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THREE MAĽCEV TYPE THEOREMS AND THEIR APPLICATION

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The aim of this paper is to prove three Maľcev type theorems for three special properties of the congruence lattice, namely for weak n-distributivity, l-modularity and dual l-modularity. By means of the second and the third of these theorems we prove that any congruence l-modular equational class as well as any congruence dual l-modular equational class is a congruence modular equational class.

1. Preliminaries

In this paper we shall understand the fundamental notions of universal algebra in the sense of Grätzer's book [3]. We shall not distinguish between an algebra and its base set and between a polynomial symbol and the polynomial induced by it. The symbols $\vee$, $\wedge$ ($\cup$, $\cap$) will denote lattice (set-theoretic) operations.

Let A be an algebra and H be a subset of A. By $\Theta(H)$ we shall denote the smallest congruence relation of A containing $H \times H$. If H_i, $i = 0, 1, \ldots, n$, are subsets of the set A, then instead of $\bigvee_{i=0}^{n} \Theta(H_i)$ we shall write $\Theta(H_0; H_1; \ldots; H_n)$. In the case of $H_i = \{a_{i0}, \ldots, a_{im_i}\}$ we shall abbreviate this symbol to

$$\Theta(a_{00}, \ldots, a_{0m_0}; \ldots; a_{n0}, \ldots, a_{nm_n}).$$

By an equational class of algebras we shall always understand a nontrivial equational class. We shall use the symbol $F_K(X)$ for a free algebra over K with the generating family X. We shall often use the following theorem.

Theorem 1.1. [9, p. 64]. *Let K be an equational class of algebras and $X = \{e_0, e_1, \ldots, e_{n-1}\}$ be a set. Let φ be a permutation of the set $\{0, 1, 2, \ldots, n-1\}$. Let $x, y \in F_K(X)$ and p, q be polynomial symbols such that*

$$p(e_0, e_1, \ldots, e_{n-1}) = x \quad q(e_0, e_1, \ldots, e_{n-1}) = y.$$

Then the following conditions are equivalent.

(i) $(x, y) \in \Theta(e_{0\varphi}, \ldots, e_{h_1\varphi}; \ldots; e_{h_{s-1}\varphi}, \ldots, e_{h_s\varphi})$

(ii) *For every algebra* $A \in K$ *and for every* $a_0, a_1, \ldots, a_{n-1} \in A$ *satisfying*

$$a_{0\varphi} = a_{1\varphi} = \ldots = a_{h_1\varphi}$$
$$a_{(h_1+1)\varphi} = \ldots = a_{h_2\varphi}$$
$$\ldots\ldots$$
$$a_{(h_{s-1}+1)\varphi} = \ldots = a_{h_s\varphi}$$

the equation $p(a_0, a_1, \ldots, a_{n-1}) = q(a_0, a_1, \ldots, a_{n-1})$ *is true.*

2. Three Mal'cev type theorems

Definition 2.1. *Let n be a positive integer. A lattice L is said to be weakly distributive of the order n if for every $x, y_0, \ldots, y_n \in L$ the following identity*

$$x \wedge \bigvee_{i=0}^{n} y_i = \bigvee_{j=0}^{n} (x \wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i) \tag{2.1}$$

holds.

Remark. If the lattice is weakly distributive of the order n and modular it is said to be n-distributive. (See [5]).

It is easy to see that the weak distributivity of the order 1 and the usual distributivity coincide. We start with

Theorem 2.1. *Let n be a positive integer. For an equational class K of algebras the following two conditions are equivalent.*

(i) *For every algebra $A \in K$ the lattice of all congruences of A is weakly distributive of the order n.*

(ii) *There exist $(n+2)$-ary polynomial symbols $w_0, \ldots, w_k$ such that for every algebra $A \in K$ and every $a_0, \ldots, a_{n+1} \in A$ we have*

(W_1) $w_0(a_0, a_1, \ldots, a_{n+1}) = a_0 \quad w_k(a_0, a_1, \ldots, a_{n+1}) = a_{n+1}$

(W_2) $w_i(a_0, a_1, \ldots, a_n, a_0) = a_0$ for $0 \leqslant i \leqslant k$

$w_i(a_0, a_1, \ldots, a_1) = w_{i+1}(a_0, a_1, \ldots, a_1)$ *for* $i \equiv 0 \bmod (n+1)$

$\ldots\ldots$

$w_i(\underbrace{a_0, \ldots, a_0}_{j+1}, a_1, \ldots, a_1) = w_{i+1}(\underbrace{a_0, \ldots, a_0}_{j+1}, a_1, \ldots, a_1)$

(W_3) *for* $i \equiv j \bmod (n+1)$

$\ldots\ldots$

$$w_i(a_0, a_0, \ldots, a_0, a_1) = w_{i+1}(a_0, a_0, \ldots, a_0, a_1) \qquad \textit{for} \quad i \equiv n \bmod (n+1)$$

and for $0 \leqslant i < k$.

