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THE SCHAUDER-TYCHONOFF FIXED POINT THEOREM AND APPLICATIONS

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1. Introduction

This paper has two main sections both concerned with the Schauder—Tychonoff fixed point theorems. The second section deals with an application of the strong version of the Schauder fixed point theorem to find criteria for the existence of "positive" solutions of non-linear vector ordinary differential equations. In the first section we give an exposition of the Schauder—Tychonoff fixed point theorems and prove a new strong version of the Tychonoff fixed point theorem.

2. Fixed point theorems

We begin by stating Schauder's theorem.

Schauder's Theorem. *Let A be a compact convex subset of a Banach space and f a continuous map of A into itself. Then f has a fixed point.*

This theorem is a special case of Tychonoff's theorem.

Tychonoff's Theorem. *Let A be a compact convex subset of a locally convex (linear topological) space and f a continuous map of A into itself. Then f has a fixed point.*

As noted in Dugundji [2], Tychonoff's fixed point theorem is not immediately applicable in analysis because the domain is required to be compact, a situation rarely met in practice.

However using Mazur's lemma

Mazur's Lemma. *The closed convex hull of a compact subset of a Banach space is compact.*

Schauder's theorem can be extended to say

Strong version of Schauder's Theorem. *Let A be a closed convex subset of a Banach space and f a continuous map of A into a compact subset of A. Then f has a fixed point.*

This version is particularly useful, an example is given in §3, and so we are led to ask: Is there a "strong version" of the Tychonoff fixed point theorem? More precisely, can the word "Banach" in the strong version of Schauder's theorem be replaced by "locally convex space"? Even the extension from "Banach space" to "normed vector space" is not obvious since Mazur's lemma would be false if "Banach" were replaced by "normed vector space" in it. Nevertheless by modifying Tychonoff's original proof [7] we obtain our main

Theorem A. *Let A be a convex subset of a locally convex space and f a continuous map of A into a compact subset of A. Then f has a fixed point.*

This theorem certainly generalizes all the previously stated theorems. Indeed, even in the case of a Banach space it is stronger than the strong version of Schauder's theorem since we do not have to assume that the set A is closed.

We now prove the theorem. We begin with some preliminary results — the first of which is well-known and appears in [6].

Theorem 2.1. *Every n-dimensional Hausdorff linear topological space is linearly homeomorphic to the Euclidean n-space E^n.*

Corollary 2.2. *Given $x_1, \ldots, x_n \in R$, a Hausdorff linear topological space, the set*

$$S = \{x : x = \sum_{i=1}^{n} \alpha_i x_i,\ \sum_{i=1}^{n} \alpha_i = 1,\ \alpha_i \geq 0\}$$

with the induced topology is homeomorphic to a closed simplex in E^n of dimension $r \leq n - 1$.

The lemma below follows easily from the Brouwer fixed point theorem together with Corollary 2.2 and its proof is therefore omitted.

Lemma 2.3. *Let the closed simplex L in a Hausdorff linear topological space be the union of a finite number of disjoint closed simplicies with vertices $g_1, \ldots, g_n$. Further, let φ be a mapping of $\{g_1, \ldots, g_n\}$ into L. Then the linear interpolation Φ of φ given by $\Phi(\sum_{i=1}^{n} \alpha_i g_i) = \sum_{i=1}^{n} \alpha_i \varphi(g_i)$ is a continuous mapping of L into itself. Also there exists a point x in L such that $\Phi(x) = x$.*

Note that the closed simplices S_1 and S_2 are said to be disjoint if their intersection is their common face (which may be the empty set).

Definition. *Let $\{W_\alpha : \alpha \in I\}$ be an open covering of a subset A of a topological space. Then the open covering $\{U_\beta : \beta \in J\}$ of A is said to be a two-fold refinement of $\{W_\alpha : \alpha \in I\}$ if for each $\beta \in J$ there is an $\alpha \in I$ such that $U_\beta \subseteq W_\alpha$ and if $U_{\beta'} \cap U_\beta \neq \emptyset$, $\beta' \in J$, then $U_{\beta'} \subseteq W_\alpha$.*

The following lemma, whilst not explicitly stated, can be found in [7].

