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ON THE EXTENSION OF GRAPHS WITH A GIVEN DIAMETER WITHOUT SUPERFLUOUS EDGES

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Graphs of a diameter r (especially for $r = 2$) were considered in [2] and [3] first of all from the point of view of different estimates of the number of edges in a graph with a given diameter and of the number of vertices in graphs with some properties. For example it was required from a graph to contain certain subgraphs, to have a given maximal degree and other properties. We shall investigate in this paper the so-called graphs without superfluous edges i. e. such graphs that an deleting of an arbitrary edge enlarges their diameter. We shall consider only graphs with a diameter r without k-gons fo $3 \leqslant k \leqslant \leqslant r + 1$. These graphs form a subclass of graphs without superfluous edges. By constructive means we solve the question how a graph with the required properties can be enlarged or reduced respectively by one vertex in the same class of graphs. For solving this question we have introduced some concepts, e. g. the μ-set and the μ-system (the set of all μ-sets of a given graph). We have also given the estimations for a number of bases for arbitrary graphs.

We wish to thank Prof. Kotzig for his suggestions, used in this paper.

I. DEFINITIONS AND DENOTATIONS

We suppose that the fundamental concepts are well known and therefore we shall not define these and use them as in [1].

We understand in this paper under a graph a graph without loops and multiple edges; if we do not emphasize the opposite a graph is also connected and finite. We denote it by $G = (U, H)$, where U is a vertex set and H is the set of edges of the graph G. If A is a set, then by $|A|$ we denote the cardinal number of A. An edge that is incident with the vertices x, y we denote by (x, y). A path from the vertex $x \in U$ to $y \in U$ $(x \neq y)$ in a graph G we denote by w (x, y). The length of the path $w(x, y)$ we denote by $\lambda[w(x, y)]$. Let $\varrho_G(x, y)$ denote the distance (i. e. the length of the shortest path) in a graph $G = (U,H)$ between the vertices $x, y \in U$. The diameter $d(G)$ of a graph $G = (U, H)$ is a number defined as follows: $d(G) = \max\limits_{x, y \in U} \varrho_G(x, y)$.

The neighbourhood of a vertex $u \in U$ is a set $\Omega(u) = \{x | x \in U \wedge \varrho(u, x) = 1\}$.

Let $G = (U, H)$ be a graph. Let $A \subset U$. A set A is denoted as a base of the graph G if it satisfies the following conditions:

a) if $x \in A$ then $\Omega(x) \cap A = \emptyset$,

b) if $x \notin A$ then $\Omega(x) \cap A \neq \emptyset$.

An n-gon is a subgraph formed by the sequence of vertices $\{v_1, v_2, \ldots, v_n\}$ where:

1) $(v_i, v_{i+1}) \in H$ for all $i = 0, 1, \ldots, n - 1$;

2) $v_0 = v_n$;

3) $v_i \neq v_j$ for all $i \neq j$; $i, j = 0, 1, \ldots, n$ except the case $i = 0, j = n$.

A graph $G = (U, H)$ is bipartite if there exist sets M, N such that $N \subset U$, $U - N = M$ and $H = \{(x, y) | x \in M, y \in N\}$.

Let $A \subset U$. Then denote $\varrho(A) = \min\limits_{\substack{x,y \in A \\ x \neq y}} \varrho(x, y)$ and $\varrho(x, A) = \min\limits_{z \in A} \varrho(x, z)$.

Definition 1. *An edge (u, v) in a graph $G = (U, H)$ is superfluous if for the subgraph $G_1 = (U, H_1)$, where $H_1 = H - \{(u, v)\}$ we have $d(G) = d(G_1)$.*

A graph G is without superfluous edges if every edge is not superfluous.

