ii) *x is a characteristic function of an observable y, i.e. $x=X_E(y)$, $E\in\mathscr{B}(R)$.*
(iii) $x^2=x$.

We say that x is an *idempotent* if $x^2=x$. If y is a proposition observable and $y(\{1\})=a$, we denote y by x_a.

Let $\tilde{\mathscr{L}}$ be the set of all propostion observables in $\mathcal{O}$. Then $\tilde{\mathscr{L}}$ will be a σ-orthocomplemented poset if we set (i) $x\leqslant y$ if $m(x)=m(y)$ for all $m\in\mathscr{M}$ and (ii) $x'=(1-i)(x)$, where $1(t)\equiv 1$ and $i(t)=t$, $t\in R$. If $a\in\mathscr{L}$, then because $\mathcal{O}$ is full, we have $a=x(E)$ for some $x\in\mathcal{O}$ and $E\in\mathscr{B}(R)$. Then $X_E(x)(\{1\}) = =x(X_E^{-1}\{1\}) = x(E)=a$. We see that the map $a\mapsto x_a$ from $\mathscr{L}$ to $\tilde{\mathscr{L}}$ is one-to-one and $a\leqslant b$ if and only if $x_a\leqslant x_b$, owing to $m(x_a) = m(a)$ for all $a\in\mathscr{L}$. It can be easily seen that $(x_a)' = x_{a'}$. Indeed, $(x_a)'(\{1\}) = (1-i)(x_a)(\{1\}) =$

$= x_a((1-i)^{-1}\{1\}) = x_a(\{0\}) = a'$. Thus we get that the σ-orthocomplemented poset $\tilde{\mathscr{L}}$ is isomorphic with $\mathscr{L}$.

The Segal model for quantum mechanics is described in [11] and [12]. A set $\mathscr{X}$ is called a *system of observables* (or a *system*) if $\mathscr{X}$ satisfies the following postulates.

1) $\mathscr{X}$ is a linear space over the real numbers R.

2) There exists in $\mathscr{X}$ an identity element I and for every $u \in \mathscr{X}$ and integer $n \geqslant 0$ an element $u^n \in \mathscr{X}$, which satisfies the following. If f, g and h are real polynomials, and if $f(g(\alpha)) = h(\alpha)$ for all $\alpha \in R$, then $f(g(u)) = h(u)$, where

$$f(u) = \beta_0 I + \sum_{k=1}^{n} \beta_k u^k \quad \text{if} \quad f(\alpha) = \sum_{k=0}^{n} \beta_k \alpha^k .$$

3) There is defined for each observable u a real number $\|u\| \geqslant 0$ such that the pair $(\mathscr{X}, \|\,.\,\|)$ is a real Banach space.

4) $\|u^2 - v^2\| \leqslant \max(\|u\|^2, \|v\|^2)$ and $\|u^2\| = \|u\|^2$.

5) u^2 is a continuous function of u.

A *state* on $\mathscr{X}$ is a real valued function m on $\mathscr{X}$ such that $m(u^2) \geqslant 0$ for all $u \in \mathscr{X}$ and $m(I) = 1$.

A collection of states $\mathscr{M}$ on $\mathscr{X}$ is *full* if $m(u) = m(v)$ for all $m \in \mathscr{M}$ implies $u = v$, where $u, v \in \mathscr{X}$. Segal [11] has shown that any system of observables has a full set of states and that $\|u\| = \sup\{|m(u)| : m \in \mathscr{M}\}$ for all $u \in \mathscr{X}$. We can define the partial ordering on $\mathscr{X}$ if we set $u \leqslant v$ if $m(v-u) \geqslant 0$ for all $m \in \mathscr{M}$.

For any two observables u, $v \in \mathscr{X}$ Segal has defined the *formal product* $u \circ v = \frac{1}{4}[(u+v)^2 - (u-v)^2]$. A system is *commutative* if the formal product is associative, distributive (relative to addition) and homogeneous (relative to scalar multiplication).