Lemma 2.4. *If A is a compact subset of a regular topological space R and $\{W_\alpha : \alpha \in I\}$ is an open covering of A, then there exists a two-fold refinement $\{U_1, \dots, U_p\}$ of $\{W_\alpha : \alpha \in I\}$. In particular this is the case when R is a linear topological space.*

Proof. As A is compact there is a finite subcovering $\{W_1, \dots, W_n\}$ of $\{W_\alpha : \alpha \in I\}$. Since R is regular, for each point $x \in A$ we can choose a neighbourhood $V(x)$ of x such that $\overline{V(x)} \subseteq W_i$, for some i. Choose a finite subcovering $\{V_1, \dots, V_m\}$ of $\{V(x) : x \in A\}$.

$$\text{(1)} \qquad \text{Put } U(x) = [\bigcap_{x \in V_k} V_k] \cap [\bigcap_{x \in R - \overline{V}_k} (R - \overline{V}_k)] \cap [\bigcap_{x \in W_k} W_k] .$$

Clearly $U(x)$ is a neighbourhood of x. Further, there are only a finite number of distinct $U's$.

Now consider $U(x)$. Choose a V_k out of the first intersection in (1). Then $U(x) \subseteq V_k \subseteq W_i$, for some i. Let $U(x') \cap U(x) \neq \emptyset$, for some $x' \in A$. Then $x' \in \overline{V}_k$, since if $x' \in A - \overline{V}_k$, $U(x') \subseteq A - \overline{V}_k$ which implies $U(x) \cap U(x') = \emptyset$. Thus $x' \in \overline{V}_k \subseteq W_i$, which implies $U(x') \subseteq W_i$.

The following lemma can easily be proved using Ch. XX §2 of [4] and its proof is therefore omitted.

Lemma 2.5. *If $\{V_1, \dots, V_k\}$ form an open covering of the closed simplex L in a Hausdorff linear topological space, then L can be expressed as the union of a finite number of disjoint closed simplices $S_1, \dots, S_m$ such that each $S_k \subseteq V_i$ for some i.*

Theorem 2.6. (Strong version of the Tychonoff theorem)

Let f be a continuous mapping of a convex subset F of a Hausdorff locally convex linear topological space R into a compact subset A of F. Then there is at least one fixed point.

Proof. We will work in the induced topology on F. Thus by a neighbourhood of $x \in F$ we mean a subset of F which contains x and is open in the induced topology on F.

Suppose there are no fixed points; that is $y = f(x) \neq x$ for all x in F. Since R is Hausdorff there are neighbourhoods $W(x)$ and $W(y)$ of x and y respectively such that

$$\text{(1)} \qquad W(x) \cap W(y) = \emptyset$$

Let $G(x) = f^{-1}[W(y)] \cap W(x)$. Then $G(x)$ is a neighbourhood of x, $G(x) \subseteq W(x)$ and $f[G(x)] \subseteq W(y)$. Hence by (1) we have that

$$\text{(2)} \qquad f[G(x)] \cap G(x) = \emptyset .$$

Since R is locally convex and F is convex, there exists a convex neighbourhood $C(x)$ of x such that $C(x) \subseteq G(x)$. Clearly then by (2)

$$f[C(x)] \cap C(x) = \emptyset\,. \tag{3}$$

For each point $x \in A$ choose a corresponding $C(x)$. Then $\{C(x) : x \in A\}$ is an open covering of A. By Lemma 2.4 there exists a two-fold refinement $\{U_1, \dots, U_p\}$ of $\{C(x) : x \in A\}$. Without loss of generality $U_i \cap A \neq \emptyset$, $i = 1, \dots, p$. Since f is continuous, $\{V_i, i = 1, \dots, p : V_i = f^{-1}(U_i)\}$ is an open covering of F. Without loss of generality $V_i \neq \emptyset$ for any i.