Definition 2. a) *β_r — graph is a graph $G = (U, H)$ for which the following holds:*

1) *$d(G) = r$,*

2) *every edge is not superfluous.*

b) *δ_r-graph is a graph $G = (U, H)$ for which:*

1) *$d(G) = r$,*

2) *G does not contain any k-gon for $3 \leqslant k \leqslant r + 1$.*

Definition 3. *Let $G = (U, H)$, $d(G) = d$. A μ-set of the graph G is a set of vertices $A \subset U$ with the following properties:*

1) *$\varrho(A) = d$, if $|A| > 1$,*

2) *for every $x \in U - A$ there exist $y \in A$ such that $\varrho(x, y) < d$.*

Definition 4. *A μ-system of the graph G is a set of all μ-sets of the graph G. We denote it by $\mu(G)$.*

Definition 5. *Let us have some graphs $G_1 = (U_1, H_1)$, $G_2 = (U_2, H_2)$; $|U_1| = n$. With every $x_i \in U_1$ there is associated a set $X_i \subset U_2$. Denote it by $\mathcal{X} = \{X_i\}_{i=1}^n$. Then we define the union of graphs G_1, G_2 through the system $\mathcal{X} = \{X_i\}_{i=1}^n$ as a graph $G = (U_1 \cup U_2, H_1 \cup H_2 \cup H')$, where $H' = \{(x_i, z) | x_i \in U_1, z \in X_i\}$.*

Especially, if $U_1 = \{y\}$, i. e. $|U_1| = 1$, then G is a ν-extension of the graph G_2 by the vertex y through the set of vertices Y ($Y \subset U_2$). G_2 is called the ν-reduction of the graph G.

Definition 6. *Let a graph G_1 be the v-extension of a graph $G = (U, H)$ with a diameter r by the vertex v through the set $A \in \mu(G)$. Then*

a) *such a v-extension is called a μ-extension,*

b) *G is a μ-reduction of the graph G_1,*

c) *if graphs G, G_1 are both δ_r-graphs, then the vertex v is a μ-reducible vertex,*

d) *if graphs G, G_1 are both δ_2-graphs and moreover $A = \Omega(x)$ for some vertex $x \in U$, then such a v-extension is called an η-extension; such a v-reduction of graph G_1 is called an η-reduction. The vertex v is an η-reducible vertex.*

Definition 7. *A graph is μ-irreducible if every vertex is not μ-reducible; a graph is η-irreducible if every vertex is not η-reducible.*

II. RESULTS

First of all we prove the existence of a μ-set, of the μ-system and construct some relations between the different notions.

Theorem 1. *Let $G = (U, H)$, $d(G) = r$ and $u \in U$. Then there exists at least one μ-set A such that $u \in A$.*

Proof. Let us denote $M_0 = \{u\}$. If $y \in U - M_0$ for every $\varrho(u, y) < r$, then $A = M_0$. If not, then there exists a vertex $x_1 \in U - M_0$, for which $\varrho(u, x_1) = r$. Let us put $M_1 = M_0 \cup \{x_1\}$, and we will again verify that $\varrho(y, M_1) < r$ for all $y \in U - M_1$ etc. Let us suppose that we have constructed the set $M_i = \{u, x_1, \ldots, x_i\}$; if there exists $x_{i+1} \in U - M_i$ such that $\varrho(x, x_{i+1}) = r$ for all $x \in M_i$ then we denote $M_{i+1} = M_i \cup \{x_{i+1}\}$. If such x_{i+1} does not exist then $A = M_i$. The described process is finite because the set U is finite.

Remark. The algorithm for finding out the μ-system of the graph G: Let $G = (U, H)$, $d(G) = d$ and $|U| = n$. From the definition of μ-set follows the assertion:

I. Let $X = \{x_1, x_2, \ldots, x_p\}$; $X \subset U$, $\varrho(X) = d$. Then $X \in \mu(G)$ if and only if $\bigcup_{i=1}^{p} B_i = U$, where $B_i = \{x | x \in U \wedge \varrho(x_i, x) < d\}$.