Remark. We can write (W_3) in a shorter form.

$$w_i(b_0, \ldots, b_{n+1}) = w_{i+1}(b_0, \ldots, b_{n+1})$$

for any i, j satisfying $i = j \bmod (n+1)$, $0 \leqslant j \leqslant n$, $0 \leqslant i < k$ such that $b_0 \quad b_1 = \ldots = b_j = a_0$ and $b_{j+1} = \ldots = b_{n+1} = a_1$.

Proof. First we shall prove (i) $\rightarrow$ (ii). Consider a free algebra $F_K(X)$, where $X = \{e_0, e_1, \ldots, e_{n+1}\}$. We denote $\varphi = \Theta(e_0, e_{n+1})$ and $\psi_i = \Theta(e_i, e_{i+1})$ for $i \quad 0, 1, \ldots, n$. Then we have

$$(e_0, e_{n+1}) \in \varphi \wedge \bigvee_{i=0}^{n} \psi_i = \bigvee_{j-0}^{n} (\varphi \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} \psi_i)$$

so that

$$(e_0, e_{n+1}) \in \bigvee_{j=0}^{n} (\varphi \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} \psi_i).$$

Therefore there exist elements $d_0, d_1, \ldots, d_k \in F_K(X)$ with $d_0 = e_0$, $d_k = e_{n+1}$ such that

$$(d_m, d_{m+1}) \in \varphi \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} \psi_i \tag{2.2}$$

for $0 \leqslant m < k$, where $j \in \{0, 1, \ldots, n\}$ and $m \equiv j \bmod (n+1)$. Since $F_K(X)$ is generated by X, there are some $(n+2)$-ary polynomial symbols $w_0, w_1, \ldots, w_k$ such that $d_i = w_i(e_0, e_1, \ldots, e_{n+1})$ for $0 \leqslant i \leqslant k$. We prove that these polynomial symbols satisfy the condition (ii). Putting $i = 0$ or $i = k$, we get

$$w_0(e_0, \ldots, e_{n+1}) = e_0 \quad w_k(e_0, \ldots, e_{n+1}) = e_{n+1}. \tag{2.3}$$

Thus we have proved (W_1). (W_2) follows from Theorem 1.1 and $(d_m, d_{m+1}) \in \varphi = \Theta(e_0, e_{n+1})$, $(0 \leqslant m < k)$.

We have still to show (W_3). Let $0 \leqslant m < k$, $m = j \bmod (n+1)$, where $j \in \{0, 1, \ldots, \mathrm{n}\}$. Then, in accordance with (2.2),

$$(d_m, d_{m+1}) \in \bigvee_{\substack{i=0 \\ i \neq j}}^{n} \psi_i = \bigvee_{\substack{i=0 \\ i \neq j}}^{n} \Theta(e_i, e_{i+1}) = \Theta(e_0, e_1, \ldots, e_j; e_{j+1}, \ldots, e_{n+1}).$$

So we have

$$(w_m(e_0, \ldots, e_{n+1}),\ w_{m+1}(e_0, \ldots, e_{n+1})) \in$$

$$\in \Theta(e_0, e_1, \ldots, e_j;\ e_{j+1}, \ldots, e_{n+1})$$

and we get (W_3) by simple applying Theorem 1.1.

Conversely assume that the condition (ii) is valid. It is enough to show

$$\varphi \wedge \bigvee_{i=0}^{n} \psi_i \subset \bigvee_{j=0}^{n} (\varphi \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} \psi_i) \tag{2.4}$$

for any congruences $\varphi, \psi_0, \psi_1, \ldots, \psi_n$ of $A \in K$.

Lemma. *Let n be the integer from Theorem 2.1 and $A \in K$. Let α_i, $i = 0, 1, \ldots, n$ be reflexive relations on A having the substitution property with respect to all operations of A. If we denote*

$$\prod_{i=0}^{n} \alpha_i = \alpha_0 \,.\, \alpha_1 \,.\, \ldots \,.\, \alpha_n$$