Let L be the convex hull of $\{x_1, \dots, x_p\}$, where x_i is an arbitrarily chosen point of U_i for each $i = 1, \dots, p$. By Corollary 2.2, L is homeomorphic to an r-dimensional closed simplex ($r \leq p - 1$). Since F is convex, $L \subseteq F$ and thus $\{V_i, i = 1, \dots, p\}$ is an open covering of L. Then by Lemma 2.5, L can be expressed as the union of disjoint closed simplices $S_i^j, j = 1, \dots, \mathrm{l}(i)$; $i = 1, \dots, p$, where $S_i^j \subseteq V_i$ for all $j = 1, \dots, \mathrm{l}(i)$. Thus

$$f(S_i^j) \subseteq f(V_i) \subseteq U_i, \quad j = 1, \dots, \mathrm{l}(i). \tag{4}$$

Let $g_1, \dots, g_n$ be the vertices of these closed simplices. Define a mapping φ of $\{g_1, \dots, g_n\}$ into L in the following manner:

There exists a U_k such that $f(g_i) \in U_k$. Choose one such U_k and let $\varphi(g_i) = x_k$, where x_k is the previously chosen element of U_k. By Lemma 2.3 the linear interpolation Φ of φ maps L into itself and has a fixed point x_0 in L.

We now show that for each x in L there is a $C \in \{C(x) : x \in A\}$ such that $f(x)$ and $\Phi(x)$ are in C. Then, in particular $f(x_0) \in f(C) \cap C$ which contradicts (3).

Let $x \in L$. Without loss of generality $x \in S_1^1$ with vertices $g_1, \dots, g_m$. Now by (4) $f(S_1^1) \subseteq U_1$ which implies $f(x)$ and $f(g_i) \in U_1$. By definition of φ, $f(g_i)$ and $\varphi(g_i) \in U_k$, for some k. Denote this U_k by U^i. Then $f(g_i) \in U_1 \cap U^i$. Thus $U_1 \cap U^i \neq \emptyset$ for $i = 1, \dots, m$. Since $\{U_1, \dots, U_p\}$ is a two-fold refinement of $\{C(x) : x \in A\}$, there exists $C \in \{C(x) : x \in A\}$ such that $U_1 \subseteq C$ and $U^i \subseteq C$ for all $i = 1, \dots, m$. Thus $\Phi(g_i) \in C$, $i = 1, \dots, m$ and since C is convex and Φ is linear, $\Phi(x) \in C$. That is $f(x) \in U_1 \subseteq C$ and also $\Phi(x) \in C$. The theorem is proved.

3. Non-oscillation criteria

Consider the quasi-linear vector differential equation

$$Lu = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}(t, u)\frac{\mathrm{d}u}{\mathrm{d}t}\right) + \boldsymbol{F}(t, u)u\,, \quad t \geq 0 \tag{1}$$

where u is a vector function with values $u(t) \in E^m$, $t \geq 0$. The coefficients

$\mathbf{A}(t, \xi)$, $\mathbf{F}(t, \xi)$ are $m \times m$ functions of class $C^1([0, \infty) \times E^m)$. The matrix $\mathbf{A}(t, \xi)$ is assumed to be non-singular for all $(t, \xi) \in [0, \infty) \times E^m$.

In the oscillation theory of differential equations one of the important problems is to find sufficient conditions on the coefficients of the differential operator for the existence of non-oscillatory solutions; that is, solutions $u(t)$ which are eventually positive.

In this section we use the strong version of the Schauder fixed point theorem to obtain sufficient conditions for the existence of non-oscillatory solutions of equation (1). The method used is a generalization of a method used by Domslak [1].