By u_1 we denote some vertex of the graph G. Let A be a set of vertices $x \in U$, for which $\varrho(u_1, x) < d$. Let $|A| = s$; by symbols $u_2, u_3, \ldots, u_{s+1}$ we denote the vertices of the graph G which are from the set A. Let u_{s+2}, $u_{s+3}, \ldots, u_n$ be the denotation of the remaining vertices of the graph G. The algorithm consists of the following steps:

1. At the beginning let i have a zero value.
2. Increase the value of i by one.
3. If $i \leqslant s + 1$, then go to 4. Otherwise we should have found already all μ-sets of the graph G.

4. Put $k = 1$, $X_k = \{u_i\}$, $q_k = i$.

5. Find out according to I which elements from X_k are μ-sets and the system of elements from X_k which are not μ-sets and denote them by Y_k.

6. If $Y_k = \emptyset$ then go to 2. Otherwise put $k = k + 1$ and construct the system of sets:

$X_k = \{Z \cup \{u_{q_k}\} \mid u_{q_k} \in U_1 \wedge Z \in Y_{k-1} \wedge q_{k-1} < q_k \wedge \varrho(u_{q_k}, Z) = d\}$; go to 5.

We can order the μ-sets as follows: Let $M, N \in \mu(G)$, where $M = \{u_{r_1}, u_{r_2}, \dots, u_{r_k}\}$, $N = \{u_{p_1}, u_{p_2}, \dots, u_{p_m}\}$, $r_1 < r_2 < \dots < r_k$ and $p_1 < p_2 \dots < p_m$. If $r_1 = p_1$, $r_2 = p_2, \dots, r_q = p_q$, ($q < \min (k, m)$) and $r_{q+1} < p_{q+1}$, then the algorithm generates in the first place the μ-set M. It is clear that the algorithm generates all μ-sets and every μ-set only once because there does not exist a μ-set that begins with the element j for $j > s + 1$.

Lemma 1. *Let $G = (U, H)$ be a δ_r-graph. Then G is a β_r-graph.*

Proof. Let $u, v \in U$. If $\varrho(u, v) = 1$; then there cannot exist a path from u to v of the length 1, $2 \leqq l \leqq r$, because the vertices u and v would belong to the k-gon for $k \leqslant r + 1$. Thus after omitting the edge (u, v) we would have $\varrho(u, v) > r$, i. e. $d(G) > r$. So G is a β_r-graph.

Lemma 2. *Let G be a δ_2-graph. Then $\Omega(u) \in \mu(G)$ for every vertex $u \in U$. (I. e. an η-extension is a special case of μ-extension of δ_2-graphs).*

Proof. If $x_1, x_2 \in \Omega(u)$, $x_1 \neq x_2$ then $\varrho(x_1, x_2) = 2$. For every $y \in U - \Omega(u)$ there exists a $z \in \Omega(u)$ such that $\varrho(y, z) = 1$, because otherwise we should have $\varrho(y, u) > 2$.

The following two Theorems are dealing with a μ-extension of δ_r-graphs.

Theorem 2. *Let $G = (U, H)$ be a δ_r-graph, $G_1 = (U_1, H_1)$ its μ-extension by the vertex v through a set $M \in \mu(G)$. Then:*

(1) *G_1 is a δ_s-graph and $s \leqslant r$ holds.*

(2) *The relation $s = r$ holds if and only if at least one of the following conditions is satisfied:*

A) *There exists $x \in U - M$ such that $\varrho_G(x, M) = r - 1$.*

B) *There exist $x_1, x_2 \in U$ such that $\varrho_G(x_1, x_2) = r$ and also $\varrho_G(x_1, M) + \varrho_G(x_2, M) \geqslant r - 2$.*

Proof. 1. After a μ-extension of a δ_r-graph G there cannot occur a k-gon for $3 \leqslant k \leqslant r + 1$, because in the opposite case there would exist two vertices $x, y \in M$ such that $\varrho_G(x, y) \leqslant r - 1$. It is a contradiction with the assumption that $M \in \mu(G)$.

