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ENTROPY ON EFFECT ALGEBRAS WITH THE RIESZ DECOMPOSITION PROPERTY I: BASIC PROPERTIES

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We define the entropy, lower and upper entropy, and the conditional entropy of a dynamical system consisting of an effect algebra with the Riesz decomposition property, a state, and a transformation. Such effect algebras allow many refinements of two partitions. We present the basic properties of these entropies and these notions are illustrated by many examples. Entropy on MV-algebras is postponed to Part II.

Keywords: effect algebra, Riesz decomposition property, MV-algebra, state, entropy

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1. INTRODUCTION

Suppose that $(\Omega, \mathcal{S}, P)$ is a probability space. We recall that the entropy of a measurable partition $\mathcal{A} = \{A_1, \dots, A_n\}$ of Ω is the number

$$H(\mathcal{A}) = -\sum_{i=1}^{n} P(A_i)\log(P(A_i)).$$

If $T: \Omega \to \Omega$ is a measure preserving transformation, and if $\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})$ denotes the common refinement of the partitions $\mathcal{A}, T^{-1}(\mathcal{A}), \dots, T^{-(n-1)}(\mathcal{A})$, then there is a finite limit

$$h(\mathcal{A}, T) := \lim_n \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})\right).$$

The Kolmogorov–Sinai entropy is the expression

$$h(T) = \sup\{h(\mathcal{A}, T):\ \mathcal{A} \text{ is a measurable partition of } \Omega\}.$$

The Kolmogorov–Sinai entropy was introduced to distinguish two dynamical systems in the classical probability theory: Every two isomorphic dynamical systems have the same entropy (see e.g. [19, Sec. 10]).

This notion was generalized in many directions ([12, 15, 16, 18, 19], etc). A great problem appears when we take into account a system of fuzzy sets instead of a σ-algebra of sets. The crucial notion of entropy is a finite partition and the refinement of two or more partitions. In the classical probability theory the common refinement of $\mathcal{A} = \{A_1, \dots, A_m\}$ and $\mathcal{B} = \{B_1, \dots, B_n\}$ is simply $\mathcal{C} = \{A_i \cap B_j : 1 \le i \le m,\ 1 \le j \le n\}$. This way cannot be used in more general structures containing fuzzy sets or effect algebras or MV-algebras. For example, if we take two fuzzy (crisp) sets χ_A and χ_B, then for $C = A \cap B$ we have at least three same expressions $\chi_C = \chi_A \cdot \chi_B = \min\{\chi_A, \chi_B\} = \max\{\chi_A + \chi_B - 1, 0\}$. For non-crisp fuzzy sets we can obtain three different fuzzy sets. We recall that the main idea of entropy suitable for these more general cases of fuzzy sets allowing many joint partitions was for the first time suggested in [12].

In [19, Sec. 10] the authors defined the refinement simply as the product of fuzzy sets assuming that the system of fuzzy sets is closed under natural product, and in [18, Sec. 4.2] it is defined on MV-algebras with product. In such a case, the refinement is uniquely defined and is unique.. Riečan in [17] defined the entropy in MV-algebras using the well-known fact that they have the Riesz decomposition property (RDP) which is well-known in theory of ℓ-groups. (RDP) is a kind of distributivity of $+$ and $\wedge$. For this case we have more, sometimes infinitely many refinements, and Riečan gave only the basic properties of entropy.

In the present paper, we generalize the notion of entropy for situations when our probability space is an effect algebra. Effect algebras were introduced by Foulis and Bennett [8] (see also [11]) and they play a very important role in the theory of quantum structures. A crucial class of effect algebras are those having (RDP) (they admit a po-group representation [20]). A special class of effect algebras are MV-algebras introduced by Chang [1].

The paper is divided into two parts. In the first one, we introduce effect algebras and partitions (Section 2). The entropy, lower and upper entropies of partitions with respect to a state (= probability measure) are studied in Section 3. Section 4 is dedicated to entropy of dynamical systems connected with effect algebras. In Section 5, we present many examples calculating their entropies. Boolean partitions roughly speaking are connected with crisp fuzzy sets, Section 6. The elements of conditional entropies are presented in Section 7. Due to many possible refinements, the known results cannot be always generalized to our case.

The second part deals mainly with the state space of effect algebras and entropies on MV-algebras. Some results known only for product MV-algebras from [18] are generalized to all σ-complete MV-algebras without any product, simultaneously we present some solution to open Problem 7 from [18] and extend it also for effect algebras with (RDP) asking how we can proceed with entropy not assuming the product on the MV-algebra.

2. PARTITIONS OF EFFECT ALGEBRAS

The probability space in our situation will be modelled by effect algebras.

An *effect algebra* ([8]) is a partial algebra $E = (E; +, 0, 1)$ with a partially defined operation $+$ and two constant elements 0 and 1 such that, for all $a, b, c \in E$,

(i) $a + b$ is defined in E iff $b + a$ is defined, and in such the case $a + b = b + a$;

(ii) $a + b, (a + b) + c$ are defined iff $b + c$ and $a + (b + c)$ are defined, and in such the case $(a + b) + c = a + (b + c)$;

(iii) for any $a \in E$, there exists a unique element $a' \in E$ such that $a + a' = 1$;

(iv) if $a + 1$ is defined in E, then $a = 0$.

If we define $a \leq b$ iff there exists an element $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering, and we write $c := b - a$. It is clear that $a' = 1 - a$ for any $a \in E$.

For example, if (G, u) is an Abelian unital po-group with a strong unit u, [1] and if $\Gamma(G, u) := \{g \in G : 0 \leq g \leq u\}$ is endowed with the restriction of the group addition $+$, then $(\Gamma(G, u); +, 0, u)$ is an effect algebra.

We say that an effect algebra E satisfies (i) the *Riesz interpolation property*, (RIP) for short, if, for all x_1, x_2, y_1, y_2 in E, $x_i \leq y_j$ for all i, j implies there exists an element $z \in E$ such that $x_i \leq z \leq y_j$ for all i, j; (ii) the *Riesz decomposition property*, (RDP) for short, if $x \leq y_1 + y_2$ implies that there exist two elements $x_1, x_2 \in E$ with $x_1 \leq y_1$ and $x_2 \leq y_2$ such that $x = x_1 + x_2$.

We recall that (1) if E is a lattice, then E has trivially (RIP); the converse is not true. (2) E has (RDP) iff, [7, Lem 1.7.5], $x_1 + x_2 = y_1 + y_2$ implies there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $x_1 = c_{11} + c_{12}$, $x_2 = c_{21} + c_{22}$, $y_1 = c_{11} + c_{21}$, and $y_2 = c_{12} + c_{22}$. (3) (RDP) implies (RIP), but the converse is not true (e.g. if $E = L(H)$, the system of all closed subspaces of a Hilbert space H, then E is a complete lattice but without (RDP)). On the other hand, every finite poset with (RIP) is a lattice.

Ravindran [20] ([7, Thm. 1.7.17]) proved the following important result which is analogical to Mundici's representation of MV-algebras [14].

Theorem 2.1. Let E be an effect algebra with the Riesz decomposition property. Then there exists a unital interpolation group (G, u) with a strong unit u such that $\Gamma(G, u)$ is isomorphic to E.

Moreover, if ϕ^* is an isomorphism of the effect algebra E with (RDP) onto $\Gamma(G, u)$ and if $\phi : E \to H$ is a mapping preserving $+$, and H an Abelian group, then there is a group homomorphism $\gamma : G \to H$ such that $\phi = \gamma \circ \phi^*$. This γ is unique.

In addition, there is a categorical equivalence, Γ, between the category of unital po-groups with interpolation and the category of effect algebras with (RDP) given by $\Gamma : (G, u) \mapsto \Gamma(G, u)$, see [6].

A most important example of effect algebras with (RDP) is the class of MV-algebra introduced by Chang [1].