and if φ is a congruence of A then we have

$$\begin{aligned} \varphi \cap \prod_{i=0}^{n} \alpha_i \subset\ & (\varphi \cap \alpha_1 \,.\, \alpha_2 \,.\, \ldots \,.\, \alpha_n) \,.\, (\varphi \cap \alpha_n^{1} \,.\, \ldots \,.\, \alpha_1^{-1}) \,. \\ & .\, (\varphi \cap \alpha_0^{-1} \,.\, \alpha_2 \,.\, \ldots \,.\, \alpha_n) \,.\, (\varphi \cap \alpha_0 \,.\, \alpha_n^{-1} \,.\, \ldots \,.\, \alpha_2^{1}) \,.\, \ldots \,. \\ & .\, (\varphi \cap \alpha_{j-1}^{-1} \,.\, \ldots \,.\, \alpha_0^{1} \,.\, \alpha_{j+1} \,.\, \ldots \,.\, \alpha_n) \,. \\ & .\, (\varphi \cap \alpha_0 \,.\, \ldots \,.\, \alpha_{j-1} \,.\, \alpha_n^{-1} \,.\, \ldots \,.\, \alpha_{j+1}^{-1}) \,.\, \ldots \,. \\ & .\, (\varphi \cap \alpha_{n-1}^{-1} \,.\, \ldots \,.\, \alpha_0^{1}) \,.\, (\varphi \cap \alpha_0 \,.\, \ldots \,.\, \alpha_{n-1}) \,. \\ & .\, (\varphi \cap \alpha_1 \,.\, \ldots \,.\, \alpha_n) \,.\, (\varphi \cap \alpha_n^{1} \,.\, \ldots \,.\, \alpha_1^{1}) \,.\, \ldots, \end{aligned} \tag{2.5}$$

where on the right-hand side there are $2k + 2$ factors and k is the integer from (ii).

Proof of the lemma. Let $(x, y) \in \varphi \cap \prod_{i=0}^{n} \alpha_i$. Then there are some elements $c_0, c_1, \ldots, c_{n+1} \in A$ such that $c_0 = x$, $c_{n+1} = y$, $(c_0, c_{n+1}) \in \varphi$ and $(c_i, c_{i+1}) \in \alpha_i$ for $i = 0, 1, \ldots$, n. Put $d_i = w_i(c_0, c_1, \ldots, c_{n+1})$ for $0 \leqq i \leqslant k$. By (W_1) we have $d_0 = c_0$ and $d_k = c_{n+1}$. $(c_0, c_{n+1}) \in \varphi$ and (W_2) imply

$$d_i = w_i(c_0, \ldots, c_{n+1})\, \varphi w_i(c_0, c_1, \ldots, c_n, c_0) = c_0 =$$

$$w_{i+1}(c_0, c_1, \ldots, c_n, c_0)\, \varphi w_{i+1}(c_0, c_1, \ldots, c_n, c_{n+1}) = d_{i+1}.$$

Therefore $(d_i, d_{i+1}) \in \varphi$ for $0 \leqslant i < k$. Further it is true that

$$d_i = w_i(c_0, \ldots, c_{n+1})\, \varphi w_i(c_0, c_1, \ldots, c_n, c_0) = c_0 = \tag{2.6}$$

$$= w_i(c_0, c_0, \ldots, c_0)\, \varphi w_i(\underbrace{c_0, \ldots, c_0}_{j+1}, c_{n+1}, \ldots, c_{n+1})$$

for any $0 \leqslant j < n+1$ and any $0 \leqslant i \leqslant k$.

Take now some i, $0 \leqslant i < k$. For $j \in \{0, 1, \ldots, n\}$ such that $i \equiv j \bmod (n + + 1)$ we have

$$d_i = w_i(c_0, \ldots, c_{n+1})\, \alpha_{j-1}^{-1} \cdot \ldots \cdot \alpha_0^{-1} \cdot$$

$$\cdot\, \alpha_{j+1} \cdot \ldots \cdot \alpha_n\, w_i(\underbrace{c_0, \ldots, c_0}_{j+1}, c_{n+1}, \ldots, c_{n+1})$$

because

$$\begin{gathered} (c_1, c_0) \in \alpha_0^{1} \\ \ldots\ldots \\ (c_j, c_0) \in \alpha_{j-1}^{-1} \cdot \ldots \cdot \alpha_0^{-1} \\ (c_{j+1}, c_{n+1}) \in \alpha_{j+1} \cdot \ldots \cdot \alpha_n \\ \ldots\ldots \\ (c_n, c_{n+1}) \in \alpha_n \end{gathered}$$

and α_i are reflexive and have the substitution property. The same reasoning and (W_3) imply

$$w_i(\underbrace{c_0, \ldots, c_0}_{j+1}, c_{n+1}, \ldots, c_{n+1}) =$$

$$= w_{i+1}(\underbrace{c_0, \ldots, c_0}_{j+1}, c_{n+1}, \ldots, c_{n+1})\, \alpha_0 \cdot \ldots \cdot \alpha_{j-1} \cdot$$

$$\cdot\, \alpha_n^{1} \cdot \ldots \cdot \alpha_{j+1}^{1}\, w_{i+1}(c_0, \ldots, c_{n+1}) = d_{i+1}.$$

Hence by (2.6) we obtain

$$(d_i, d_{i+1}) \in (\varphi \cap \alpha_{j\ 1}^{\ 1} \cdot \ldots \cdot \alpha_0^{-1} \cdot \alpha_{j+1} \cdot \ldots \cdot \alpha_n) \cdot$$

$$\cdot\, (\varphi \cap \alpha_0 \cdot \ldots \cdot \alpha_{j-1} \cdot \alpha_n^{1} \cdot \ldots \cdot \alpha_{j+1}^{-1})$$

for $i - j \bmod (n+1)$ and this implies (2.5).