For all $x \in U$ either $x \in M$ holds and then $\varrho_{G_1}(v, x) = 1$, or $x \notin M$ and then there exists a vertex $z \in M$ such that $\varrho_G(z, x) \leqslant r - 1$. So $\varrho_{G_1}(v, x) \leqslant r$, i. e. $d(G_1) \leqslant d(G)$. This is a proof of the assertion (1).

2. If the condition A) is satisfied, $\varrho_{G_1}(x, v) = r$, hence $d(G) = d(G_1)$. If it fulfils the condition B) then $\varrho_{G_1}(x_1, x_2) = r$ and again $d(G_1) = d(G)$.

Let $d(G_1) = d(G) = r$. Then there must exist vertices $x, y \in U$ such that $\varrho_G(x, y) = r$. There are two possibilities:

a) $x \neq v$, $y = v$, then $x \notin M$ and $\varrho_G(x, M) = r - 1$ must hold because otherwise we would have $\varrho_{G_1}(x, v) < r$.

b) $x \neq v$, $y \neq v$, then $\varrho_G(x, M) + \varrho_G(y, M) \geqslant r - 2$, because otherwise we would have $\varrho_{G_1}(x, y) < r$. This completes the proof of Theorem 2.

Theorem 3. *Let G_1 be a ν-extension of a δ_r-graph $G = (U, H)$ by the vertex v through a set M. If G is a δ_r-graph then this ν-extension is a μ-extension.*

Proof. For $x, y \in M$, $x \neq y$, is $\varrho_G(x, y) \leqslant r$ (since $M \subset U$). G does not contain a k-gon for $3 \leqslant k \leqslant r + 1$, hence $\varrho_G(x, y) = r$, i. e. $\varrho(M) = r$. If there exists a vertex $z \in U - M$ such that $\varrho_G(z, M) = r$, then it would be $\varrho_{G_1}(z, v) > > r$, thus $d(G_1) > r$. Hence $M \in \mu(G)$. must hold.

Remark. One can see from Figure 1 that it is possible to construct a δ_r-graph from a δ_s-graph also by a different extension than a μ-extension.

Corollary 1. *Let G be a δ_2-graph and its ν-reduction G_1 be also a δ_2-graph. Then this ν-reduction is a μ-reduction.*

Corollary 2. *Let G be a δ_2-graph and let G_1 be a ν-extension of a graph G by the vertex v through a set M. Then G_1 is a δ_2-graph if and only if this ν-extension is a μ-extension.*

Proof. If G_1 is a δ_2-graph then by Theorem 3 $M \in \mu(G)$. If $M \in \mu(G)$, then from Theorem 2 it follows that G_1 is a δ_2-graph because here the condition A) is satisfied (for every $x \in M$ in graph G $\Omega(x) \neq \emptyset$ holds).

Corollary 3. *Let G be a δ_2-graph and G_1 be its ν-reduction by the vertex v*

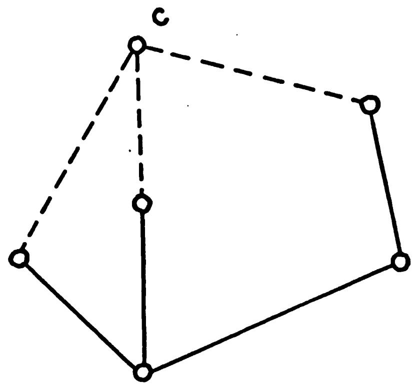

Fig. 1. From δ_3-graph there is constructed a δ_2-graph by a ν-extension of the vertex c.

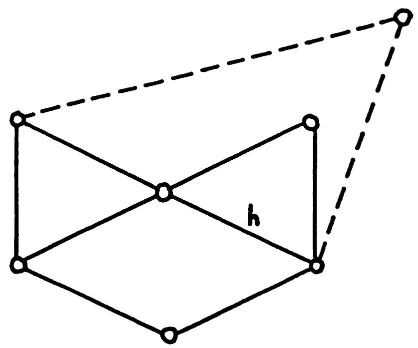

Fig. 2.

through a set M. Then G_1 is a δ_2-graph if and only if this v-reduction is a μ-reduction.