Let $M = (M; \oplus, ^*, 0, 1)$ $(0 \neq 1)$ be an MV-algebra, that is an algebra of type (2,1,0,0) such that, for all $a, b, c \in M$, we have

[1] An element $u \in G^+$ is said to be a *strong unit* for a po-group G, if given an element $g \in G$, there is an integer $n \geq 1$ such that $-nu \leq g \leq nu$; the couple (G, u) is said to be *unital po-group*. If (RIP) holds for elements of G^+, G is said to be an *interpolation group*.

(i) $a \oplus b = b \oplus a$;

(ii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(iii) $a \oplus 0 = a$;

(iv) $a \oplus 1 = 1$;

(v) $(a^*)^* = a$;

(vi) $a \oplus a^* = 1$;

(vii) $0^* = 1$;

(viii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

We note that one of the most important examples of MV-algebras is the MV-algebra $[0,1] = \Gamma(\mathbb{R}, 1)$.

We recall that according to the famous result of Mundici [14], every MV-algebra $M = \Gamma(G, u)$, where (G, u) is an ℓ-group with strong unit u, and $a \oplus b = (a+b) \wedge u$, $a^* = u - a$, $1 = u$.

Two elements a and b of M are said to be *summable* if $a \leq b^*$, and in such a case, we set $a + b = a \oplus b$. It is possible to show that $(a+b)+c = a+(b+c)$ whenever one of the sides exists (see e.g. [7, 11]). Then $(M; +, 0, 1)$ is an effect algebra with (RDP), [7]. It is possible to show that an effect algebra E can be converted into an MV-algebra iff E is a lattice satisfying (RDP).

Let E be an effect algebra. A finite sequence $\mathcal{A} = \{a_i\}_{i=1}^m$ of elements of E is a *partition* of unity 1 if $a_1 + \cdots + a_m = 1$.

A partition $\mathcal{B} = \{b_j\}_{j=1}^n$ is a *refinement* of a partition $\mathcal{A} = \{a_i\}_{i=1}^m$, and we write $\mathcal{A} \prec \mathcal{B}$, if for any element a_i $(i = 1, \dots, m)$ there is a subset $\alpha_i \subseteq \{1, \dots, n\}$ such that $a_i = \sum_{j \in \alpha_i} b_j$, $\bigcup_{i=1}^m \alpha_i = \{1, \dots, n\}$ and $\alpha_i \cap \alpha_k = \emptyset$ for $i \neq k$.

Let $\mathcal{A} = \{a_i\}_{i=1}^m$ and $\mathcal{B} = \{b_j\}_{j=1}^n$ be two partitions of 1 in an effect algebra E with (RDP). Due to (RDP), there is a joint refinement $\mathcal{C} = \{c_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ of $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j=1}^n$ such that, for all $1 \leq i \leq m$ and all $1 \leq j \leq n$, we have

$$\begin{aligned} a_i &= c_{i1} + \cdots + c_{in}, \\ b_j &= c_{1j} + \cdots + c_{mj}. \end{aligned}$$

Any such refinement $\mathcal{C} = \{c_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is said to be a *Riesz refinement* of $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j=1}^n$.

Moreover, if E is a lattice (i.e., E is an MV-algebra), we may assume that

$$(c_{i+1,j} + \cdots + c_{mj}) \wedge (c_{i,j+1} + \cdots + c_{in}) = 0 \tag{2.1}$$

for all $i < m$ and all $j < n$, and under this condition the c_{ij}'s are uniquely determined (with respect to the given orders of elements in $\mathcal{A}$ and $\mathcal{B}$).

Let $\mathcal{A}_k = \{a_j^k\}_{j=1}^{m_k}$ for $k = 1, \dots, n$ be partitions of unity in an effect algebra E with (RDP). Using the Riesz decomposition property, there is a refinement $\mathcal{C} = \{c_{i_1 \dots i_n} : 1 \leq i_1 \leq m_1, \dots, 1 \leq i_n \leq m_n\}$ of all $\mathcal{A}_k$'s such that

$$a_j^k = \sum \{c_{i_1 \dots i_n} : i_k = j,\ 1 \leq i_1 \leq m_1, \dots, 1 \leq i_n \leq m_n\} \tag{2.2}$$

for $1 \leq j \leq n_k$ and $k = 1, \ldots, n$. This $\mathcal{C}$ is said to be a Riesz refinement of $\mathcal{A}_1, \ldots, \mathcal{A}_n$.

We denote by $\mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and $\mathrm{Ref}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ the set of all Riesz refinements and the set of all refinements of the partitions $\mathcal{A}_1, \ldots, \mathcal{A}_n$, respectively.

It is worth recalling that if E is a Boolean algebra, then a Riesz refinement of two partitions $\mathcal{A} = \{a_i\}_{i=1}^m$ and $\mathcal{B} = \{b_j\}_{j=1}^n$ consists only of all intersections $\mathcal{C} = \{a_i \wedge b_j : 1 \leq i \leq m, 1 \leq j \leq n\}$, and is unique (see also Section 6). For a general case of effect algebras with (RDP) it can happen that $\mathrm{Ref}_{\mathcal{R}}(\mathcal{A}, \mathcal{B})$ may have more elements (even infinitely many) as the following example shows.

Example 2.2. Let $\mathcal{A} = (0.2, 0.8)$ ($\tilde{\mathcal{A}} = (0.8, 0.2)$) and $\mathcal{B} = (0.3, 0.7)$ be two partitions of 1 in the MV-algebra $[0, 1]$. Then the partitions $\mathcal{C} = (0.2, 0, 0.1, 0.7)$ and $\tilde{\mathcal{C}} = (0.3, 0.5, 0, 0.2)$ give Riesz refinements of $\mathcal{A}$, $\mathcal{B}$ and of $\tilde{\mathcal{A}}$, $\mathcal{B}$, respectively, such that $c_{12} \wedge c_{21} = 0 = \tilde{c}_{12} \wedge \tilde{c}_{21}$ but $\mathcal{C} \neq \tilde{\mathcal{C}}$.

3. ENTROPY

The notion of a probability measure is replaced by a state for effect algebras.

Let E be an effect algebra. A mapping $s : E \to [0, 1]$ is said to be a *state* if (i) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in E, and (ii) $s(1) = 1$. A state on MV-algebras was introduced in [2] and [14]. It is well-known [10, 7] that not every effect algebra possesses a state. However if E satisfies (RDP) and $0 \neq 1$, then E has at least one state [7], in particular, every non-degenerate MV-algebra (i. e., $0 \neq 1$) has a state.

We define a real-valued function $\phi : [0, 1] \to \mathbb{R}^+$ by

$$\phi(x) = \begin{cases} -x \log x & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

Then ϕ is a concave continuous function. It is clear that the function $\sum_{i=1}^n \phi(x_i)$ defined on $[0, 1]^n$ takes it maximum, $\log n$, under the condition $\sum_{i=1}^n x_i = 1$ in $x_i = 1/n$ for $i = 1, \ldots, n$.

Let s be a state on E. The *entropy* of the partition $\mathcal{A} = \{a_i\}_{i=1}^m$ of 1 in the state s is the expression

$$H(\mathcal{A}) := \sum_{i=1}^m \phi(s(a_i)) = -\sum_{i=1}^m s(a_i) \log(s(a_i)). \tag{3.1}$$

Proposition 3.1. Let $\mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Then

$$\max\{H(\mathcal{A}_1), \ldots, H(\mathcal{A}_n)\} \leq H(\mathcal{C}) \leq H(\mathcal{A}_1) + \cdots + H(\mathcal{A}_n). \tag{3.2}$$

Proof. Let $n = 2$ and set $\Omega = \{1, \ldots, m\} \times \{1, \ldots, k\}$, $\mathcal{S} = 2^\Omega$, and let P be a probability measure on $\mathcal{S}$ defined by $P(\{i, j\}) = s(c_{ij})$ for all i and j. Then

$(\Omega, \mathcal{S}, P)$ is a probability space which easily proves (3.2) due to entropy on the classical probability spaces. The general case can be obtained by induction. □

We recall (1) that in view of Example 2.2, different Riesz refinements of $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j=1}^n$ can give different values of its entropies. (2) Let $a_i = 1/m$ and $b_j = 1/n$. Then for their Riesz refinement $c_{ij} = 1/(mn)$ we have $H(\{c_{ij}\}) = \log mn = H(\{a_i\}) + H(\{b_j\})$.