A collection of observables are said to *commute* if the subsystem generated by the collection is commutative.

Segal [11] has proved that a commutative system is isomorphic alebraically and metrically with the system $C(\Gamma)$ of all real valued continuous functions on a compact Hausdorff space Γ. The operations in $C(\Gamma)$ are defined in the usual way and the norm is a supremum norm. If m is a state on $\mathscr{X}$, then there is a regular probability measure μ on Γ such that $m(f) = \int_\Gamma f d\mu$ for all $f \in C(\Gamma)$.

An observable $u \in \mathscr{X}$ is an idempotent if $u^2 = u$. Let $\mathscr{P}$ be the set of all idempotents in $\mathscr{X}$. Clearly, 0 and I are idempotents. For $a, b \in \mathscr{P}$ we define $a \leqslant b$ if $m(a) \leqslant m(b)$ for all $m \in \mathscr{M}$, where $\mathscr{M}$ is the full set of states on $\mathscr{X}$, and $a' = I - a$. Then $(\mathscr{P}, \leqslant, ')$ is a partially ordered set which satisfies the properties (a), (b) and (d) from the definition of the σ-orthocomplemented poset [1]. The elements $a, b \in \mathscr{P}$ are orthogonal if and only if $a + b \leqslant I$.

2. The imbedding of a Segal system into a Maskey system

Let $\mathscr{X}$ be a Segal system. $\mathscr{M}$ the full set of states on it and let $\mathscr{P}$ denote the set of all idempotents in $\mathscr{X}$.

Definition. *We shall say that $\mathscr{X}$ is imbedded into a Mackey system if there exist a full set of observables $\mathcal{O}$ on a σ-orthocomplemented poset $\mathscr{L}$ with the full set of states $\mathscr{N}$ which is isomorphic with $\mathscr{M}$ as a convex set and a one-to-one map τ from $\mathscr{X}$ into $\mathcal{O}$ such that:*

(i) $m'(\tau x)=m(x)$ *for all $x\in\mathscr{X}$ and all $m\in\mathscr{M}$, where $m\mapsto m'$ is the isomorphism of $\mathscr{M}$ onto $\mathscr{N}$,*

(ii) $\tau(x^n)=(\tau x)^n$ *for all $x\in\mathscr{X}$ and all integers $n\geq 0$,*

(iii) *if $m(x)=m'(y)$ for all $m\in\mathscr{M}$, where $x\in\mathscr{X}$, $y\in\mathcal{O}$, then $y=\tau x$.*

It is clear that τ preserves the norms. By (ii), from $\tau[\mathscr{X}]\subset\mathcal{O}$ it follows that $\tau[\mathscr{P}]\subset\tilde{\mathscr{L}}$, where $\tilde{\mathscr{L}}$ is the set of all proposition observables in $\mathcal{O}$. By (i), $\tau[\mathscr{P}]$ is isomorphic with $\mathscr{P}$. The property (iii) ensures the uniqueness of the sums $u+v$ in $\mathcal{O}$ if $u=\tau x$ and $v=\tau y$ for some $x, y\in\mathscr{X}$. In addition, we have that $m'(\tau(x+y)) = m(x+y) = m(x) + m(y) = m'(\tau x) + m'(\tau y) = m'(\tau x+\tau y)$, so that $\tau x+\tau y = \tau(x+y)$.

Now let $\mathscr{K}$ be any *σ-orthocomplemented poset and $\mathscr{S}$* its full set of states. Then each member $a\in\mathscr{K}$ gives rise to the function $\bar{a}$: $\mathscr{S}\to[0, 1]$ defined by $\bar{a}(m)=m(a)$ for all $m\in\mathscr{S}$. Let $\mathscr{S}'$ be the set of all such functions, i.e. the *dual* of $\mathscr{S}$. By [8], $\mathscr{S}'$ is the σ-orthocomplemented poset with respect to the natural ordering of functions ($\bar{a}\leq\bar{b}$ iff $\bar{a}(x)\leq b'(x)$ for all $x\in\mathscr{S}$), with the complementation $a'=1-a$, where *1* denotes the function $1(x)=1$ for all $x\in\mathscr{S}$, and $(\mathscr{K}, \leq, ')$ is isomorphic with $(\mathscr{S}, \leq, ')$.