A natural way of generalizing the notion of a non-oscillatory scalar function to vector functions is to require one or more of the components of the vector function to be eventually positive. This leads us to the following definition.

Definitions. *Let h be a non-zero vector in E^m. A vector solution $u(t)$ is said to be h-non-oscillatory in $[0, \infty)$ if for some $t_1 \geq 0$ the scalar function $(u(t), h) < 0$ for all $t \geq t_1$, where $(,)$ denotes the usual scalar product in E^m.*

Let $C(0, \infty; E^m)$ denote the vector space of all continuous vector valued functions on $(0, \infty)$ with values in E^m. Let $|\ |$ be the norm defined on C by $|u| = \sup_{t \geq 0} \| u(t)\|$, where $\|\ \|$ is the Euclidean norm in E^m.

Theorem B. *The differential equation* (1) *has h-non-oscillatory solutions if the matrices $\mathbf{A}^{-1}$ and $\mathbf{F}$ satisfy the following conditions*:

(i) *for any non-zero vector $u \in E^m$, there exist positive constants K_1, K_2 which may depend on u, such that for all v $\|v\| \leq \|u\|$ and $t \geq 0$.*
   (a) $\|\mathbf{A}^{-1}(t, v)\| \leq K_1 \|\mathbf{A}^{-1}(t, u)\|$
   (b) $\|\mathbf{F}(t, v)\| \leq K_2 \|\mathbf{F}(t, u)\|$

(ii) $\int_0^\infty \int_0^S \|\mathbf{A}^{-1}(t, h)\| \, \|\mathbf{F}(s, h)\| \, ds \, dt < \infty$.

Proof. Consider the integral equation

$$u(t) = \frac{h}{2} + \left(\int_{t_1}^{t} \mathbf{A}^{-1}(s,u(s))ds\right)\int_t^\infty \mathbf{F}(s, u(s))u(s)\, ds)$$

$$+ \int_{t_1}^{t}\int_{t_1}^{s} \mathbf{A}^{-1}(\tau,u(\tau))\, \mathbf{F}(s, u(s))u(s)\, ds\, d\tau.$$

Every solution of this integral equation satisfies the differential equation (1).

Let S_{t_1} be the closed convex subset of $C(0, \infty; E^m)$ defined by

$$S_{t_1} = \{u(t) \in C : (u(t), h) \geq \frac{\|h\|^2}{4} < 0, \quad \|u\| \leq \|h\| \text{ for } t \geq t_1\}.$$

Define the operator $T : C(0, \infty; E^m) \to C(0, \infty; E^m)$ by

$$Tu = \frac{h}{2} + (\int_{t_1}^{t} \mathbf{A}^{-1}(s, u(s))\, ds)(\int_{t}^{\infty} \mathbf{F}(s, u(s))u(s)\, ds)$$

$$+ \int_{t_1}^{t}\int_{t_1}^{s} \mathbf{A}^{-1}(\tau, u(\tau))\mathbf{F}(s, u(s))u(s)\, ds\, d\tau$$

Then for any $u \in S_{t_1}$, we have using condition (i) that

$$||Tu|| \leq \frac{||h||}{2} + K_1K_2(\int_{t_1}^{t} ||\mathbf{A}^{-1}(s, h)||\, ds)(\int_{t}^{\infty} ||\mathbf{F}(s, h)||\, ds)\, ||h||$$

$$+ K_1K_2(\int_{t_1}^{t}\int_{t_1}^{s} ||\mathbf{A}^{-1}(\tau, h)||\, ||\mathbf{F}(s, h)||\, ds\, d\tau)\, ||h||$$

where K_1, K_2 are positive constants. Condition (ii) then shows that for sufficiently large t_1, $||Tu|| \leq ||h||$ for all $u \in S_{t_1}$.