Proposition 3.2. Let $\mathcal{A} \prec \mathcal{B}$. Then

$$H(\mathcal{A}) \leq H(\mathcal{B}). \tag{3.3}$$

Proof. Assume that $\mathcal{A} = \{a_i\}_{i=1}^m$ and $\mathcal{B} = \{b_j\}_{j=1}^n$. Without loss of generality, we can assume that all α_i's from the definition of refinements satisfy $\alpha_1 = \{1, \dots, n_1\}, \alpha_2 = \{n_1+1, \dots, n_2\}, \dots, \alpha_m = \{n_{m-1}+1, \dots, n_m\}$ where $n_m = n$. We define a Riesz refinement $\mathcal{C} = \{c_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ of $\mathcal{A}$ and $\mathcal{B}$ such that

$$c_{ij} = \begin{cases} b_j & \text{if } \; j \in \alpha_i \\ 0 & \text{otherwise.} \end{cases}$$

Then by Proposition 3.1, we have $H(\mathcal{A}), H(\mathcal{B}) \leq H(\mathcal{C})$. But it is clear that $H(\mathcal{C}) = H(\mathcal{B})$. □

We now show that the right-hand inequality of (3.2) does not hold for any refinement $\mathcal{C}$ of $\mathcal{A}_1, \dots, \mathcal{A}_n$, i.e., (3.2) holds only if $\mathcal{C}$ is a Riesz refinement of $\mathcal{A}_1, \dots, \mathcal{A}_n$.

Example 3.3. If $\mathcal{C}$ is an arbitrary refinement of $\mathcal{A}$ and $\mathcal{B}$, then it can happen that $H(\mathcal{C}) > H(\mathcal{A}) + H(\mathcal{B})$.

Indeed, let $M = [0,1]$ be the standard MV-algebra of the real interval $[0,1]$. Take two partitions $\mathcal{A} = \{1/n, \dots, 1/n\}$ and $\mathcal{B} = \{1/m, \dots, 1/m\}$, where $n, m \geq 1$ are integers. For any integer $k \geq 2$, we define $\mathcal{C}_k = \{1/(knm), \dots, 1/(knm)\}$. Then any $\mathcal{C}_k$ is not a Riesz refinement of $\mathcal{A}$ and $\mathcal{B}$, and $H(\mathcal{C}_k) = \log k + \log n + \log m > \log n + \log m = H(\mathcal{A}) + H(\mathcal{B})$.

In analogy with the classical case ([19]), for given partitions $\mathcal{A}_1, \dots, \mathcal{A}_n$ of unity 1, we define two pairs of expressions, $H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n)$ and $H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n)$, and $H_*(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n)$ and $H^*(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n)$, respectively, defined by

$$\begin{aligned}
H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n) &:= \inf\{H(\mathcal{C}) : \mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n)\}, \\
H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n) &:= \sup\{H(\mathcal{C}) : \mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n\}, \\
H_*(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n) &:= \inf\{H(\mathcal{C}) : \mathcal{C} \in \mathrm{Ref}(\mathcal{A}_1, \dots, \mathcal{A}_n)\}, \\
H^*(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n) &:= \sup\{H(\mathcal{C}) : \mathcal{C} \in \mathrm{Ref}(\mathcal{A}_1, \dots, \mathcal{A}_n)\}.
\end{aligned}$$

In view of (3.2), $H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n)$ and $H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n)$ are finite,

$$\begin{aligned}
\max\{H(\mathcal{A}_1), \dots, H(\mathcal{A}_n)\} &\leq H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n) \leq H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \dots \vee \mathcal{A}_n) \\
&\leq H(\mathcal{A}_1) + \dots + H(\mathcal{A}_n),
\end{aligned} \tag{3.4}$$

$H_*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n) \le H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n)$ but $H^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n)$ can attain the infinite value $+\infty$, see Example 5.2.

We recall that in view of Proposition 3.2, if $\mathcal{A} \prec \mathcal{B}$, then

$$H(\mathcal{A}) \le H_*^{\mathcal{R}}(\mathcal{A} \vee \mathcal{B}) = H(\mathcal{B}).$$

Proposition 3.4. Let $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}$ be partitions of unity in an effect algebra E with (RDP). Then

$$\begin{aligned} H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n) &\le H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}), \\ H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n) &\le H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}). \end{aligned}$$

Proof. Take $\mathcal{C} = \{c_{i_1\cdots i_n j}\} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B})$. If $\tilde{c}_{i_1\cdots i_n} := \sum_j c_{i_1\cdots i_n j}$, then $\tilde{\mathcal{C}} = \{\tilde{c}_{i_1\cdots i_n}\} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n\}$. Hence, $H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n) \le H(\tilde{\mathcal{C}}) \le H(\mathcal{C})$ which yields $H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n) \le H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B})$.

In a similar way we prove the second inequality. □

4. ENTROPY OF DYNAMICAL SYSTEMS

A mapping $T : E \to E$ is said to be a *transformation* of an effect algebra E if (i) $T(a+b) = T(a) + T(b)$ whenever $a+b$ is defined in E, and (ii) $T(1) = 1$.[2] A transformation T is said to be *preserving* the state s (or s-preserving) if $s(T(a)) = s(a)$ for any $a \in E$. Let s be a state on an effect-algebra E. A triple (E, s, T) is said to be a *dynamical system* if T is a transformation of E preserving the state s. We recall that every effect algebra E with (RDP) has a state, and, for any state s, there is an s-preserving transformation (e. g., the identity of E). In what follows, we will assume that T is s-preserving.

If $\mathcal{A} = \{a_i\}_{i=1}^m$ is a partition of unity, so is $T(\mathcal{A}) := \{T(a_i)\}_{i=1}^m$, and $H(\mathcal{A}) = H(T(\mathcal{A}))$.

For any partition $\mathcal{A}$ of unity 1 and for any integer $n \ge 1$, we define

$$\begin{aligned} H_*^n(\mathcal{A}, T)_{\mathcal{R}} &:= H_*^{\mathcal{R}}(\mathcal{A} \vee T(\mathcal{A}) \vee \cdots \vee T^{n-1}(\mathcal{A})), \\ H_n^*(\mathcal{A}, T)_{\mathcal{R}} &:= H_{\mathcal{R}}^*(\mathcal{A} \vee T(\mathcal{A}) \vee \cdots \vee T^{n-1}(\mathcal{A})), \\ H_*^n(\mathcal{A}, T) &:= H_*(\mathcal{A} \vee T(\mathcal{A}) \vee \cdots \vee T^{n-1}(\mathcal{A})), \\ H_n^*(\mathcal{A}, T) &:= H^*(\mathcal{A} \vee T(\mathcal{A}) \vee \cdots \vee T^{n-1}(\mathcal{A})). \end{aligned}$$

In view of (3.2) – (3.3), we have, similarly as for MV-algebras in [17],

$$0 \le H(\mathcal{A}) \le H_*^n(\mathcal{A}, T) \le H_*^n(\mathcal{A}, T)_{\mathcal{R}} \le H_n^*(\mathcal{A}, T)_{\mathcal{R}} \le nH(\mathcal{A}). \tag{4.1}$$

[2] If $(\Omega, \mathcal{S}, P, T)$ is a classical dynamical system, then the mapping $T^{-1} : \mathcal{S} \to \mathcal{S}$ is our transformation.