Remark. A β_r-graph, in general, need not be a β_r-graph after a μ-extension as it can be seen from Figure 2 (the edge h is after a μ-extension superflouus).

Now we shall deal with the cardinal numbers of μ-systems and the number of bases.

Theorem 4. *Let $G = (U, H)$ be a δ_r-graph and G_1 be its μ-extension by the vertex v such that $d(G_1) = r$. Then*

a) $|\mu(G)| \leqslant |\mu(G_1)|$

b) *If in addition $r = 2$ and G_1 is the η-extension of a graph G, then* $|\mu(G)| = |\mu(G_1)|$.

Proof. a) Let $N \in \mu(G)$, then either $\varrho(v, N) < r$, hence $N \in \mu(G_1)$, or $\varrho(v, N) = r$, hence $N \cup \{v\} \in \mu(G)$.

b) Let G_1 be an η-extension of a graph G over $\Omega(a)$, $a \in U$. Let $N_1 \in \mu(G_1)$. Then:

1. If there exists $x \in \Omega(a)$ and also $x \in N_1$, then $v \notin N_1$ and so $N_1 \in \mu(G)$.
2. If there does not exist such vertex x then $v \in N_1$, $a \in N_1$ (because for all $x \in \Omega(a)$ we have $x \notin N_1$), thus $N_1 = \{v\} \cup L$ where $a \in L$ and there exists $Q \in \mu(G)$ such that $L \subset Q$. We shall show that $L = Q$. Let be $Q - L = \emptyset$. Then there exists $y \in Q$, $y \notin L$ such that $\varrho(y, L) = 2$. Then $\varrho(y, a) = 2$ and $\varrho(y, v) = 2$. Hence $\varrho(y, N_1) = 2$ in contradiction with the definition of a μ-set ($N_1 \in \mu(G_1)$).

Remark. The second part of Theorem 4 does not hold for a μ-extension in general as it can be seen from the example on Figure 3. Let the extension of Peterson's graph G by vertex c through the set $\{1, 7, 8, 4\}$ be a graph G_1. If we consider the μ-sets of graph $G_1\{2, 6, 10, c\}$, $\{3, 5, 6, c\}$, $\{3, 9, 10, c\}$ and $\{2, 5, 9, c\}$ then we find that after the μ-reduction of vertex c none of the subsets of the set mentioned above is a μ-set of Peterson's graph. Hence $|\mu(G_1)| \geqslant |\mu(G)| + 4$.

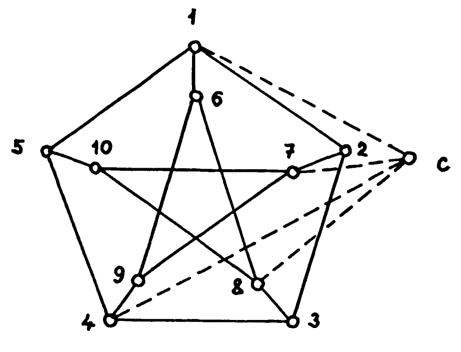

Fig. 3.

Lemma 3. *Let $G = (U, H)$ be a δ_2-graph, let $M \in \mu(G)$. Then there exists $N \in \mu(G)$ such that $M \cap N = \emptyset$.*

Proof. Let be $x \in M$; we put $N = \Omega(x)$. Obviously $N \in \mu(G)$; since $x \in M$, for every $y \in \Omega(x)$ we have $y \notin M$, so $M \cap N = \emptyset$.

Corollary. *Let G be a δ_2-graph, then there exists $A \in \mu(G)$ such that $|A| \leqslant \left[\frac{n}{2}\right]$, where n is the cardinal number of the set of vertices of the graph G.*

Theorem 5. *Let $G = (U, H)$ be a graph with diameter r. Then:*

A) *If there exists a subgraph $G_1 = (U_1, H_1)$ of the graph G such that $d(G_1) \leqslant r - 1$, then $|U_1| \leqslant |\mu(G)|$ holds.*

B) $|\mu(G)| \geqslant r$.