We also have

$$(Tu, h) \geq \frac{||h||^2}{2} - K_1K_2||h||^2(\int_{t_1}^{t} ||\mathbf{A}^{-1}(s, h)||\, ds)\, (\int_{t}^{\infty} ||\mathbf{F}(s, h)||\, ds)$$

$$- K_1K_2||h||^2 \int_{t_1}^{t}\int_{t_1}^{s} ||\mathbf{A}^{-1}(\tau, h)||\, ||\mathbf{F}(s, h)||\, d\tau\, ds.$$

By condition (ii), then t_1 can be chosen sufficiently large that $(Tu, h) > \dfrac{||h||^2}{4}$.

Hence T is a continuous transformation of S_{t_1} into itself. Using condition (ii) it is easily verified that $T(S_{t_1})$ is precompact. So T has at least one fixed point in S_{t_1}. This means that the integral equation has a solution in S_{t_1} which is a h-non-oscillatory solution of the differential equation (1). This completes the proof of Theorem B.

Remark. In the special case $\mathbf{A}(t, \xi) = I$, the identity matrix and $\mathbf{F}(t, \xi)$ is of the form $\mathbf{F}(t, \xi) = f(t, \xi)I$ our theorem reduces to a recent result of Domšlak [1]. Actually. Domšlak also requires that $f(t, 0) = 0$ for all t and $f(t, \xi) > 0$ for $\xi \neq 0$.

The following example is one where Domšlak's result is not applicable but ours is.

Example. Consider the fourth order non-linear differential equation

$$(2) \qquad (p(x, u)u'')'' + q(x, u)u = 0, \qquad p(x, u) > 0$$

Let $u = y_1$, $p(x, u)u'' = y_2$, $v = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then (2) is equivalent to the vector equation

$$v'' + \begin{pmatrix} 0 & , -\dfrac{1}{p(x, u)} \\ q(x, u), & 0 \end{pmatrix} v = 0.$$

$$\text{Let } g(x, u) = \left\| \begin{pmatrix} 0 & , -\dfrac{1}{p(x, u)} \\ q(x, u), & 0 \end{pmatrix} \right\|$$

$$= \max \left\{ |q(x, u)|, \quad \frac{1}{p(x, u)} \right\}$$

Write $g(x, u) = g(x, ev)$, where e is the row vector $(1,0)$.

Condition (i) in the above theorem becomes: for any non-zero vector $u \in E^m$, there exists a constant $k > 0$ such that for all v such that $\|v\| \leq \|u\|$

$$g(x, ev) \leq g(x, eu) \tag{3}$$

If we choose $h = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $(v, h) = u$. Hence by Theorem B, equation (2) will have bounded non-oscillatory solutions if condition (3) is satisfied and

$$\int_0^\infty xg(x, 1)\, \mathrm{d}x < \infty.$$

Remarks. For the class of differential equations satisfying conditions (a) and (b) of Theorem B, condition (ii) is not a necessary condition for the existence of h-non-oscillatory solutions. For example, let $\boldsymbol{A} = \boldsymbol{I}$, the identity matrix, $\boldsymbol{F} = \dfrac{1}{8t^2}\boldsymbol{I}$ and $h = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then condition (ii) is not satisfied but we still have an h-non-oscillatory solution, by Hille [3].

However condition (ii) of Theorem B cannot be replaced by the weaker condition (ii') $\int_0^\infty \int_0^s (\boldsymbol{A}(t, h)h, h)\ |(\boldsymbol{F}(s, h)h, h)|\, \mathrm{d}t\, \mathrm{d}s < \infty$. For example, consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{pmatrix} 2, & -1 \\ 1, & 0 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right] + \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0.$$

Condition (ii') is satisfied trivially for $h = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. However there are no h-non-oscillatory solutions. To see this, suppose $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is an h-non-oscillatory solution.

Then u_1 is eventually a positive solution of the scalar equation $\frac{d^2u_1}{dt^2} + u_1 = 0$, which is clearly impossible.

Finally we note that necessary conditions of the integral type for the existence of h-non-oscillatory solutions are given in [5].

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