Theorem 4.1. Let (E,s,T) be a dynamical system connected with an effect algebra E with (RDP). For any partition $\mathcal{A}$, there exist limits

$$h_*^{\mathcal{R}}(\mathcal{A},T) := \lim_n \frac{1}{n} H_*^n(\mathcal{A},T)_{\mathcal{R}}, \quad h_{\mathcal{R}}^*(\mathcal{A},T) := \lim_n \frac{1}{n} H_n^*(\mathcal{A},T)_{\mathcal{R}}.$$

Proof. First of all, we show that $H_*^{n+m}(\mathcal{A},T)_{\mathcal{R}} \le H_*^n(\mathcal{A},T)_{\mathcal{R}} + H_*^m(\mathcal{A},T)_{\mathcal{R}}$ holds for all positive integers n and m.

Assume that the partition $\mathcal{A} = \{a_i\}_{i=1}^k$. Let $\mathcal{C}$ be a Riesz refinement of partitions $\mathcal{A}, T(\mathcal{A}), \ldots, T^{n-1}(\mathcal{A})$, and $\mathcal{D}$ be a Riesz refinement of $\mathcal{A}, T(\mathcal{A}), \ldots, T^{m-1}(\mathcal{A})$, respectively. They consist of k^n and k^m elements, respectively. Then $T^n(\mathcal{D})$ is a Riesz refinement of $T^n(\mathcal{A}), T^{n+1}(\mathcal{A}), \ldots, T^{n+m-1}(\mathcal{A})$.

Let now $\mathcal{E}$ be any Riesz refinement of $\mathcal{C}$ and $T^n(\mathcal{D})$ consisting of $k^n k^m$ elements. Then $\mathcal{E} \succ \mathcal{A}, \mathcal{E} \succ T(\mathcal{A}), \ldots, \mathcal{E} \succ T^{n+m-1}(\mathcal{A})$ and, in addition, $\mathcal{C}$ is their Riesz refinement. By (3.2), we have

$$H_*^{n+m}(\mathcal{A},T)_{\mathcal{R}} \le H(\mathcal{E}) \le H(\mathcal{C}) + H(T^n(\mathcal{D})) = H(\mathcal{C}) + H(\mathcal{D}).$$

Since $\mathcal{D}$ is arbitrary, $H_*^{n+m}(\mathcal{A},T)_{\mathcal{R}} - H(\mathcal{D}) \le H(\mathcal{C})$, so that $H_*^{n+m}(\mathcal{A},T)_{\mathcal{R}} - H(\mathcal{D}) \le H_*^n(\mathcal{A},T)_{\mathcal{R}}$ while $\mathcal{C}$ is a Riesz refinement of $\mathcal{A}, T(\mathcal{A}), \ldots, T^{n-1}(\mathcal{A})$. By a similar argument we have $H_*^{n+m}(\mathcal{A},T)_{\mathcal{R}} - H_*^n(\mathcal{A},T)_{\mathcal{R}} \le H_*^m(\mathcal{A},T)_{\mathcal{R}}$.

By a well known argument, if a sequence of non-negative numbers, $\{a_n\}$, has the property $a_{n+m} \le a_n + a_m$, then there is a finite limit $\lim_n a_n/n$.

The second limit can be proved by analogous reasoning, we only stress that in view of $H(\mathcal{E}) \le H(\mathcal{C}) + H(\mathcal{D})$, we have $H(\mathcal{E}) \le H_n^*(\mathcal{A},T)_{\mathcal{R}} + H_m^*(\mathcal{A},T)_{\mathcal{R}}$. $\square$

In analogy with Theorem 4.1, we can introduce also $h_*(\mathcal{A},T)$ as a limit of $H_*^n(\mathcal{A},T)$ whenever it exists.

The lower and upper entropy, $h_*^{\mathcal{R}}(T)$ and $h_{\mathcal{R}}^*(T)$, of a dynamical system (E,s,T), where E satisfies (RDP), are defined as follows

$$\begin{aligned} h_*^{\mathcal{R}}(T) &:= \sup\{h_*^{\mathcal{R}}(\mathcal{A},T) : \mathcal{A} \text{ is a Riesz partition of } E\}, \\ h_{\mathcal{R}}^*(T) &:= \sup\{h_{\mathcal{R}}^*(\mathcal{A},T) : \mathcal{A} \text{ is a Riesz partition of } E\}. \end{aligned}$$

The notion of entropy of a dynamical system was introduced by Kolmogorov and Sinai. Their aim was to characterize isomorphic dynamical systems, and they proved: Two isomorphic dynamical systems have the same entropy. They showed that some Bernoulli schemes are not isomorphic. An analogical result can be proved also for effect algebras.

We recall that two dynamical systems (E_1,s_1,T_1) and (E_2,s_2,T_2) are *isomorphic* if there exists a bijective mapping $\psi : E_1 \to E_2$ such that $\psi(a) + \psi(b) = \psi(c)$ iff $a + b = c$, $\psi(1) = 1$, and $s_2(\psi(a)) = s_1(a)$ and $T_2(\psi(a)) = \psi(T_1(a))$ hold for all $a \in E_1$.

Theorem 4.2. If (E_1, s_1, T_1) and (E_2, s_2, T_2) are isomorphic dynamical systems, where E_1 and E_2 have (RDP), then $h_*^{\mathcal{R}}(T_1) = h_*^{\mathcal{R}}(T_2)$ and $h_{\mathcal{R}}^*(T_1) = h_{\mathcal{R}}^*(T_2)$.

Proof. If $\mathcal{A}$ is a partition in E_1, so is $\psi(\mathcal{A})$ in E_2 and vice versa, and $H(\mathcal{A}) = H(\psi(\mathcal{A}))$. If a refinement $\mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}, T_1(\mathcal{A}), \dots, T_1^{n-1}(\mathcal{A}))$, then a refinement $\psi(\mathcal{C}) \in \mathrm{Ref}_{\mathcal{R}}(\psi(\mathcal{A}), T_2(\psi(\mathcal{A})), \dots, T_2^{n-1}(\psi(\mathcal{A})))$ and vice versa. Therefore, it is easy to prove that $H_*^n(\mathcal{A}, T_1)_{\mathcal{R}} = H_*^n(\psi(\mathcal{A}), T_2)_{\mathcal{R}}$ and consequently, $h_*^{\mathcal{R}}(\mathcal{A}, T_1) = h_*^{\mathcal{R}}(\psi(\mathcal{A}), T_2)$ and $h_*^{\mathcal{R}}(T_1) = h_*^{\mathcal{R}}(T_2)$. In a similar way we can prove the second equality. $\square$

We recall that the equality $h_*^{\mathcal{R}}(T_1) = h_*^{\mathcal{R}}(T_2)$ does not imply that two dynamical systems (E_1, s_1, T_1) and (E_2, s_2, T_2) are necessarily isomorphic as examples below show.

5. EXAMPLES

In the present section, we give some examples of dynamical systems and calculate their entropies.

Example 5.1. Let $M_k = \{0, 1/k, 2/k, \dots, k/k\}$ be a finite MV-algebra, $k \geq 1$. Then M_k possesses a unique state s and a unique transformation T, namely $s(1/k) = T(1/k) = 1/k$. Then $\{1/k, \dots, 1/k\}$ is the finest refinement of unity in M_k. Therefore, $0 \leq H_*^n(\mathcal{A}, T)_{\mathcal{R}} \leq H_n^*(\mathcal{A}, T)_{\mathcal{R}} \leq H_n^*(\mathcal{A}, T) \leq \log k$, which implies $0 = h_*^{\mathcal{R}}(\mathcal{A}, T) = \lim_n H_*^n(\mathcal{A}, T)_{\mathcal{R}}/n \leq \lim_n H_n^*(\mathcal{A}, T)_{\mathcal{R}}/n \leq \lim_n (\log k)/n = 0$. So that $h_*^{\mathcal{R}}(T) = h_{\mathcal{R}}^*(T) = 0$.

