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ON SOME GENERALIZATIONS OF LIE ALGEBRAS

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§ 1. Weak Lie pseudo-algebra.

Let K be a commutative field and L a vector space over K.

Definition. *L is a weak Lie pseudo-algebra (w. L. p.-a) if, to every pair of vectors x, y in L, there corresponds a vector $z \in L$, called the product of x and y, $z = [x, y]$, satisfying the following axioms:*

1°. $[x, y + z] = [x, y] + [x, z] - [x, o]$

2°. $[x + y, z] = [x, z] + [y, z] - [o, z]$

3°. $[x, \alpha y] = \alpha[x, y] - (\alpha - 1)[x, o] + \langle x, \alpha \rangle y, \quad \langle x, \alpha \rangle \in K$

4°. $[x, x] = o$

5°. *There exist $\mu, \nu \in K$ such that for every $x, y, z \in L$,*
$[[x, y], z] + [[y, z], x] + [[z, x], y] - \mu\{[[x, y], o] +$
$+ [[y, z], o] + [[z, x], o]\} - \nu([o, x] + [o, y] + [o, z]) = o.$

CONSEQUENCES FROM THE DEFINITION

1. From the axioms 1, 2, 4 it is possible to infer the anticommutativity of the multiplication:

$$[x, y] = - [y, x], \tag{1}$$

if the characteristic of the field K is different from two.

2. The consequence 1 and the third axiom involve:

$$[\alpha x, y] = \alpha[x, y] - (\alpha - 1)[o, y] - \langle y, \alpha \rangle x. \tag{2}$$

3. If we take in the 5-th axiom $x = y = o$, we obtain

$$(1 - \nu)[o, z] = 0.$$

Since in general $[o, z] \neq o$, it follows that $\nu = 1$. In the 5-th axiom there rests only the constant μ called „the constant of type".

4. $$\left[\sum_{i=1}^{n} x_i, \sum_{j=1}^{m} y_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} [x_i, y_j] - (n - 1) \sum_{j=1}^{m} [o, y_j] -$$

$$- (m - 1) \sum_{i=1}^{n} [x_i, o].$$

§ 2. Special classes of w. L. p.-a and examples

If $[x, o] = [o, x] = o$, $\forall\ x \in L$, we obtain Lie's pseudo-algebras introduced by Herz [2] and when $\langle x, \alpha \rangle = o$, $\forall\ x \in L$, $\forall\, \alpha \in K$, the weak Lie's algebra (w. L. a).

Finally, if $[x, o] = [o, x] = o$ and $\langle x, \alpha \rangle = o$, $\forall\, x \in L$, $\forall\, \alpha \in K$, we obtain the usual Lie algebras.

We shall give some examples of w. L. a and w. L. p.-a.

1. Let $\mathscr{A}$ be a weak algebra [1], that means, a linear algebra in which the distributivity laws are replaced by:

$$x(y + z) = xy + xz - x \cdot o,$$

$$(x + y)z = xz + yz - o \cdot z.$$

If $\mathscr{A}$ is endowed with the product:

$$[x, y] = xy - yx, \tag{3}$$

we obtain a w. L. a with the constant of the type $\mu = o$. This is the weak Lie algebra associated to a weak algebra.

2. Let Φ be the set of functions:

$$f : [GF(2)]^n \to [GF(2)].$$

By using the inner operation:

$$[f, g] = f(x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_n)) \oplus g(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_n)), \tag{4}$$

where the sign $\oplus$ means the addition modulo 2, we obtain a w. L. a.

Indeed, using the common notations from the logical algebra, every f belonging to Φ may be written as:

$$f = x_n f_1 \oplus (1 \oplus x_n) f_0 = x_n (f_1 \oplus f_0) \oplus f_0,$$

where f_1 and f_0 are respectively $f(x_1, \ldots, x_{n-1}, o)$ and $f(x_1, \ldots, x_{n-1}, 1)$.