Let $\mathscr{P}$ be the set of all idempotents in a Segal system $\mathscr{X}$. Let $\mathscr{P}^\circ$ denote the set of alls function $\bar{a}(m)=m(a)$ for all $m\in\mathscr{M}$, where $a\in\mathscr{P}$.

Theorem 1. *The necessary condition for imbedding the Segal system $\mathscr{X}$ with the full set of states $\mathscr{M}$ into a Mackey system is the existence of a set $\mathscr{L}_1$ of functions from the set $\mathscr{M}$ into* $[0, 1]$ *satisfying the following conditions:*

(i) *The zero function belongs to $\mathscr{L}_1$.*

(ii) $f\in\mathscr{L}_1$ *implies* $1-f\in\mathscr{L}_1$.

(iii) *For any sequence $f_1, f_2, \dots$ of members of $\mathscr{L}_1$ satisfying $f_i+f_j\leq 1$ for $i\neq j$ we have $f_1+f_2+\dots\in\mathscr{L}_1$.*

(iv) $\mathscr{P}^\circ\subset\mathscr{L}_1$.

Proof. It is clear that $\mathscr{P}^\circ$ satisfies the conditions (i) and (ii). From the Definition it follows that there is a set $\mathscr{N}$ isomorphic with $\mathscr{M}$, which is a full set of states on a σ-orthoposet $\mathscr{L}$. Let $\mathscr{N}'$ be the dual of $\mathscr{N}$. To each $\bar{a}\in\mathscr{N}'$ let f_a be the function on $\mathscr{M}$ defined by $f_a(m)=\bar{a}(m')$, where $m\mapsto m'$ is the isomorphism of $\mathscr{M}$ onto $\mathscr{N}$. Let $\mathscr{L}_1$ be the set of all such functions. Since by [8], Theorem 1,

$\mathcal{N}'$ satisfies (i)—(iii), so does $\mathcal{L}_1$. To show (iv), let $\bar{c} \in \mathcal{P}^\circ$. Then $\bar{c}(m) = m(c) = m'(\tau c)$. As $\tau c \in \tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}$ is isomorphic with $\mathcal{L}$, there is a $c' \in \mathcal{L}$ such that $n(\tau c) = n(c')$ for all $n \in \mathcal{N}$. Thus we ge $m'(\tau c) = m'(c') = \bar{c}'(m') = f_{c'}(m)$, i.e. $\bar{c} = f_{c'} \in \mathcal{L}_1$.

The following theorem is the consequence of Theorem 1.

Theorem 2. *The necessary condition for imbedding the Segal system $\mathcal{X}$ into a Mackey system is as follows:*

(α) If $a_1, a_2, \ldots$ is a sequence of elements of $\mathcal{P}$ such that $a_i + a_j \leq I$ for $i \neq j$ and if there is an $a \in \mathcal{X}$ such that $m(a) = \sum_{i=1}^{\infty} m(a_i)$ for all $m \in \mathcal{M}$, then $a \in \mathcal{P}$.