Let us return to the proof of (2.4). It is clear that

$$\bigvee_{i-0}^{n} \psi_i = \bigcup_{s=n}^{\infty} \{\prod_{i-0}^{s} \alpha_i : \alpha_i \in \{\psi_0, \ldots, \psi_n\}\},$$

where on the right-hand side $\cup$ means the set-theoretic union. Thus

$$\varphi \wedge \bigvee_{i=0}^{n} \psi_i = \bigcup_{s=n}^{\infty} \{\varphi \cap \prod_{i=0}^{s} \alpha_i : \alpha_i \in \{\psi_0, \ldots, \psi_n\}\}.$$

Hence it is enough to show

$$\varphi \cap \prod_{i-0} \alpha_i \subset \bigvee_{j\ 0}^{n} (\varphi \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} \psi_i)$$

for any $\alpha_0, \ldots, \alpha_s \in \{\psi_0, \ldots, \psi_n)$ and for any $s \geqslant n$. We prove this statement by induction on s. Let $s = n$. Then by (2.5) $\varphi \cap \prod_{i-0}^{n} \alpha_i$ is contained in a superposition of relations of the form $\varphi \cap \beta_1 . \beta_2 . \ldots . \beta_n$, where $\beta_j \in \{\psi_0, \ldots, \psi_n\}$. For any such relation there exists ψ_j $(0 \leqslant j \leqslant n)$ such that $\psi_j \notin \{\beta_1, \ldots, \beta_n\}$. But then

$$\varphi \cap \beta_1 . \ldots . \beta_n \subset \varphi \wedge \bigvee_{\substack{i\ 0 \\ i \neq j}}^{n} \psi_j \subset \bigvee_{j-0}^{n} (\varphi \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} \psi_i).$$

So by transitivity

$$\varphi \cap \prod_{i=0}^{n} \alpha_i \subset \bigvee_{i=0}^{n} (\varphi \wedge \bigvee_{\substack{j=0 \\ i \neq j}}^{n} \psi_i).$$

Now let our result hold for some $s \geqslant n$. We prove its validity for $s + 1$. Consider $\varphi \cap \prod_{i=0}^{s+1} \alpha_i$, where $\alpha_i \in \{\psi_0, \ldots, \psi_n\}$. Denoting

$$\beta_0 = \prod_{i=0}^{s+1-n} \alpha_i$$

$$\beta_i = \alpha_{s+1-n+i} \quad \text{for} \quad 1 \leqslant i \leqslant n$$

we can write $\varphi \cap \prod_{i=0}^{s+1} \alpha_i = \varphi \cap \prod_{i=0}^{n} \beta_i$. The relations β_i are evidently reflexive and the have substitution property with respect to the operations of the considered algebra. So we can apply the lemma to the expression $\varphi \cap \prod_{i\ 0}^{n} \beta_i$. By (2.5) and by the definition of β_i we get that $\varphi \cap \prod_{i=0}^{n} \beta_i$ is a subset of the superposition of relations each of them being of the form $\varphi \cap \prod_{i\ 0}^{s} \gamma_i, \gamma_i \in \in \{\psi_0, \ldots, \psi_n\}$. The induction assumption and transitivity of $\bigvee_{j\ 0}^{n} (\varphi \wedge \bigvee_{\substack{i-0 \\ i \neq j}}^{n} \psi_i)$ imply that our relation holds true for $s + 1$. So (2.4) holds and therefore (ii) implies (i).

Remark 1. As a special case of Theorem 2.1 we get, for $n = 1$, Jónsson's theorem [6] characterizing classes of algebras with distributive congruence lattices.

Remark 2. A. Day [1] characterized modularity by Mal'cev type theorem. Thus n-distributivity can be characterized by Mal'cev type theorem as well.

Definition 2.2. *We shall call a lattice L l-modular if for every $x, y_1, y_2 \in L$*

$$(2.7) \qquad (x \vee y_1) \wedge (x \vee y_2) = x \vee ((x \vee y_1) \wedge (x \vee y_2) \wedge (y_1 \vee y_2))$$

holds. The lattice L is called dually l-modular if L satisfies the dual identity with (2.7).

Identity (2.7) and the dual identity have been introduced by McKenzie [7].