Then:

$$[f, g] = (x_n(g_1 \oplus g_0) \oplus g_0)(f_1 \oplus f_0) \oplus f_0 \oplus$$

$$\oplus\ (x_n(f_1 \oplus f_0) \oplus f_0)(g_1 \oplus g_0) \oplus g_0 = g_0 f_1 \oplus f_0 g_1 \oplus f_0 \oplus g_0 .$$

We obtain immediately the relations:

$$[f, g \oplus h] = [f, g] \oplus [f, h] \oplus [f, o],$$
$$[f \oplus g, h] = [f, h] \oplus [g, h] \oplus [o, h],$$
$$[f, f] = o,$$

and

$$[[f, g], h] \oplus [[g, h], f] \oplus [[h, f], g] \oplus [f, o] \oplus [g, o] \oplus [h, o] = o,$$
$$(\text{Here } a = -a \ (\text{mod } 2) \text{ and } [f, o] = f_0).$$

Using the properties of this algebra we obtained some applications in the switching theory [5, 6].

3. Let L be a distributive algebra in the usual sense.

If we introduce the product

$$[x, y] = xy - yx + x - y,$$

we obtain a w. L. a.

We have:

$$[x, y + z] = [x, y] + [x, z] - [x, o],$$
$$[x + y, z] = [x, z] + [y, z] - [o, z].$$
$$[x, x] = o; \quad [x, o] = x,$$

and

$$[[x, y], z] + [[y, z], x] + [[z, x], y] + \{[[x, y], o] + [[x, z], o] +$$
$$+ [[z, x], o]\} - ([x, o] + [y, o] + [z, o]) = o.$$

For this w. L. a the constant of type is $\mu = 1$.

4. Let $\mathscr{F}(\tau)$ be the field of meromorphic functions of the variable τ in a simply connected region and R_n an n-dimensional vector space of n-tuples, defined over $\mathscr{F}(\tau)$.

Let $\{u_i\}$, $i = \overline{1, n}$ be a basis in R_n. We suppose that R_n is differentiably closed, namely the vectors u_i verify the equations

$$\frac{\mathrm{d}u_i}{\mathrm{d}\tau} = \alpha_{ij} u_j(\tau). \tag{5}$$

If $u \in R_n$, then $u = u_i \alpha_i$, $\alpha_i \in \mathscr{F}(\tau)$.

We introduce the mapping $T : R_n \to R_n$, by

$$Tu = \frac{du_i}{d\tau}\alpha_i + u_i\frac{d\alpha_i}{d\tau}. \tag{6}$$

It is clear that:

$$T(u + v) = Tu + Tv.$$

Then

$$T(\lambda u) = \lambda Tu + \frac{d\lambda}{d\tau}u.$$

Because formula (6) contains the term $\dfrac{du_i}{d\tau}\alpha_i$ the transformation T depends on the basis chosen in R_n.

It is clear that there are many transformations T. Let us consider the matrix $\boldsymbol{U}$ of the basis, $\boldsymbol{U} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$; because u_i are vectors the $\boldsymbol{U}$-type is $n \times n$.

If we take a new basis, related to the basis $\boldsymbol{U}$ by the relation

$$\boldsymbol{U} = AV,$$

where $A = (A_{ij})$ is an invertible $n \times n$ matrix, from (5) it follows that:

$$\frac{dA_{ij}}{d\tau}v_j + A_{ij}\frac{dv_j}{d\tau} = \alpha_{ih}A_{hk}v_k,$$

or:

$$A_{ij}\frac{dv_j}{d\tau} = \left(-\frac{dA_{ik}}{d\tau} + \alpha_{ih}A_{hk}\right)v_k,$$

and if we denote by $B = (B_{ji})$ the inverse of the matrix A, we obtain the new coefficients of the equation (5):

$$\tilde{\alpha}_{ij} = B_{ip}\left(-\frac{dA_{pj}}{d\tau} + \alpha_{ph}A_{hj}\right).$$

Denote by $\mathscr{T}$ the set of all transformations T.

Let us consider the operation

$$[\cdot,\cdot] : \mathscr{T} \times \mathscr{T} \to \mathscr{T},$$

defined by

$$[T, U] = T \circ U - U \circ T + T - U, \tag{7}$$

where

$$(T \circ U)u = T(Uu).$$

To make $\mathcal{T}$ a w. L. p.-a. we interpret addition and scalar multiplication as the ordinary addition of two transformations and multiplication of a transformation by a scalar and use the product defined by (7).