Proof. Let $\mathcal{L}_1 \subset [0, 1]^{\mathcal{M}}$ be the set satisfying the conditions (i)—(iv). Let $a_1, a_2, \ldots$ be a sequence of elements of $\mathcal{P}$ satisfying $a_i + a_j \leq I$ for $i \neq j$ and let $m(a) = \sum_{i=1}^{\infty} m(a_i)$ for all $m \in \mathcal{M}$, where $a \in \mathcal{X}$. Then by (iv) from Theorem 1 we have $\bar{a}_1, \bar{a}_2, \ldots \in \mathcal{L}_1$ and by (iii) $\bar{a}_i + \bar{a}_2 + \ldots \in \mathcal{L}_1$. If we set $\bar{a}(m) = m(a)$ for all $m \in \mathcal{M}$, then $\bar{a} = \bar{a}_1 + \bar{a}_2 + \ldots \in \mathcal{L}_1$. As $\mathcal{L}_1$ is isomorphic with the set $\mathcal{L}$ of all propostion observables, we have $m(a) = \bar{a}(m) = m'(b) = m'(x_b)$, for some $b \in \mathcal{L}$, i.e. by (iii) from the Definition, $x_b = \tau a$, so that by (ii) from the Definition, a is an idempotent.

It can be shown that if (α) is fulfiled, then a is the supremum of $a_1, a_2, \ldots$ in $\mathcal{P}$, i.e. $a = a_1 \cup a_2 \cup \ldots$. Indeed, $a_i \leq a$ for all $i = 1, 2, \ldots$ and if $g \in \mathcal{P}$ such that $a_i \leq g$, $i = 1, 2, \ldots$ then $a_i + I - g \leq I$, $i = 1, 2, \ldots$. Since $m(a + I - g) = \sum_{i=1}^{\infty} m(a_i) + m(I - g)$ for all $m \in \mathcal{M}$, we have by (α) that $a + I - g \in \mathcal{P}$, but then $a + I - g \leq I$, i.e. $a \leq g$. For the finite sequence $a_1, a_2, \ldots, a_n$ of orthogonal elements of $\mathcal{P}$ we get that $a_1 \cup a_2 \cup \ldots \cup a_n = a_1 + a_2 \ldots + a_n \in \mathcal{P}$.

Theorem 3. *Condition (α) from* Theorem 2 *is equivalent to the following two conditions:*

(β) If $a, b \in \mathcal{P}$ such that $a + b \leqq I$, then $a + b = a \cup b \in \mathcal{P}$.

(γ) If $s_1, s_2, \ldots \in \mathcal{P}$ such that $s_1 \leq s_2 \leq \ldots$ and $\lim_{n \to \infty} m(s_n) = m(s)$ for each $m \in M$, where $s \in \mathcal{X}$, then $s \in \mathcal{P}$.

Proof. Let α) be fulfilled. Then (β) is obvious. Let $s_1 \leq s_2 \leqq \ldots$ be a sequence of elements of $\mathcal{P}$ such that $\lim_{n \to \infty} m(s_n) = m(s)$ for each $m \in \mathcal{M}$, where $s \in \mathcal{X}$. Let us set $a_1 = s_1$, $a_s = s_n - s_{n-1}$ for $n = 2, 3, \ldots$. From $s_{n-1} \leq s_n$ it follows that $s_{n-1} + I - s_n \leq I$ and by (α) $s_{n-1} + I - s_n \in \mathcal{P}$. Then also $I - (s_{n-1} + I - s_n) = s_n - s_{n-1} \in \mathcal{P}$ and $\sum_{i=1}^{\infty} m(a_i) = \lim_{n \to \infty} \sum_{i=1}^{n} m(a_i) = \lim_{n \to \infty} m(s_n) = m(s)$ for all $m \in \mathcal{M}$. Since $m(s_n) \leq 1$ for each m, there is also $m(s) = \lim_{n\to\infty} m(s_n) \leqslant 1$ for each m, i.e. $s \leqslant I$. Then $a_k + a_l \leqslant s \leqslant I$ and by α), $s \in \mathcal{P}$.