Theorem 2.2. *For an equational class K of algebras the following two conditions are equivalent.*

(i) *For every algebra $A \in K$ the lattice of all congruences of A is l-modular.*

(ii) *There exist 6-ary polynomial symbols $w_0, \ldots, w_n$ such that for every algebra $A \in K$ and for every $a, b, c, d, e, f \in A$ we have*

(L_1) $w_0(a, b, c, d, e, f) = a \quad w_n(a, b, c, d, e, f) = f$

(L_2) $w_i(a, a, a, b, b, a) = w_i(a, b, b, a, a, a) = a$ *for* $0 \leqslant i \leqslant n$

(L_3) $w_i(a, b, b, c, c, d) = w_{i+1}(a, b, b, c, c, d)$ *for* $0 \leqslant i < n$, *i odd*

$w_i(a, a, b, a, b, b) = w_{i+1}(a, a, b, a, b, b)$ *for* $0 \leqslant i < n$, *i even.*

Proof. (i) implies (ii). Consider the free algebra $F_K(X)$ where $X = \{e_0, e_1, \ldots, e_5\}$. If we put $\varphi = \Theta(e_1, e_2;\ e_3, e_4)$, $\psi_1 = \Theta(e_0, e_1;\ e_2, e_5)$ and $\psi_2 = \Theta(e_0, e_3;\ e_4, e_5)$ then

$$(e_0, e_5) \in (\varphi \vee \psi_1) \wedge (\varphi \vee \psi_2) = \\ = \varphi \vee ((\varphi \vee \psi_1) \wedge (\varphi \vee \psi_2) \wedge (\psi_1 \vee \psi_2)).$$

There exist elements $d_0, \ldots, d_n \in F_K(\mathrm{X})$ such that

$$(2.8) \qquad d_0 = e_0 \quad d_n = e_5$$

$$(2.9) \qquad (d_i, d_{i+1}) \in \varphi = \Theta(e_1, e_2;\ e_3, e_4) \quad \text{for} \quad 0 \leqslant i < n,\ i \text{ odd}$$

$$(2.10) \qquad (d_i, d_{i+1}) \in (\varphi \vee \psi_1) \wedge (\varphi \vee \psi_2) \wedge (\psi_1 \vee \psi_2) = \\ = \Theta(e_0, e_1, e_2, e_5;\ e_3, e_4) \wedge \Theta(e_0, e_3, e_4, e_5;\ e_1, e_2) \wedge \\ \wedge \Theta(e_0, e_1, e_3;\ e_2, e_4, e_5) \quad \text{for} \quad 0 \leqslant i < n,\ i \text{ even}.$$

Since $F_K(X)$ is generated by X, there exist 6-ary polynomial symbols $w_0, \ldots, w_n$ such that

$$(2.11) \qquad d_i = w_i(e_0, \ldots, e_5) \quad \text{for} \quad 0 \leqslant i \leqslant n.$$

The validity of (ii) then follows from (2.8), (2.9), (2.10) and (2.11) by using Theorem 1.1 analogously as in the proof of Theorem 2.1.

(ii) implies (i). We have to prove that any congruence lattice $C(A)$ fulfils the identity (2.7) for each $A \in K$. It is enough to show

$$(2.12) \qquad (\varphi \vee \psi_1) \wedge (\varphi \vee \psi_2) \subset$$
$$\subset \varphi \vee ((\varphi \vee \psi_1) \wedge (\varphi \vee \psi_2) \wedge (\psi_1 \vee \psi_2)).$$

For brevity denote by P the right-hand side of (2.12). Define the sequences $s_0, s_1, \ldots$ and $t_0, t_1, \ldots$ of relations on A in the following way:

$$s_0 = \psi_1 \quad s_k = s_{k-1} \cdot \varphi \cdot s_{k-1}$$
$$t_0 = \psi_2 \quad t_k = t_{k-1} \cdot \varphi \cdot t_{k-1}.$$

The relations s_i and t_j are for every i, j reflexive, symmetric and have the substitution property with respect to the operations on A. In addition we have $\varphi \vee \psi_1 = \bigcup_{i=0}^{\infty} s_i$,

$$\varphi \vee \psi_2 = \bigcup_{j=0}^{\infty} t_j \quad \text{and} \quad (\varphi \vee \psi_1) \wedge (\varphi \vee \psi_2) = \bigcup_{i,j=0}^{\infty} (s_i \cap t_j).$$

It is enough to show (by induction)

$$(2.13) \qquad s_i \cap t_j \subset P \quad \text{for every} \quad i, j.$$

For $i = 0$ and arbitrary j we have $s_0 \cap t_j = \psi_1 \cap t_j \subset \psi_1 \quad (\varphi \quad \psi_2) \subset P$. The same holds for $j = 0$ and arbitrary i. Now let (2.13) hold for $i = i_0 - 1$, $j = j_0$ and $i = i_0$, $j = j_0 - 1$. We shall show that this relations holds for $i = i_0$ and $j = j_0$. Let $(a, f) \in s_{i_0} \cap t_{j_0} = s_{i_0-1} \cdot \varphi \cdot s_{i_0-1} \cap t_{j_0-1} \cdot \varphi \cdot t_{j_0-1}$. Then there are elements $b, c, d, e \in A$ such that

$$(2.14) \qquad (a, b) \in s_{i_0-1}, \quad (b, c) \in \varphi, \quad (c, f) \in s_{i_0-1},$$
$$(a, d) \in t_{j_0-1}, \quad (d, e) \in \varphi, \quad (e, f) \in t_{j_0-1}.$$