C) $|\mu(G)| = r$ *if and only if either $r = 2$ and G is a bipartite graph or G is a path of the length r, or G is a $2r$-gon.*

Proof. A) By Theorem 1 for every vertex $x \in U$ we can construct a set $M \in \mu(G)$ such that $x \in M$. For all vertices $y \in U_1$ we have $\varrho(x, y) \leqslant r - 1$, hence every $M \in \mu(G)$ contains at most one vertex $x \in U_1$. Thus we can construct k different μ-sets, when $k = |U_1|$.

B) The assertion follows from A).

C) From the definition of a μ-set it follows that $|\mu(G)| = r$ for the graphs in C). Let $|\mu(G)| = r > 2$. Then in the graph G there exists a path $\lambda = (u_1, u_2, \ldots, u_{r+1})$ such that $\varrho(u_1, u_{r+1}) = r$. By Theorem 1 there exist in the graph G μ-sets $A_1, A_2, \ldots, A_r$ with the following properties: $u_1 \in A_1$ and hereby $u_{r+1} \in A_1$; $u_i \in A_i$ for $i = 2, 3, \ldots, r$. It is obvious that $A_i \neq A_j$ for $i \neq j$. If one of the vertices u_k, $k = 1, 2, \ldots, r + 1$, has the degree $s > 2$ in the graph G, then there exists a vertex $v \in U$ ($v \neq u_i$, $i = 1, 2, \ldots, r + 1$) such that $\varrho(v, u_k) = 1$. By Theorem 1 there exists $B \in \mu(G)$ with the following properties:

1. if $2 \leqslant k \leqslant r$, then $v \in B$;
2. if $k = 1$, then there exist $v, w \in U$ such that $\varrho(w, u_1) = 1$ and also $\varrho(v, u_1) = 1$, $w \neq v$. Thus we may select such an A_r that $w \in A_r$ and $u_r \in A_r$. A set B may be constructed such that $v \in B$ and simultaneously $u_r \in B$;
3. if $k = r + 1$, then the μ-sets may be constructed analogously as in 2).

From this construction one can see that the set $B \neq A_i$ for $i = 1, 2, \ldots, r$; hence $|\mu(G)| \geqslant r + 1$. From the preceding it follows also that a degree of every vertex u_i of the path λ is $s_i \leqslant 2$.

Let in a graph G be a vertex $y_1 \in U$ of the degree $s \geqslant 3$ and $y_1 \neq u_i$ ($i = 1, 2, \ldots, r + 1$). Let us denote $t = \min (\varrho(y_1, u_1), \varrho(y_1, u_{r+1}))$. Let $t = \varrho(y_1, u_1)$. Then there exists a sequence $y_1, y_2, \ldots, y_{t+1} = u_1, \ldots, y_{r+1} = u_{r-t+1}$, such that:

a) $\varrho(y_i\, y_{i+1}) = 1$ for $i = 1, 2, \ldots, r + 1$,

b) $\varrho(y_1, y_{r+1}) = r$.

Thereby this case was transformed in the same way as the above mentioned one. We proved, that a degree s of every $u \in U$ is $s \leqslant 2$.

G cannot be a $(2r + 1)$-gon. (The proof may be given by a construction of $r + 1$ different μ-sets).

Let $|\mu(G)| = r = 2$. G cannot contain a triangle following the part A of this Theorem. Let us denote $\mu(G) = \{M, N\}$. By Lemma 3 $M \cap N = \emptyset$. Let $U - (M \cup N) \neq \emptyset$. Then for $x \in U - (M \cup N)$ we can construct $P \in \mu(G)$ such that $x \in P$, $P \neq N$, which is a contradiction. Hence there is $U = M \cup N$. If $x \in M$, $y \in N$ then there must be $\varrho(x, y) = 1$. Hence G is a bipartite graph. This completes the proof of Theorem 5.