Now let (β) and (γ) be fulfilled. Let $a_1, a_2, \ldots$ be a sequence of mutually orthogonal elements from $\mathcal{P}$ and let $m(a) = \sum_{i=1}^{\infty} m(a_i)$ for all $m \in \mathcal{M}$, where $a \in \mathcal{X}$. Then $s_n = a_1 + a_2 + \ldots + a_n \in \mathcal{P}$. Indeed, by β) we have $a_1 + a_2 \in \mathcal{P}$ and $a_1 + a_2 = a_1 \cup a_2$. Now we proceed by induction. Let $a_1 + a_2 + \ldots + a_{n-1} = a_1 \cup a_2 \cup \ldots \cup a_{n-1} \in \mathcal{P}$. Then $a_i \leqslant I - a_n$ for $i = 1, 2, \ldots, n-1$ imply $b = a_1 + a_2 + \ldots + a_{n-1} \leqslant I - a_n$, so that $b + a_n \leqslant I$ and again by (β), $b + a_n = b \cup a_n \in \mathcal{P}$. Since $s_1 \leqslant s_2 \leqslant$ and $\lim_{n\to\infty} m(s_n) = m(a)$ for each $m \in \mathcal{M}$, we have by (γ) that $a \in \mathcal{P}$.

Lemma. *Let $\mathcal{X}$ be a Segal system and $\mathcal{P}$ its set of idempotents. Them $a, b \in \mathcal{P}$, $a + b \leqslant I$ imply $a \cap b = 0$.*

Proof. Let $0 = g \in \mathcal{P}$ be such that $g \leqslant a$, $g \leqslant b$. $a + b \leqslant I$ implies $b \leqslant I - a$, so that $g \leqslant a$ and $g \leqslant I - a$. Let $m \in \mathcal{M}$ be such that $m(g) = 1$. (From the properties of the Segal system it follows that such an m exists). Then $m(g) = 1$ implies $m(a) = 1$ and $m(I - a) = 1$, which is impossible. Thus $g = 0$.

Some authors [5, 13] have considered another form of the formal product instead of the Segal form $u \circ v = \frac{1}{4}[(u+v)^2 - (u-v)^2]$. The other form is $u \circ v = \frac{1}{2}[(u+v)^2 - u^2 - v^2]$. In the distributive Segal system both forms are equivalent. In the following we shall consider Segal system with the latter form of the formal product. In such systems if $a, b \in \mathcal{P}$ such that $a + b \leqslant I$, then $a + b \in \mathcal{P}$ is identical with $a \circ b = \frac{1}{2}[(a+b)^2 - a^2 - b^2] = 0$ and this is equivalent to $a \circ b = a \cap b$.

Theorem 4. *In the Segal system with the formal product defined by $a \circ b = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$ the condition (β) in* Theorem 3 *is equivalent to the following condition.*

δ) If a, b, c, are pairwise orthogonal elements in $\mathcal{P}$, then $(a+b) \circ c = a \circ c + b \circ c = 0$.

Proof. Let $a, c \in \mathcal{P}$ be such that $a + c \leqslant I$. As 0 is orthogonal to all elements in $\mathcal{P}$, we have $(a + 0) \circ c = a \circ c = 0$, from which it follows that $(a+c)^2 = a + c$. Now let $a + b \leqslant I$ and let $d \in \mathcal{P}$ be such that $a \leqslant d$, $b \leqslant d$. Then a, b, $I - d$ are mutually orthogonal. Consequently $(a+b) \circ (I - d) = 0$, from which $a + b + I - d \in \mathcal{P}$. But then $a + b + I - d \leqslant I$, i.e. $a + b \leqslant d$. Thus we get $a \cup b = a + b$, i.e. (δ) $\Rightarrow$ (s).

Now let a, b, c be mutually orthogonal elements from $\mathcal{P}$. By (β) $a+b$, $a+c$, $b+c$ and $a+b+c$ are idempotents, from which $(a+b)\circ c = a\circ c+b\circ c=0$.

Delyiannis [7] has shown that condition β) from Theorem 3 is fulfilled in all distributive Segal systems. He has given also an example of a non-distributive system which can be imbedded into a Mackey system. From this it follows that distributivity is not necessary for the imbedding. In his counterexample (example 2 in [7]) the only sets of pairwise orthogonal idempotents are $(0, a, I-a)$, where $a\in\mathcal{P}$. Such sets commute so that the systems generated by them are distributive and the condition β) is fulfilled. From this we see that distributivity is satisfactory for the validity of (β), but there are non-distributive systems in which (β) is also fulfilled. On the other hand, an example of a non-distributive system (a Sherman counterexample [12]) is given in [1], in which (β) is clearly not fulfilled.