Example 5.2. Let $M_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$ be the MV-algebra of all rational numbers in the real interval $[0,1]$. Then $M_{\mathbb{Q}}$ possesses a unique state s and a unique transformation T, namely $s(t) = T(t) = t$ for any $t \in M_{\mathbb{Q}}$.

Let $\mathcal{A}_k = \{1/k, \dots, 1/k\}$ for any integer $k \geq 1$. Then $H_n^*(\mathcal{A}_k, T)_{\mathcal{R}} = \sup\{H(\mathcal{C}) : \mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_k, T(\mathcal{A}_k), \dots, T^{n-1}(\mathcal{A}_k)\} = n \log k$ and $h_{\mathcal{R}}^*(\mathcal{A}_k, T) = \log k$.

$H_n^*(\mathcal{A}_k, T) = \sup\{H(\mathcal{C}) : \mathcal{C} \succ \mathcal{A}_k\} \geq \sup\{H(\{1/(mk)\}) : m \geq 1\} = \sup\{\log m + \log k) : m \geq 1\} = \infty$.

Let $\mathcal{A} = \{t_1, \dots, t_k\}$ be an arbitrary partition in $M_{\mathbb{Q}}$. Then by (3.3), $H_*^n(\mathcal{A}, T) = \inf\{H(\mathcal{C}) : \mathcal{C} \succ \mathcal{A}\} = H(\mathcal{A})$, i.e., $h_*(\mathcal{A}, T) = 0$.

Define a partition $\mathcal{D} = \{t_{i_1} \cdots t_{i_n} : t_{i_j} \in \{t_1, \dots, t_k\},\ j = 1, \dots, n\}$. Then $\mathcal{D}$ is a Riesz refinement of $\mathcal{A}, T(\mathcal{A}), \dots, T^{n-1}(\mathcal{A})$. Therefore, $nH(\mathcal{A}) \geq H_n^*(\mathcal{A}, T)_{\mathcal{R}} \geq H(\mathcal{D}) = nH(\mathcal{A})$, consequently, $H_n^*(\mathcal{A}, T)_{\mathcal{R}} = nH(\mathcal{A})$, and $h_{\mathcal{R}}^*(\mathcal{A}, T) = H(\mathcal{A})$, $h_{\mathcal{R}}^*(T) = \infty$.

We define a partition $\mathcal{C} = \{c_{i_1 \dots i_n} : 1 \leq i_j \leq k,\ j = 1, \dots, n\}$, where $c_{i_1 \dots i_n} = t_i$ if $i_1 = i_2 = \cdots = i_n = i$ and $c_{i_1 \dots i_n} = 0$ otherwise. Then $\mathcal{C}$ is a Riesz refinement of $\mathcal{A}, T(\mathcal{A}), \dots, T^{n-1}(\mathcal{A})$. Hence, $H(\mathcal{A}) \leq H_*^n(\mathcal{A}, T)_{\mathcal{R}} \leq H(\mathcal{C}) = H(\mathcal{A})$, which gives $h_*^{\mathcal{R}}(\mathcal{A}, T) = 0$ and $h_*^{\mathcal{R}}(T) = 0$.

Example 5.3. Let now $M = [0,1]$ be the standard MV-algebra of the real interval. Then M possesses a unique state s and a unique transformation T, namely $s(t) = T(t) = t$ for any $t \in M$. Then the same statements on entropies as in Example 5.2 hold, i.e. $h^*_{\mathcal{R}}(T) = \infty$, and $h^{\mathcal{R}}_*(T) = h^{\mathcal{R}}_*(\mathcal{A}, T) = 0$ for any partition $\mathcal{A}$ in M.

Since two MV-subalgebras of $[0,1]$ are isomorphic iff they are same ([3, Cor. 7.2.6]), we have infinitely many non-isomorphic MV-subalgebras of $[0,1]$, and we now calculate their entropy. We recall that each of them has a unique state, s, and a unique transformation, T, namely $s(t) = T(t) = t$ for any $t \in M$.

Example 5.4. Let $D = \{k/2^n : 0 \le k \le 2^n, n \ge 1\}$ be the MV-algebra of all dyadic numbers in $[0,1]$. Then $h^*_{\mathcal{R}}(T) = \infty$, and $h^{\mathcal{R}}_*(T) = h^{\mathcal{R}}_*(\mathcal{A}, T) = 0$ for any partition $\mathcal{A}$ in M.

Example 5.5. Let M be a multiplicative MV-subalgebra of $[0,1]$, i.e., M is an MV-algebra such that if $t_1, t_2 \in M$, then the product $t_1 t_2 \in M$. Multiplicative MV-subalgebras of $[0,1]$ are either $\{0,1\}$ or they have to be infinite. Example 5.1 is not multiplicative, and Examples 5.2–5.4 are multiplicative. Then $h^{\mathcal{R}}_*(T) = h^{\mathcal{R}}_*(\mathcal{A}, T) = 0$ for any partition $\mathcal{A}$ in M. For multiplicative MV-subalgebras of $[0,1]$ we can calculate $h^*_{\mathcal{R}}(T) = \infty$.

Example 5.6. Let α be an irrational number from $(0,1)$ and let $M(\alpha)$ be the MV-subalgebra of $[0,1]$ generated by α. Then $M(\alpha) = \{m + n\alpha : m, n \in \mathbb{Z}, 0 \le m + n\alpha \le 1\}$, [3, p. 149], is countable and dense in $[0,1]$, and $M(\alpha) = M(\beta)$ iff $\alpha = \beta$ or $\alpha = 1 - \beta$. For example, if $\alpha = \sqrt{2}/2$, then $M(\alpha)$ is not multiplicative. For any α, we have $h^{\mathcal{R}}_*(T) = h^{\mathcal{R}}_*(\mathcal{A}, T) = 0$ for any partition $\mathcal{A}$ in M.

Example 5.7. Let M be any MV-subalgebra of $[0,1]$. Then we have $h^{\mathcal{R}}_*(T) = h^{\mathcal{R}}_*(\mathcal{A}, T) = 0$ for any partition $\mathcal{A}$ in M.

Example 5.8. Let E be a finite lattice effect algebra with (RDP). Then E has a unique transformation T, the identity. This follows from the fact that E is a direct product of effect algebras (= MV-algebras) from the Example 5.1, and in every state $h^{\mathcal{R}}_*(T) = h^*_{\mathcal{R}}(T) = 0$.

Example 5.9. Let G be an interpolation directed Abelian po-group and define the lexicographical product $G(\mathbb{Z}) := \mathbb{Z} \times_{lex} G$, where $\mathbb{Z}$ is the group of all integers. Then the element $(1,0)$ is a strong unit in the po-group $G(\mathbb{Z})$ and

$$E(G) := \Gamma(G(\mathbb{Z}), (1,0)),$$

is an effect algebra with (RDP) [6]. Every element $a \in E(G)$ is of the form either $a = (1, -g)$ or $a = (0, g)$, where $g \in G^+$. $E(G)$ has a unique state s, namely $s(0,g) = 0$ and $s(1,-g) = 1$. For any integer $k \ge 1$, we set $T_k : E(G) \to E(G)$ by $T_k(0,g) = (0, kg)$ and $T_k(1,-g) = (1, -kg)$. Then every T_k is s-preserving, and $h^{\mathcal{R}}_*(T_k) = h^*_{\mathcal{R}}(T_k) = 0$.

Another s-preserving transformation is the mapping $T: E(G) \to E(G)$ defined by $T(0,g) = (0,0)$ and $T(1,-g) = (1,0)$ for any $g \in G^+$. Of course, both entropies of this T are also 0.