At last we shall prove some bounds of the number of k-coverings of the bases and of the cardinal number of the system of all μ-sets. At first we shall introduce the notion of the k-covering, which is in fact a generalization of the base.

Definition 8. *Let $G = (U, H)$ be a graph (not necessarily connected) let $G_1 = (U_1, H_1)$ be its supergraph (not necessarily finite) such that for every edge $(x, y) \in H_1 - H$ at least one of the vertices x, y is an element of the set $U_1 - U$. Let $k \geqslant 2$ be a natural number. We call the set A ($A \subset U$) a k-cove.ing of the graph G in the graph G_1 if the following holds:*

1. *$\varrho_{G_1}(x, y) \geqslant k$ for every $x, y \in A$.*
2. *For every $x \in U$ there exists $y \in A$ such that $\varrho_{G_1}(x, y) < k$.*

By $\gamma_k(G, G_1)$ we denote the system of all k-coverings of the graph G in a graph G_1.

Remark 1. Let $U = U_1$ and $d(G) = k$. Then the k-covering of the graph G in the graph G is identical with the notion of a μ-set of the graph G.

Remark 2. Let $U = U_1$, then the 2-covering of a graph G in graph G is identical with the notion of a base of a graph G. As $\gamma_2(G, G_1)$ does not depend on the graph G_1, we shall write only $\gamma_2(G)$.

Lemma 4. *Let G be a snake-like graph which is formed by the path $w = (x_1, x_2, \ldots, x_n)$, where $\varrho_G(x_1, x_n) = n - 1$ and $n \geqslant 3$. Then $|\gamma_2(G)| \leqslant A^n$ where A is the positive root of the equation $1 + x = x^3$.*

Proof (by induction).

1. For $n \leqslant 5$ one may verify the assertion directly.
2. For $n > 5$ the number of 2-coverings containing (not containing) the vertex x_2 is by the assumption at most A^{n-3} (resp. A^{n-2}). Hence the number of all 2-coverings of G is at most $A^{n-3} + A^{n-2} \leqslant A^n$.

Lemma 5. *Let G be an n-gon with vertices $x_1, x_2, \ldots, x_n$, where $\varrho(x_{i-1}, x_i) = 1$ for $i = 2, 3, \ldots, n$. Let $n \geqslant 6$. Then $|\gamma_2(G)| \leqslant 2^{\frac{n}{2}}$.*

Proof. The number of those coverings which contain the vertex $x_1(x_2,$ and x_3 respectively) is at most A^{n-3} (by Lemma 4). Hence the number of all 2-coverings is at most $3A^{n-3}$. So for $n \geqslant 4$ we have $3A^{n-3} \leqslant 2^{\frac{n}{2}}$, hence $|\gamma_2(G)| \leqslant \leqslant 2^{\frac{n}{2}}$.

Corollary. *Let $G = (U, H)$ be a connected graph without triangles with n vertices. Let the degree of every vertex be at most 2. Then $|\gamma_2(G)| \leqslant 2^{\frac{n}{2}}$.*

A proof for $n > 5$ follows from the preceding lemmas. For $n \leqslant 5$ the assertion may be verified directly.

Lemma 6. *Let G be a graph with r components $G_1, G_2, \ldots, G_r$. Then $|\gamma_2(G)| = = |\gamma_2(G_1)|\,|\gamma_2(G_2)| \ldots |\gamma_2(G_r)|$.*

The proof follows from the definition 8.

Theorem 6. *For a graph G (not necessarily connected) with n vertices and without triangles $|\gamma_2(G)| \leqslant 2^{\frac{n}{2}}$ holds.*

Proof (by induction). 1. For $n = 1$ the Theorem obviously holds.