Let $d_i = w_i(a, b, c, d, e, f)$ for $0 \leqslant i \leqslant n$. From (2.14) and ($L_2$) we have, for $0 \leqslant i \leqslant n$,

$$d_i = w_i(a, b, c, d, e, f) \, s_{i_0-1} \quad w_i(a, a, f, d, e, f)$$
$$d_i = w_i(a, b, c, d, e, f) \, t_{j_0-1} \quad w_i(d, b, c, d, e, e) \, \varphi$$
$$\varphi w_i(d, b, b, d, d, d) = d = w_i(d, d, d, d, d, d) \, \varphi$$
$$\varphi w_i(d, d, e, d, e, e) \, t_{j_0-1} \quad w_i(a, a, f, d, e, f).$$

Hence

$$(2.15) \qquad d_i (s_{i_0-1} \cap t_{j_0}) \, w_i(a, a, f, d, e, f).$$

Analogously we get

$$(2.16) \qquad w_i(a, a, f, d, e, f)\,(s_{i_0} \cap t_{j_0-1})\, w_i(a, a, f, a, f, f).$$

Using (2.15), (2.16) and (L_3) we have for every odd i, $0 \leqslant i < n$, $d_i(s_{i_0-1} \cap \cap t_{j_0})\, w_i(a, a, f, d, e, f)\,(s_{i_0} \cap t_{j_0-1})\, w_i(a, a, f, a, f, f) = w_{i+1}(a, a, f, a, f, a)\,.\,(s_{i_0} \cap t_{j_0-1})\, w_{i+1}(a, a, f, d, e, f)\,(s_{i_0-1} \cap t_{j_0})\,(s_{i_0-1} \cap t_{j_0}) d_{i+1}$.

Then the induction assumption implies that

$$(2.17) \qquad (d_i, d_{i+1}) \in P$$

holds for every i odd, $0 \leqslant i < n$.

For i even, $0 \leqslant i < n$, we have by (L_3)

$$d_i = w_i(a, b, c, d, e, f)\, \varphi w_i(a, b, b, d, d, f) = w_{i+1}(a, b, b, d, d, f)\, \varphi$$
$$\varphi d_{i+1}.$$

So $d_i \varphi d_{i+1}$. But $\varphi \subset P$ and therefore we have

$$(2.18) \qquad (d_i, d_{i+1}) \in P.$$

By (L_1) $d_0 = a$ and $d_n = f$. P is transitive. Thus (2.17) and (2.18) imply $(a, f) \in P$. So (2.13) is proved for $C(A)$, $A \in K$.

The proof of the following theorem is very similar to the proof of Theorem 2.2. Therefore we shall do it in a shorter form.

Theorem 2.3. *For an equational class K of algebras the following two conditions are equivalent.*

(i) *For every algebra $A \in K$ the lattice of all congruences of A is dually l-modular, i.e., for any congruences φ, ψ_1, ψ_2 of A*

$$(2.19) \qquad \varphi \wedge ((\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2) \vee (\psi_1 \wedge \psi_2)) =$$
$$= (\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2)$$

is true.

(ii) *There exist 7-ary polynomial symbols $w_0, \ldots, w_n$ such that for every algebra $A \in K$ and for every $a, b, c, d, e, f, g \in A$ we have*

(DL_1) $w_0(a, b, c, d, e, f, g) = a \quad w_n(a, b, c, d, e, f, g) = g$

(DL_2) $w_i(a, b, b, d, e, e, a) = a$ *for* $0 \leqslant i \leqslant n$

(DL_3) $w_i(a, a, a, a, a, b, b) = w_{i+1}(a, a, a, a, a, b, b)$ *for* $0 \leqslant i < n$, i *even*

$w_i(a, a, b, b, b, b, b) = w_{i+1}(a, a, b, b, b, b, b)$ *for* $0 \leqslant i < n$, i *odd*

Proof. (i) implies (ii). Consider $F_K(X)$, where $X = \{e_0, \ldots, e_6\}$. Put $\varphi = \Theta(e_0, e_6;\ e_1, e_2;\ e_4, e_5)$, $\psi_1 = \Theta(e_0, e_1, e_2, e_3, e_4;\ e_5, e_6)$ and $\psi_2 = \Theta(e_0, e_1;\ e_2, e_3, e_4, e_5, e_6)$. Then we can write

$$(e_0, e_6) \in \varphi \wedge ((\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2) \vee (\psi_1 \wedge \psi_2)) =$$

$$= (\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2).$$

The condition (ii) can be derived from this relation in the same way as in the case of Theorem 2.2.