Finally we show a property of the distributive systems which gives a partial answer to the question mentioned in [5].

Theorem 5. *Let $\mathcal{X}$ be a distributive Segal system. Let $a, b\in\mathcal{P}$. Then $a\circ b = a\cap b$ if and only if $a\leftrightarrow b$. In this case, $a\cup b = (a+b) - (a\circ b)$.*

Proof. Let $a\circ b=a\cap b$. Let us set $a=(a-a\circ b)+a\circ b$, $b=(b-a\circ b)+a\circ b$. As β) is valid in a distributive system, from $a\circ b\leqslant a$ we have $a-a\circ b\in\mathcal{P}$ and, analogously, $b-a\circ b\in\mathcal{P}$. As $a\leqslant I$ and $b\leqslant I$, $a\circ b$ is orthogonal to $(a-a\circ b)$ and to $(b-a\circ b)$, so that $a=(a-a\circ b)\cup a\circ b$, $b=(b-a\circ b)\cup a\circ b$. We have to show that $(a-a\circ b)+(b-a\circ b)\leqslant I$. But $(a-a\circ b)+(b-a\circ b) = a+b-2(a\circ b) = (a-b)^2$. From the properties of Segal system it follows that $\|a^2-b^2\|\leqslant\max(\|a\|^2, \|b\|^2)$. Then $\|(a-b)^2\|=\|a-b\|^2= \|a^2-b^2\|^2\leqslant 1$, so that $(a-b)^2\leqslant I$. Now let $a\leftrightarrow b$. Then there exists a_1, b_1, $c\in\mathcal{P}$, mutually orthogonal and such that $a=a_1\cup c = a_1+c$ and $b = b_1\cup c = b_1+c$. It can be easily seen that (β) implies the orthomodularity property. Indeed, let $x, y\in\mathcal{P}$ be such that $x\leqslant y$, then x is orthogonal to $I-y$ and $x+I-y = x\cup(I-y)\in\mathcal{P}$. From $x + (I-[x+(I-y)]) \leqslant I$ It follows that $y=x\cup(I-[x\cup(I-y)])=x\cup(x\cup y')'$. Then by [14] $c=a\cap b$. On the other hand, from the distributivity and Theorem 4 it follows that $a\circ b=(a_1+c)\circ(b_1+c)=c$.

Now we have to show that $a\cup b=a+b-a\circ b$. The condition (β) and the distributivity imply that from $x\leqslant y$, where $x, y\in\mathcal{P}$, it follows that $x\circ(I-y)= x-x\circ y=0$, i.e. $x\circ y=x$. Consequently, $a\circ(b-a\circ b) = a\circ b-a\circ b=0$, i.e. $a+(b-a\circ b)\in\mathcal{P}$. Clearly, $a\leqslant a + (b-a\circ b)$ and $b\leqslant b + (a-a\circ b)$. Now let $g\in\mathcal{P}$ be such that $a\leqslant g$, $b\leqslant g$. Then $a\circ b=a\cap b\leqslant g$, so that $[(a+b) - a\circ b)]\circ(I-g) = (a+b)\circ(I-g) - (a\circ b)\circ(I-g) = 0$, which implies $(a+b-a\circ b) + (I-g)\in\mathcal{P}$, i.e. $a+b-a\circ b\leqslant g$.

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ЗАМЕЧАНИЕ О СРАВНЕНИИ МОДЕЛЕЙ СИГАЛА И МАККИ

Сильвия Пулманнова

Резюме

В данной статье исследуются две системы аксиом для квантовой механики: система Сигала и система Макки. В работе показазаны необходимые условия для включения системы Сигала и систему Макки.