2. a) if G is not connected, then according to Lemma 6 and to the assumption the theorem holds.

b) if G is connected and there exists a vertex of a degree $k \geqq 3$ (in the opposite case the assertion follows from Lemma 5) then by the induction hypothesis the number of all 2-coverings which contain this vertex is at most $2^{\frac{n-k-1}{2}}$; the number of coverings which does not contain this vertex is at most $2^{\frac{n-1}{2}}$. Hence for $k \geqq 3$ we have $2^{\frac{n-k-1}{2}} + 2^{\frac{n-1}{2}} \leqslant 2^{\frac{n}{2}}$, that is $|\gamma_2(G)| \leqslant 2^{\frac{n}{2}}$.

Lemma 7. *For a graph (which need not be connected) with n vertices $|\gamma_2(G)| \leqslant \leqslant 3^{\frac{n}{3}}$ holds.*

Proof. 1. One may directly verify that for a triangle the assertion holds; for the rest of the graphs which do not contain a vertex of degree $k \geqslant 3$ the assertion follows from the corollary of Lemma 5.

2. Let for the vertex x be $|\Omega(x)| = k \geqslant 3$, then it can be shown analogously as in the proof of Theorem 6 that $|\gamma_2(G)| \leqslant 3^{\frac{n-k-1}{3}} + 3^{\frac{n-1}{3}} \leqslant 3^{\frac{n}{3}}$.

Theorem 7. *Let $G = (U, H)$ be a graph with n vertices which need not be connected. Let $k \geqslant 2$. Then $|\gamma_k(G, G_1)| \leqslant 3^{\frac{n}{3}}$ for every graph G_1.*

Proof (by induction). 1. For $k = 2$ the assertion holds by Lemma 7. Let $k > 2$. The theorem holds for every graph G for which $d(G) < k$. (Here $|\gamma_k(G,G_1) \leqslant n \leqslant 3^{\frac{n}{3}}$).

2. If $d(G) \geqslant k$, then there exists in G a path w with the vertices $v_0, v_1, \ldots, v_k$ whereby $\varrho_G(v_0,v_k) = k$. Let us consider the graphs $G' = (U', H')$; $G'' = = (U'', H'')$ (see Figure 4) with the properties: $U' = U - \{v_1\}$; $U'' = U - M$,

where $M = \{x \mid x \in U \wedge \varrho_G(v_1, x) < k\}$. A set H' (or H'') contains all edges of G with endpoints in the set U' (or U''). For every k-covering of G the following holds: either it contains a vertex v_1 and then it does not contain any vertex from the set $M - \{v_1\}$ or it contains the vertex from the set $M - \{v_1\}$ and then it does not contain the vertex v_1. By using the induction the following hypothesis holds: $|\gamma_k(G, G_1)| \leqslant 3^{\frac{n-p}{3}} + 3^{\frac{n-1}{3}}$, where $p = |M|$. From Figure 4 one may see that $p \geqslant k + 1 \geqslant 4$. Then we have: $3^{\frac{n-p}{3}} + 3^{\frac{n-1}{3}} \leqslant 3^{\frac{n-4}{3}} + 3^{\frac{n-1}{3}} \leqslant 3^{\frac{n}{3}}$

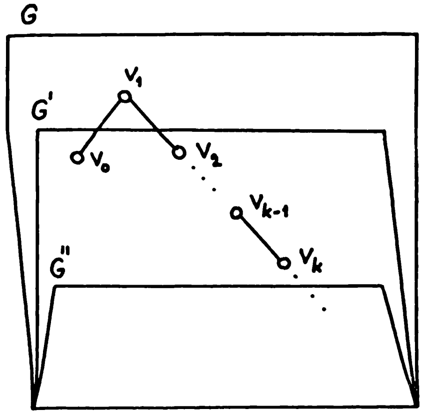

Fig. 4.

Corollary.

1. *If G is connected, then* $|\mu(G)| \leqslant 3^{\frac{n}{3}}$.

2. *If G is a δ_2-graph, then* $|\mu(G)| \leqslant 2^{\frac{n}{2}}$.

Remark. The equality in Theorem 7 is achieved for the graphs which consist of the triangles as components for every n of the form $n = 3p$. Analogously the equality in Theorem 6 is achieved for graphs, the components of which are connected graphs with two vertices.

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