(ii) implies (i). Suppose φ, ψ_1, ψ_2 are congruences of $A \in K$. We show that (2.19) is satisfied by these congruences. It is enough to show that the left-hand side of this relation is a subset of its right-hand side. Define sequences $s_0, s_1, \ldots$ and $t_0, t_1, \ldots$ of relations on A as follows:

$$s_0 = \psi_1 \cap \psi_2 \quad s_k = s_{k-1} . (\varphi \cap \psi_1) . s_{k-1}$$

$$t_0 = \psi_1 \cap \psi_2 \quad t_k = t_{k-1} . (\varphi \cap \psi_2) . t_{k-1}.$$

Then the expression on the left-hand side of (2.19) is equal to $\bigcup_{i,j=0}^{\infty} (\varphi \cap s_i . t_j)$.

If we denote the right-hand side of (2.19) by P, we show that for every integer i, j we have $\varphi \cap s_i . t_j \subset P$. We do it again by induction. It is easy to see that $\varphi \cap s_0 . t_j \subset P$ and $\varphi \cap s_i . t_0 \subset$ P for any i, j. Suppose $\varphi \cap s_{i_0-1} . . t_{j_0} \subset P$ and $\varphi \cap s_{i_0} . t_{j_0-1} \subset P$. Let $(a, g) \in \varphi \cap s_{i_0} . t_{j_0}$. Then there exist elements $b, c, d, e, f \in A$ satisfying the following relations

$$(a, b) \in s_{i_0-1}, \quad (b, c) \in \varphi \cap \psi_1, \quad (c, d) \in s_{i_0-1} \tag{2.20}$$

$$(d, e) \in t_{j_0-1}, \quad (e, f) \in \varphi \cap \psi_2, \quad (f, g) \in t_{j_0-1}$$

$$(a, g) \in \varphi.$$

If we put $d_i = w_i(a, b, c, d, e, f, g)$ for $0 \leqslant i \leqslant n$, then from (2.20) we can, similarly as in Theorem 2.2, derive

$$d_i(\varphi \cap s_{i_0-1} . t_{j_0})\, w_i(a, a, g, g, g, g, g) =$$

$$= w_{i+1}(a, a, g, g, g, g, g)\, (\varphi \cap t_{j_0} . s_{i_0-1})\, d_{i+1}$$

for $0 \leqslant i < n$, i odd. By the induction assumption we have

$$\varphi \cap s_{i_0-1} . t_{j_0} \subset P$$

and therefore

$$(\varphi \cap s_{i_0-1} . t_{j_0})^{-1} = \varphi \cap t_{j_0} . s_{i_0-1} \subset P.$$

Therefore $(d_i, d_{i+1}) \in P$.

For i even, $0 \leqslant i < n$, we obtain analogously

$$d_i(\varphi \cap t_{j_0-1} . s_{i_0})\, w_i(a, a, a, a, a, g, g) =$$

$$w_{i+1}(a, a, a, a, a, g, g)\, (\varphi \cap s_{i_0} . t_{j_0-1})\, d_{i+1}$$

and this implies $(d_i, d_{i+1}) \in P$. Since $d_0 = a$ and $d_n = g$, we can write $(a, g) \in \in P$.

Remark. Theorems 2.1, 2.2 and 2.3 have been first proved in [8].

3. l-modularity and dual l- modularity in equational classes

A. Day [2] has proved the following result.

Theorem 3.1. *If the congruence lattice of every algebra A of an equational class K is p-modular, then the congruence lattice of every algebra $A \in K$ is modular.*

The proof of this theorem is based on the Mal'cev type theorem characterizing a congruence lattice as p-modular, which has been proved by E. Gedeonova in [4] and in the following theorem.

Theorem 3.2. *For an equational class K of algebras the following two conditions are equivalent.*

(i) *For every algebra $A \in K$ the lattice of all congruences of A is modular.*

(ii) *There exist 4-ary polynomial symbols $m_0, \ldots, m_n$ such that for every algebra $A \in K$ and for every $a, b, c, d \in A$ we have*

(M1) $m_0(a, b, c, d) = a \quad m_n(a, b, c, d) = d$
(M2) $m_i(a, b, b, a) = a \quad \text{for} \quad 0 \leqslant i \leqslant n$
(M3) $m_i(a, b, b, d) = m_{i+1}(a, b, b, d) \quad \text{for} \quad 0 \leqslant i < n, \ i \text{ odd}$
(M4) $m_i(a, a, d, d) = m_{i+1}(a, a, d, d) \quad \text{for} \quad 0 \leqslant i < n, \ i \text{ even}.$

This theorem characterizes modularity and has been proved by A. Day [1].

In this part we derive two similar theorems.

Theorem 3.3. *Let K be an equational class of algebras. If the congruence lattices of all algebras of K are l-modular, then they are modular.*

Proof. Let the congruence lattice of every algebra $A \in K$ be l-modular. By means of polynomial symbols w_i, $i = 0, 1, \ldots, n$, the existence of which follows from Theorem 2.2, we shall construct polynomial symbols which will satisfy the conditions of Theorem 3.2. Put

$$u_i = w_i(x_0, x_1, x_2, x_0, x_3, x_3)$$

$$v_i = w_i(x_0, x_0, x_0, x_1, x_2, x_3)$$

$$t_i = w_i(x_0, x_1, x_1, x_0, x_3, x_3)$$

for $i = 0, 1, \ldots, n$.