Thus

$$\begin{aligned}[T + U, V]u &= ((T + U) \circ V)u - (V \circ (T + U))u + \\ &+ (T + U)u - Vu = (T \circ V)u + (U \circ V)u - (V \circ T)u - \\ &- (V \circ U)u + Tu + Uu - Vu = \\ &= (T \circ V - V \circ T + T - V)u + (U \circ V - V \circ U + U - V)u + \\ &+ Vu = ([T, V] + [U, T])u + Vu.\end{aligned}$$

Since

$$[0, V] = -V,$$

we obtain

$$[T + U, V] = [T, V] + [U, V] - [0, V].$$

Now, by using (7) and (6), we have:

$$\begin{aligned}[\alpha T, U]u &= (\alpha T \circ U)u - (U \circ \alpha T)u + \alpha Tu - Uu = \\ &= ((\alpha T \circ U)u - (\alpha U \circ T)u + \mathrm{d}\alpha/\mathrm{d}\tau\, Tu) + \alpha Tu - Uu = \\ &= (\alpha T \circ U)u - (\alpha U \circ T)u - \mathrm{d}\alpha/\mathrm{d}\tau\, Tu + \alpha Tu - Uu = \\ &= (\alpha T \circ U)u - (\alpha U \circ T)u + \alpha Tu - \alpha Uu + \alpha Uu - \\ &- \mathrm{d}\alpha/\mathrm{d}\tau\, Tu - Uu = \\ &= \alpha[T, U]u - (\alpha - 1)[0, U]u - \langle U, \alpha\rangle Tu.\end{aligned}$$

For $\forall T \in \mathcal{T}$ by comparing with the relation (2), we have

$$\langle T, \alpha\rangle = \mathrm{d}\alpha/\mathrm{d}\tau.$$

Finally, for the axiom 5, let us calculate:

$$\begin{aligned}&[[T, U], V] + [[U, V], T] + [[V, T], U] =^{1)} \\ &= [TU - UT + T - U, V] + [UV - VU + U - V, T] + \\ &+ [VT - TV + V - T, U] = \\ &= (TU - UT + T - U)V - V(TU - UT + T - U) + \\ &TU - UT + T - U - V + \\ &+ (UV - VU + U - V)T - T(UV - VU + U - V) + \\ &+ UV - VU + U - V - T + \\ &+ (VT - TV + V - T)U - U(VT - TV + V - T) +\end{aligned}$$

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[1] Here for simplicity we write TU for $T \circ U$.

$$+ VT - TV + V - T - U =$$
$$= VU - TU + TV - VT + UT - UV - U - T - V =$$
$$= TV - VT - T - V + VU - UV + V - U + UT - TU +$$
$$+ U - T - U - T - V =$$
$$= [T, V] + [V, U] + [U, T] + [0, T] + [0, U] + [0, V].$$

Thus we obtain

$$[[T, U], V] + [[U, V], T] + [[V, T], U] + [T, 0] + [U, 0] + [V, 0] +$$
$$+ [[T, U], 0] + [[U, V], 0] + [[V, T], 0] = 0.$$

§ 3. The transformation $\langle .,. \rangle$ in a w. L. p.-a

For the transformation $\langle .,. \rangle : L \times K \to K$, the w. L. p.-a acts like a domain of operators over the field K.

Now, we shall deal with some properties of these transformations.