Example 5.10. Let $G = \mathbb{Q} \times_{lex} \mathbb{Z}$, where $\mathbb{Q}$ is the group of all rational numbers with the usual ordering and $\mathbb{Z}$ is the group of all integers with the discrete ordering. Then G is an interpolation group with strong unit $(1,0)$, and the effect algebra $E = \Gamma(G,(1,0))$ satisfies (RDP) and is not a lattice.

E has a unique state s, namely $s(q,n) = q$. Let T be the identity, and $\mathcal{A} = \{(1/k,0)\}$. Then, for any $n \geq 1$, $\mathcal{C} = \{(1/(kn),0)\}$ is its Riesz refinement, and hence $H_n^*(\mathcal{A},T)_{\mathcal{R}} = \log nk$, so that $h_{\mathcal{R}}^*(\mathcal{A},T) = 0 = h_{\mathcal{R}}^*(T)$.

6. BOOLEAN PARTITIONS AND ENTROPY

Let E be an effect algebra. For an element $e \in E$, we denote by $[0,e] := \{x \in E : 0 \leq x \leq e\}$. Then $[0,e]$ endowed with $+$ restricted to $[0,e] \times [0,e]$ is an effect algebra $[0,e] = ([0,e];+,0,e)$, and, for any $x \in [0,e]$, we have $x' := e - x$.

According to [7] or [5], an element e of an effect algebra E is said to be *central* (or *Boolean*) if there exists an isomorphism

$$f_e : E \to [0,e] \times [0,e']$$

such that $f_e(e) = (e,0)$ and if $f_e(x) = (x_1,x_2)$, then $x = x_1 + x_2$ for any $x \in E$.

We denote by $C(E)$ the set of all central elements of E. A partition $\mathcal{A} = \{e_i\}$ in E such that every element e_i is central is said to be *Boolean*. We have (i) $0,1 \in C(E)$, and if $e \in C(E)$, then $e' \in C(E)$; (ii) $C(E)$ is a Boolean algebra; (iii) if $x \in E$ and $e \in C(E)$, then $x \wedge e \in E$; (iv) if $\{e_i\}_{i=1}^n$ is a Boolean partition in E, then for every element $x \in E$ we have $x = x \wedge e_1 + \cdots + x \wedge e_n$; (v) if E is with (RDP), then $e \in C(E)$ iff $e \wedge e' = 0$.

Let $\mathcal{A}_k = \{e_j^k\}_{j=1}^{m_k}$ for $k = 1,\dots,n$ be Boolean partitions of unity in an effect algebra. Then $\mathcal{E} = \{e_{i1}^1 \wedge \cdots \wedge e_{i_n}^n : 1 \leq i_1 \leq m_1,\dots,1 \leq i_n \leq m_n\}$ is a unique Riesz refinement of $\mathcal{A}_1,\dots,\mathcal{A}_n$. In fact, if $\mathcal{C} = \{c_{i_1\cdots i_n}\}$ is a Riesz refinement of $\mathcal{A}_1,\dots,\mathcal{A}_n$, then by (2.2) $c_{i_1\cdots i_n} \leq e_{i1}^1 \wedge \cdots \wedge e_{i_n}^n$, which easily implies their equality. We recall that $\mathcal{E}$ is also a Boolean partition.

Consequently,

$$H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n) = H(\mathcal{E}) = H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n). \tag{6.1}$$

If, in particular, T preserves the central elements (this can happen e.g. if T is an automorphism of E or if it preserves all existing finite infima and suprema in E) we can define entropy, $h_B(T)$, when we restrict T and s to $C(E)$, defined by

$$h_B(T) := \sup\{h_{\mathcal{R}}^*(\mathcal{A},T) : \mathcal{A} \text{ is a Boolean partition}\}.$$

Then $h_B(T) = \sup\{h_*^{\mathcal{R}}(\mathcal{A},T) : \mathcal{A}$ is a Boolean partition$\}$ and

$$h_B(T) \leq h_*^{\mathcal{R}}(T). \tag{6.2}$$

It is interesting to exhibit when in (6.2) we have the identity. Of course, this is true whenever $h_*^{\mathcal{R}}(T) = 0$. Such a situation happens e. g. in Theorem 9.4 [4].

7. CONDITIONAL ENTROPY

Let $\mathcal{C} = \{c_{ij} : 1 \le i \le m,\ 1 \le j \le n\}$ be a Riesz refinement of two partitions $\mathcal{A} = \{a_i\}_{i=1}^m$ and $\mathcal{B} = \{b_j\}_{j=1}^n$. We define a conditional entropy, $H_{\mathcal{C}}(\mathcal{A}|\mathcal{B})$, in a state s on an effect algebra E by

$$H_{\mathcal{C}}(\mathcal{A}|\mathcal{B}) := \sum_{ij} \left\{ s(b_j)\phi\left(\frac{s(c_{ij})}{s(b_j)}\right) : s(b_j) > 0 \right\}. \tag{7.1}$$

Proposition 7.1. Let $\mathcal{C}$ be a Riesz refinement of partitions $\mathcal{A}$ and $\mathcal{B}$. Then

$$H_{\mathcal{C}}(\mathcal{A}|\mathcal{B}) \le H(\mathcal{A}). \tag{7.2}$$

Proof. To avoid technical details, we can assume that $s(b_j) > 0$ for every j. Since the function ϕ is concave, we have

$$\begin{aligned} H_{\mathcal{C}}(\mathcal{A}|\mathcal{B}) &= \sum_{i=1}^m \sum_{j=1}^n s(b_j)\phi\left(\frac{s(c_{ij})}{s(b_j)}\right) \le \sum_{i=1}^m \phi\left(\sum_{j=1}^n \frac{s(b_j)s(c_{ij})}{s(b_j)}\right) \\ &= \sum_{i=1}^m \phi(s(a_i)) = H(\mathcal{A}). \end{aligned}$$

□

Proposition 7.2. Let $\mathcal{C}$ be a Riesz refinement of partitions $\mathcal{A}$ and $\mathcal{B}$. Then

$$H(\mathcal{C}) = H(\mathcal{A}) + H_{\mathcal{C}}(\mathcal{B}|\mathcal{A}). \tag{7.3}$$

Proof. Without loss of generality, we can assume $s(a_i) > 0$ for all i. Then

$$\begin{aligned} H(\mathcal{C}) &= \sum_{i=1}^m \sum_{j=1}^n \phi(s(c_{ij})) = \sum_{i=1}^m \sum_{j=1}^n \phi\left(\frac{s(a_i)s(c_{ij})}{s(a_i)}\right) \\ &= -\sum_{i=1}^m \sum_{j=1}^n s(c_{ij})\log s(a_i) - \sum_{i=1}^m \sum_{j=1}^n s(c_{ij})\log\frac{s(c_{ij})}{s(a_i)} \\ &= -\sum_{i=1}^m s(a_i)\log s(a_i) + \sum_{i=1}^m \sum_{j=1}^n s(a_i)\phi(s(c_{ij})/s(a_i)) \\ &= H(\mathcal{A}) + H_{\mathcal{C}}(\mathcal{B}|\mathcal{A}). \end{aligned}$$

□

According to (7.2), given two partitions $\mathcal{A}$ and $\mathcal{B}$, the following two functions, called *lower* and *upper conditional entropies* of $\mathcal{A}$ with respect to $\mathcal{B}$, are finite

$$H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}) := \inf\{H_{\mathcal{C}}(\mathcal{A}|\mathcal{B}) : \ \mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A},\mathcal{B})\},$$
$$H_{\mathcal{R}}^*(\mathcal{A}|\mathcal{B}) := \sup\{H_{\mathcal{C}}(\mathcal{A}|\mathcal{B}) : \ \mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A},\mathcal{B})\}.$$

In particular, if $\mathcal{B} = \{1\}$, then $H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}) = H_{\mathcal{R}}^*(\mathcal{A}|\mathcal{B}) = H(\mathcal{A})$.