Let $d_0, d_1, \ldots, d_r$ be the following sequence.

$$u_0, u_1, t_1, v_1, v_2, t_2, u_2, u_3, t_3, v_3, v_4, t_4, u_4, u_5, t_5, v_5, \ldots .$$

This is a sequence of 4-ary polynomial symbols. We show that this sequence satisfies $(M_1)-(M_4)$ from Theorem 3.2. Let $A \in K$ and $a, b, c, d \in A$. (M_1) is evidently satisfied. Every d_i has the form u_k or v_k or t_k. Therefore $d_i(a, b, b, a) = u_k(a, b, b, a) = w_k(a, b, b, a, a, a) = a$ or $d_i(a, b, b, a) = v_k(a, b, b, a) = w_k(a, a, a, b, b, a) = a$ or $d_i(a, b, b, a) = t_k(a, b, b, a) = w_k(a, b, b, a, a, a) = a$. So (M_2) holds. Let i be odd. Then $d_i = u_k$, $d_{i+1} = t_k$ or $d_i = v_k$, $d_{i+1} = v_{k+1}$, k odd, or $d_i = t_k$, $d_{i+1} = u_k$. It is easy to see that in every case $d_i(a, b, b, d) = d_{i+1}(a, b, b, d)$. Really, in the first case

$$d_i(a, b, b, d) = u_k(a, b, b, d) = w_k(a, b, b, a, d, d) =$$
$$= t_k(a, b, b, d) = d_{i+1}(a, b, b, d).$$

In the second case

$$d_i(a, b, b, d) = v_k(a, b, b, d) = w_k(a, a, a, b, b, d).$$

Since k is odd and (L_3) holds, we have

$$w_k(a, a, a, b, b, d) = w_{k+1}(a, a, a, b, b, d) = v_{k+1}(a, b, b, d) =$$
$$= d_{i+1}(a, b, b, d).$$

In the third case

$$d_i(a, b, b, d) = t_k(a, b, b, d) = w_k(a, b, b, a, d, d) =$$
$$= u_k(a, b, b\ d) = d_{i+1}(a, b, b, d).$$

Thus (M_3) is satisfied. Assume i even.

Then $d_i = u_k$, $d_{i+1} = u_{k+1}$, k even, or $d_i = t_k$, $d_{i+1} = v_k$ or $d_i = v_k$, $d_{i+1} = t_k$. In the same way as in the case of i odd it follows that $d_i(a, a, d, d) - = d_{i+1}(a, a, d, d)$. So (M_4) holds and therefore the congruence lattice of every algebra of K is modular.

Analogously one can prove:

Theorem 3.4. *Let K be an equational class of algebras. If the congruence lattices of all algebras of K are dually l-modular, then they are modular.*

Proof. If the congruence lattice of every algebra is dually l-modular, then by Theorem 2.3 there are 7-ary polynomial symbols $w_0, \ldots, w_m$ satisfying relations $(DL_1)-(DL_3)$. Put

$$u_i = w_i(x_0, x_1, x_1, x_1, x_1, x_2, x_3)$$
$$v_i = w_i(x_0, x_1, x_2, x_2, x_2, x_2, x_3)$$

for $0 \leqslant i \leqslant n$.

Then the sequence

$$u_0, u_1, v_1, v_2, u_2, u_3, v_3, v_4, u_4, u_5, v_5, v_6, u_6, \ldots$$

satisfies the condition (ii) of Theorem 3.2 and therefore the congruence lattice of every algebra $A \in K$ is modular.

REFERENCES

[1] DAY A.: A characterization of modularity for congruence lattices of algebras, Canad. math. Bull., *12*, 1969, 167—173.

[2] DAY A.: p-modularity implies modularity in equational classes, (preprint).

[3] GRÄTZER G.: Universal algebra, Van Nostrand, Princeton N. J., 1968.

[4] GEDEONOVÁ E.: A characterization of p-modularity for congruence lattices of algebras, Acta Fac. Rcrum Natur. Univ. Comenian. Math., *28*, 1972, 99—106.

[5] HUHN A.: Schwach distributive Verbände, Acta Fac. Rerum Natur. Univ. Comenian. Math. Mim. č., 1971, 51—56.

[6] JÓNSSON B.: Algebras whose congruence lattices are distributive, Math. Scand., *21*, 1967, 110—121.

[7] McKENZIE R.: Equational bases and nonmodular lattice varieties, Trans. Amer. math. Soc., *174*, 1972, 1—43.

[8] MEDERLY P.: Maľcev type conditions for equational classes of algebras, (Slovak), Thesis, Komenský Univ., Bratislava, 1971.

[9] WILLE R.: Kongruenzklassengeometrien, Lecture notes in Math., *113*, 1970.

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