Proposition 1. *If the characteristic of the field K is not two, then $\langle .,. \rangle : L \times \times K \to K$ is a bilinear application.*

We note next that according to the axiom 2,

$$[x + y, \alpha z] = [x, \alpha z] + [y, \alpha z] - [o, \alpha z],$$

on the other hand according to the axioms 3 and 2,

$$[x + y, \alpha z] = \alpha[x + y, z] - (\alpha - 1)[x + y, o] + \langle x + y, \alpha \rangle z =$$
$$= \alpha[x, z] + \alpha[y, z] - \alpha[o, z] - (\alpha - 1)[x, o] -$$
$$- (\alpha - 1)[y, o] + \langle x + y, \alpha \rangle z =$$
$$= \alpha[x, z] - (\alpha - 1)[x, o] + \langle x, \alpha \rangle z + \alpha[y, z] -$$
$$- (\alpha - 1)[y, o] + \langle y, \alpha \rangle z - \alpha[o, z] - \langle o, \alpha \rangle z.$$

Hence,

$$\langle x + y, \alpha \rangle z = \langle x, \alpha \rangle z + \langle y, \alpha \rangle z - \langle o, \alpha \rangle z.$$

This relation holds for $\forall z \in L$, therefore:

$$\langle x + y, \alpha \rangle = \langle x, \alpha \rangle + \langle y, \alpha \rangle - \langle o, \alpha \rangle.$$

If we set $y = -x$, taking into account proposition 2, when the characteristic of the field is different from two, we obtain $\langle 0, \alpha \rangle = 0$ and so

$$\langle x + y, \alpha \rangle = \langle x, \alpha \rangle + \langle y, \alpha \rangle.$$

Hence

$$[x, (\alpha + \beta)y] = [x, \alpha y + \beta y] = [x, \alpha y] + [x, \beta y] - [x, o] =$$
$$= \alpha[x, y] - (\alpha - 1)[x, o] + \langle x, \alpha \rangle y + \beta[x, y] - (\beta - 1)[x, o] +$$

$$+ \langle x, \beta\rangle y - [x, o] =$$
$$= (\alpha + \beta)[x, y] - (\alpha + \beta - 1)[x, o] + \langle x, \alpha\rangle y + \langle x, \beta\rangle y.$$

On the other hand,

$$[x, (\alpha + \beta)y] = (\alpha + \beta)[x, y] - (\alpha + \beta - 1)[x, o] + \langle x, \alpha + \beta\rangle y,$$

therefore

$$\langle x, \alpha + \beta\rangle = \langle x, \alpha\rangle + \langle x, \beta\rangle.$$

Proposition 2. *If the characteristic of the field K is not two, then*

$$\langle \beta y, \alpha\rangle = \beta\langle y, \alpha\rangle.$$

Let us calculate the expression

$$[\alpha x, \beta y] = \beta[\alpha x, y] - (\beta - 1)[\alpha x, o] + \langle \alpha x, \beta\rangle y =$$
$$= \beta(\alpha[x, y] - (\alpha - 1)[o, y] - \langle y, \alpha\rangle x) -$$
$$- (\beta - 1)\alpha[x, o] + \langle \alpha x, \beta\rangle\, y,$$

therofore

$$[\alpha x, \beta y] = \alpha\beta[x, y] - \beta(\alpha - 1)[o, y] - \langle y, \alpha\rangle\beta x -$$
$$- \alpha(\beta - 1)[x, o] + \langle \alpha x, \beta\rangle y =$$
$$\alpha\beta[x, y] - \alpha(\beta - 1)[x, o] - \beta(\alpha - 1)[o, y] - \langle y, \alpha\rangle\beta x + \langle \alpha x, \beta\rangle y =$$
$$= \alpha\beta([x, y] - [x, o] - [o, y]) + \alpha[x, o] + \beta[o, y] -$$
$$- \langle y, \alpha\rangle\beta x + \langle \alpha x, \beta\rangle y.$$

Then

$$[\beta y, \alpha x] = - [\alpha x, \beta y] = \alpha\beta([y, x] - [y, o] - [o, x]) +$$
$$+ \beta[y, o] + \alpha[o, x] - \langle x, \beta\rangle\alpha y + \langle \beta y, \alpha\rangle x,$$

whence

$$\langle y, \alpha\rangle\beta x - \langle \alpha x, \beta\rangle y = \langle \beta y, \alpha\rangle x - \langle x, \beta\rangle\alpha y,$$

or

$$(\beta\langle y, \alpha\rangle - \langle \beta y, \alpha\rangle)x + (\alpha\langle x, \beta\rangle - \langle \alpha x, \beta\rangle)y = o.$$

As this relation holds $\forall\, x, y \in L$, we get the necessary equality.