We recall that if $\mathcal{A} = \{a_i\}_{i=1}^m$ is an arbitrary partition of unity in an effect algebra E and $\mathcal{B} = \{b_j\}_{j=1}^n$ is a Boolean partition, then, due to the properties (iii) – (iv) of central elements, $\mathcal{C} := \{a_i \wedge b_j : \ 1 \le i \le m, \ 1 \le j \le n\}$ is a unique Riesz refinement of $\mathcal{A}$ and $\mathcal{B}$. Therefore

$$H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}) = H_{\mathcal{R}}^*(\mathcal{A}|\mathcal{B}).$$

In such a case we simply write $H(\mathcal{A}|\mathcal{B}) = H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}) = H_{\mathcal{R}}^*(\mathcal{A}|\mathcal{B})$.

Let now $\mathcal{A}$ and $\mathcal{B}$ be arbitrary partitions. By (7.3), we have $H_*^{\mathcal{R}}(\mathcal{A} \vee \mathcal{B}) \le H(\mathcal{A}) + H_{\mathcal{C}}(\mathcal{B}|\mathcal{A})$ and $H_{\mathcal{R}}^*(\mathcal{A} \vee \mathcal{B}) \ge H(\mathcal{A}) + H_{\mathcal{C}}(\mathcal{B}|\mathcal{A})$, i. e.

$$\begin{aligned} H_*^{\mathcal{R}}(\mathcal{A} \vee \mathcal{B}) &\le H(\mathcal{A}) + H_*^{\mathcal{R}}(\mathcal{B}|\mathcal{A}), \\ H_{\mathcal{R}}^*(\mathcal{A} \vee \mathcal{B}) &\ge H(\mathcal{A}) + H_{\mathcal{R}}^*(\mathcal{B}|\mathcal{A}). \end{aligned}$$

On the other hand by the use of definition, $H_*^{\mathcal{R}}(\mathcal{B}|\mathcal{A}) \le H_{\mathcal{C}}(\mathcal{B}|\mathcal{A})$ and $H_{\mathcal{R}}^*(\mathcal{B}|\mathcal{A}) \ge H_{\mathcal{C}}(\mathcal{B}|\mathcal{A})$, i. e., $H_*^{\mathcal{R}}(\mathcal{B}|\mathcal{A}) + H(\mathcal{A}) \le H_{\mathcal{C}}(\mathcal{B}|\mathcal{A}) + H(\mathcal{A})$ and $H_{\mathcal{R}}^*(\mathcal{B}|\mathcal{A}) + H(\mathcal{A}) \ge H_{\mathcal{C}}(\mathcal{B}|\mathcal{A}) + H(\mathcal{A})$, which gives

$$H_*^{\mathcal{R}}(\mathcal{A} \vee \mathcal{B}) = H(\mathcal{A}) + H_*^{\mathcal{R}}(\mathcal{B}|\mathcal{A}), \tag{7.4}$$

$$H_{\mathcal{R}}^*(\mathcal{A} \vee \mathcal{B}) = H(\mathcal{A}) + H_{\mathcal{R}}^*(\mathcal{B}|\mathcal{A}). \tag{7.5}$$

Let now $\mathcal{A}_1, \dots, \mathcal{A}_n$ and $\mathcal{B}$ be partitions in an effect algebra E with (RDP). Take $\mathcal{A}_0^n \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n)$ and $\mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_0^n, \mathcal{B})$. Then by (7.3), we have $H(\mathcal{C}) = H(\mathcal{B}) + H_{\mathcal{C}}(\mathcal{A}_0^n|\mathcal{B})$, which gives $H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}) \le H(\mathcal{C})$. Hence $H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}) \le H(\mathcal{B}) + H_*^{\mathcal{R}}(\mathcal{A}_0^n|\mathcal{B})$, i. e.

$$\begin{aligned} H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}) &\le H(\mathcal{B}) + H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}), \\ H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}) &\ge H(\mathcal{B}) + H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}), \end{aligned}$$

where

$$H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}) := \inf\{H_*^{\mathcal{R}}(\mathcal{A}_0^n|\mathcal{B}) : \ \mathcal{A}_0^n \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n)\},$$
$$H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}) := \sup\{H_{\mathcal{R}}^*(\mathcal{A}_0^n|\mathcal{B}) : \ \mathcal{A}_0^n \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n)\}.$$

As in (7.4) – (7.5) we can prove

$$H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}) = H(\mathcal{B}) + H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}), \tag{7.6}$$

$$H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}) = H(\mathcal{B}) + H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}). \tag{7.7}$$

In a similar way, let $\mathcal{A}_1, \dots, \mathcal{A}_n$ and $\mathcal{B}_1, \dots, \mathcal{B}_m$ be partitions in E with (RDP). Choose $\mathcal{A}_0^n \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n)$, $\mathcal{B}_0^m \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{B}_1, \dots, \mathcal{B}_m)$, and $\mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_0^n, \mathcal{B}_0^m)$.

Then $\mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}_1, \dots, \mathcal{B}_m)$. By (7.6), we have $H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}_0^m) = H(\mathcal{B}_0^m) + H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n | \mathcal{B}_0^m)$. Hence $H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}_0^m) \geq H_*^{\mathcal{R}}(\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) + H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m)$ and

$$H_*^{\mathcal{R}}\left(\bigvee_{i=1}^{n} \mathcal{A}_i \vee \bigvee_{j=1}^{m} \mathcal{B}_j\right) \geq H_*^{\mathcal{R}}\left(\bigvee_{j=1}^{m} \mathcal{B}_j\right) + H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m).$$

In a similar way we have

$$H^*_{\mathcal{R}}\left(\bigvee_{i=1}^{n} \mathcal{A}_i \vee \bigvee_{j=1}^{m} \mathcal{B}_j\right) \leq H^*_{\mathcal{R}}\left(\bigvee_{j=1}^{m} \mathcal{B}_j\right) + H^*_{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m),$$

where

$$H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) := \inf\{H_{\mathcal{C}}(\mathcal{A}_0^n | \mathcal{B}_0^m)\},$$
$$H^*_{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) := \sup\{H_{\mathcal{C}}(\mathcal{A}_0^n | \mathcal{B}_0^m)\}.$$

We recall that it is possible to show that

$$H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) = \inf\{H_{\mathcal{C}}(\mathcal{C}_{\mathcal{A}} | \mathcal{C}_{\mathcal{B}})\},$$
$$H^*_{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) = \sup\{H_{\mathcal{C}}(\mathcal{C}_{\mathcal{A}} | \mathcal{C}_{\mathcal{B}})\},$$

where $\mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}_1, \dots, \mathcal{B}_m)$ and $\mathcal{C}_{\mathcal{A}}$, $\mathcal{C}_{\mathcal{B}}$ are the Riesz refinements of $\mathcal{A}_1, \dots, \mathcal{A}_n$ and $\mathcal{B}_1, \dots, \mathcal{B}_m$, respectively, obtained from $\mathcal{C}$.