Proposition 3. *We have*

$$\langle x, \alpha\beta\rangle = \langle x, \alpha\rangle\beta + \langle x, \beta\rangle\alpha, \tag{8}$$

hence the elements of a w. L. p.-a may be interpreted like differentiations over the field K.

Indeed

$$\begin{gathered}[x, \alpha\beta y] = [x, \alpha(\beta y)] = \alpha[x, \beta y] - (\alpha - 1)[x, o] + \\ \langle x, \alpha\rangle\beta y = \alpha(\beta[x, y] - (\beta - 1)[x, o] + \langle x, \beta\rangle y) - \\ - (\alpha - 1)[x, o] + \langle x, \alpha\rangle\beta y = \\ = \alpha\beta[x, y] - \alpha(\beta - 1)[x, o] + \langle x, \beta\rangle\alpha y - (\alpha - 1)[x, o] + \\ + \langle x, \alpha\rangle\beta y = \alpha\beta[x, y] - (\alpha\beta - 1)[x, o] + \langle x, \beta\rangle\alpha y + \langle x, \alpha\rangle\beta y.\end{gathered}$$

On the other hand

$$[x, (\alpha\beta)y] = \alpha\beta[x, y] - (\alpha\beta - 1)[x, o] + \langle x, \alpha\beta\rangle y.$$

Comparing these two expressions we obtain the relation (8).

Proposition 4. $\langle x, \pm n\rangle = o, x \in L,$ *where* $n = 1 + \ldots + 1$, *(n times), 1 being the unity in K.*

If in axiom 3 we put $\alpha = 1$, it is easy to see that $\langle x, 1\rangle\, y = o,\ \forall\, y \in L$, hence $\langle x, 1\rangle = o$.

Thus $\langle x, o\rangle = [x, o] - [x, o] = o$.

Thus $\langle x, -1\rangle = \langle x, -1\rangle + \langle x, 1\rangle = \langle x, 1 - 1\rangle = \langle x, o\rangle = o$.

Taking into account the proposition 1, we obtain P. 4.

Corollary

$$\begin{gathered}[x, -y] = [y, x] + 2[x, o] \\ [-x, y] = [y, x] + 2\,[o, y],\end{gathered}$$

Proposition 5. $\langle x, -\alpha\rangle = -\langle x, \alpha\rangle$.

Indeed:

$$\langle x, -\alpha\rangle = \langle x, (-1)\alpha\rangle = \langle x, -1\rangle\alpha + \langle x, \alpha\rangle(-1) = -\langle x, \alpha\rangle,$$

using the propositions 3 and 4.

Proposition 6. *For every* $\alpha \in K$ *and* $x, y, \in L$, *the following holds:*

(9) $\mu(\langle x, \alpha\rangle - \langle y, \alpha\rangle)[z, o] + (\langle [x, y], \alpha\rangle - \langle x, \langle y, \alpha\rangle\rangle + \langle y, \langle x, \alpha\rangle\rangle)z = o.$

Let us consider now the axiom 5:

$$\begin{gathered}[[x, y], \alpha z] + [[y, \alpha z], x] + [[\alpha z, x], y] - \mu\{[[x, y], o] + \\ [[y, \alpha z], o] + [[\alpha z, x], o]\} - ([o, x] + [o, y] + [o, \alpha z]) = o.\end{gathered}$$

Hence, since:

$$\begin{gathered}[[x, y], \alpha z] = \alpha[[x, y], z] - (\alpha - 1)[[x, y], o] + \langle [x, y], \alpha\rangle z, \\ [[y, \alpha z], x] = \alpha[[y, z], x] - \langle y, \alpha\rangle[o, x] + \langle x, \alpha\rangle[y, o] - \\ - \langle x, \alpha\rangle[y, z] + \langle y, \alpha\rangle[z, x] + (1 - \alpha)[[y, o], x] - \langle x, \langle y, \alpha\rangle\rangle z, \\ [[\alpha z, x], y] = \alpha[[z, x], y] + \langle x, \alpha\rangle[o, y] + \langle y, \alpha\rangle[o, x] +\end{gathered}$$

$$+ (1 - \alpha)[[o, x], y] - \langle y, \alpha\rangle[z, x] - \langle x, \alpha\rangle[z, y] + \langle y, \langle x, \alpha\rangle\rangle z,$$

and

$$[[y, \alpha z], o] = \alpha[[y, z], o] - (\alpha - 1)[[y, o], o] + \langle y, \alpha\rangle[z, o],$$
$$[[\alpha z, x], o] = \alpha[[z, x], o] - (\alpha - 1)[[o, x], o] - \langle x, \alpha\rangle[z, o],$$

it follows that:

$$\begin{aligned}&\alpha([[x, y], z] + [[y, z], x] + [[z, x], y]) + (\langle [x, y], \alpha\rangle - \langle x, \langle y, \alpha\rangle\rangle +\\ &+ \langle y, \langle x, \alpha\rangle\rangle)z + (1 - \alpha)([[o, x], y] + [[x, y], o] + [[y, o], x]) -\\ &- \mu([[x, y], o] + \alpha[[y, z], o] + \alpha[[z, x], o] - (\alpha - 1)[[y, o], o] -\\ &- (\alpha - 1)[[o, x], o] + \langle y, \alpha\rangle[z, o] - \langle x, \alpha\rangle [z, o]) - [o, x] -\\ &- [o, y] - \alpha [o, z] = o,\end{aligned}$$

and using axiom 5, we obtain relation (9).

Corollary. *If* $\mu = o$ *or* $[z, o] = o$ *we obtain Herz's relation*

$$\langle [x, y], \alpha\rangle = \langle x, \langle y, \alpha\rangle\rangle - \langle y, \langle x, \alpha\rangle\rangle.$$

§ 4. The adjoint transformation in a w. L. p.-a

We introduce in a w. L. p.-a the adjoint transformation by:

$$\mathrm{ad}(x)y = [x, y] - [x, o] - [o, y].$$

Proposition 7. *In a w. L. p. a.* ad *is a pseudo-linear transformation. Indeed, we have:*

$$\begin{aligned}\mathrm{ad}\,(x)\alpha y &= [(x, \alpha y] - [x, o] - [o, \alpha y] =\\ &= \alpha[x, y] - (\alpha - 1)[x, o] - [x, o] + \langle x, \alpha\rangle y -\\ &- \alpha[o, y] = \alpha[x, y] - \alpha[x, o] - \alpha[o, y] + \langle x, \alpha\rangle y =\\ &= \alpha\, \mathrm{ad}\,(x)y + \langle x, \alpha\rangle y,\end{aligned}$$

and

$$\begin{aligned}\mathrm{ad}\,(x)(y + z) &= [x, y + z] - [x, o] - [o, y + z] =\\ &= [x, y] + [x, z] - [x, o] - [x, o] - [o, y] - [o, z] =\\ &= \mathrm{ad}\,(x)y + \mathrm{ad}(x)z,\end{aligned}$$

hence

$$\mathrm{ad}\,(x)(\alpha y + \beta z) = \alpha\, \mathrm{ad}\,(x)y + \beta\, \mathrm{ad}\,(x)z + \langle x, \alpha\rangle y + \langle x, \beta\rangle z.$$

If $\langle x, \alpha\rangle = o$, $\forall\, x \in L$, then ad is a linear transformation (the case of weak Lie algebras).

Proposition 8. *There hold the following equalities:*

$$\mathrm{ad}\,(\alpha x)y = \alpha\,\mathrm{ad}\,(x)y - \langle y, \alpha\rangle x,$$

and

$$\mathrm{ad}\,(x + y)z = \mathrm{ad}\,(x)z + \mathrm{ad}\,(y)z.$$

The proof of these relations is an immediate one.

If in the set of the adjoint transformations of a weak Lie algebra we introduce the ordinary addition and multiplication by a scalar, and use the product:

$$([\mathrm{ad}\,(x), \mathrm{ad}\,(y)])z = (\mathrm{ad}\,[x, y])z,$$

then this set becomes a weak Lie algebra.

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