We recall that in the last two inequalities, there are possible cases to have proper inequalities, see the footnote in Example 7.5. However, if $\mathcal{B}_1, \dots, \mathcal{B}_m$ are Boolean partitions, then they possess a unique Riesz refinement, $\mathcal{B}_0 = \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m$, which is also Boolean, and from (7.6) and (7.7) we have the following equalities

$$H_*^{\mathcal{R}}\left(\bigvee_{i=1}^{n} \mathcal{A}_i \vee \bigvee_{j=1}^{m} \mathcal{B}_j\right) = H_*^{\mathcal{R}}\left(\bigvee_{j=1}^{m} \mathcal{B}_j\right) + H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m), \tag{7.8}$$

$$H^*_{\mathcal{R}}\left(\bigvee_{i=1}^{n} \mathcal{A}_i \vee \bigvee_{j=1}^{m} \mathcal{B}_j\right) = H^*_{\mathcal{R}}\left(\bigvee_{j=1}^{m} \mathcal{B}_j\right) + H^*_{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m). \tag{7.9}$$

Proposition 7.3. Let $\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_{m+1}$ be partitions in an effect algebra E with (RDP). Then

$$H_*^{\mathcal{R}}(\mathcal{A} | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_{m+1}) \leq H_*^{\mathcal{R}}(\mathcal{A} | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m), \tag{7.10}$$

$$H^*_{\mathcal{R}}(\mathcal{A} | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_{m+1}) \leq H^*_{\mathcal{R}}(\mathcal{A} | \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m). \tag{7.11}$$

Proof. Let $\mathcal{B}_0^{m+1} = \{b_{i_1 \cdots i_{m+1}}\} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{B}_1, \dots, \mathcal{B}_{m+1})$, $\mathcal{C}^{m+1} = \{c_{i_1 \cdots i_{m+1} k}\} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_{m+1})$. We consider partitions $\mathcal{B}_0^m = \{\tilde{b}_{i_1 \cdots i_m}\} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{B}_1, \dots, \mathcal{B}_m)$

and $\mathcal{C}^m = \{\tilde{c}_{i_1\cdots i_m k}\} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_m)$, where $\tilde{b}_{i_1\cdots i_m} = \sum_{i_{m+1}} b_{i_1\cdots i_{m+1}}$ and $\tilde{c}_{i_1\cdots i_m k} = \sum_{i_{m+1}} c_{i_1\cdots i_{m+1}k}$. Without loss of generality, we can assume that $s(b_{i_1\cdots i_{m+1}}) > 0$ for all $i_1, \ldots, i_{m+1}$. Hence

$$\begin{aligned} H_{\mathcal{C}^{m+1}}(\mathcal{A}|\mathcal{B}_0^{m+1}) &= \sum_{i_1\cdots i_{m+1}k} s(b_{i_1\cdots i_{m+1}})\phi\left(\frac{s(c_{i_1\cdots i_{m+1}k})}{s(b_{i_1\cdots i_{m+1}})}\right) \\ &= \sum_{i_1\cdots i_m k}\sum_{i_{m+1}} s(\tilde{b}_{i_1\cdots i_m})\frac{s(b_{i_1\cdots i_{m+1}})}{s(\tilde{b}_{i_1\cdots i_m})}\phi\left(\frac{s(c_{i_1\cdots i_{m+1}k})}{s(b_{i_1\cdots i_{m+1}})}\right) \\ &\leq \sum_{i_1\cdots i_m k} s(\tilde{b}_{i_1\cdots i_m})\phi\left(\sum_{i_{m+1}} \frac{s(b_{i_1\cdots i_{m+1}})}{s(\tilde{b}_{i_1\cdots i_m})}\frac{s(c_{i_1\cdots i_{m+1}k})}{s(b_{i_1\cdots i_{m+1}})}\right) \\ &= H_{\mathcal{C}^m}(\mathcal{A}|\mathcal{B}_0^m). \end{aligned}$$

□

We recall that (7.10)–(7.11) hold also for partitions $\mathcal{A}_1, \ldots, \mathcal{A}_n$ instead of one partition $\mathcal{A}$.

In addition, we have

$$H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) \leq H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_{n+1}|\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m), \quad (7.12)$$

$$H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) \leq H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_{n+1}|\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m). \quad (7.13)$$

Indeed, due to (7.3) the following two sets

$$\{H_{\mathcal{C}}(\mathcal{A}_0^n|\mathcal{B}_0^m) : \mathcal{A}_0^n \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \ldots, \mathcal{A}_n), \mathcal{B}_0^m \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{B}_1, \ldots, \mathcal{B}_m), \\ \mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_0^n, \mathcal{B}_0^m)\}$$

and

$$\{H(\mathcal{E}) - H(\mathcal{D}) : \mathcal{D} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{B}_1, \ldots, \mathcal{B}_m), \mathcal{E} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{D})\}$$

are equal. Therefore,

$$\begin{aligned} & H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) \\ = \; & \inf\{H_{\mathcal{C}}(\mathcal{A}_0^n|\mathcal{B}_0^m) : \mathcal{A}_0^n \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \ldots, \mathcal{A}_n), \\ & \qquad \mathcal{B}_0^m \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{B}_1, \ldots, \mathcal{B}_m), \mathcal{C} \in \mathrm{Ref}(\mathcal{A}_0^n, \mathcal{B}_0^m)\} \\ = \; & \inf\{H(\mathcal{E}) - H(\mathcal{D}) : \mathcal{D} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{B}_1, \ldots, \mathcal{B}_m), \mathcal{E} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{D})\} \\ \leq \; & \inf\{H(\mathcal{E}') - H(\mathcal{D}') : \mathcal{D}' \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{B}_1, \ldots, \mathcal{B}_m), \mathcal{E}' \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}, \mathcal{D}')\} \\ = \; & H_{\mathcal{R}}^*(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_{n+1}|\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m). \end{aligned}$$

In a similar way we prove (7.13).

Proposition 7.4. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be partitions in an effect algebra E with (RDP). Then

$$H_*^{\mathcal{R}}(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) \geq H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{C}) + H_*^{\mathcal{R}}(\mathcal{B}|\mathcal{A} \vee \mathcal{C}), \tag{7.14}$$

$$H^*_{\mathcal{R}}(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) \leq H^*_{\mathcal{R}}(\mathcal{A}|\mathcal{C}) + H^*_{\mathcal{R}}(\mathcal{B}|\mathcal{A} \vee \mathcal{C}). \tag{7.15}$$

If $\mathcal{C}$ is a Boolean partition, then in (7.14) – (7.15) we have the equalities.

Proof. Suppose that $\mathcal{A} = \{a_i\}$, $\mathcal{B} = \{b_j\}$ and $\mathcal{C} = \{c_k\}$. Choose $\mathcal{E} = \{e_{ijk}\} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and set $\mathcal{D} = \{d_{ij} := \sum_k e_{ijk}\}$, and $\mathcal{F} = \{f_{ik} := \sum_j e_{ijk}\}$. Then $\mathcal{D} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}, \mathcal{B})$ and $\mathcal{F} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}, \mathcal{C})$.

Calculate

$$\begin{aligned}
H_{\mathcal{E}}(\mathcal{D}|\mathcal{C}) &= \sum_{ijk} s(c_k)\phi\left(\frac{s(e_{ijk})}{s(c_k)}\right) = \sum_{ijk} s(c_k)\phi\left(\frac{s(e_{ijk})}{s(f_{ik})}\frac{s(f_{ik})}{s(c_k)}\right) \\
&= -\sum_{ijk} s(e_{ijk}) \log \frac{s(e_{ijk})}{s(f_{ik})} - \sum_{ijk} s(e_{ijk}) \log \frac{s(f_{ik})}{s(c_k)} \\
&= \sum_{ijk} s(f_{ik})\phi\left(\frac{s(e_{ijk})}{s(f_{ik})}\right) + \sum_{ik} s(c_k)\phi\left(\frac{s(f_{ik})}{s(c_k)}\right) \\
&= H_{\mathcal{E}}(\mathcal{B}|\mathcal{F}) + H_{\mathcal{F}}(\mathcal{A}|\mathcal{C}),
\end{aligned}$$

which yields $H_{\mathcal{E}}(\mathcal{D}|\mathcal{C}) \geq H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{C}) + H_*^{\mathcal{R}}(\mathcal{B}|\mathcal{A} \vee \mathcal{C})$ and $H_*^{\mathcal{R}}(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) \geq H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{C}) + H_*^{\mathcal{R}}(\mathcal{B}|\mathcal{A} \vee \mathcal{C})$.

The equality in (7.14) is evident if $\mathcal{C}$ is a Boolean partition while in this case $\mathcal{F}$ is a unique Riesz refinement of $\mathcal{A}$ and $\mathcal{C}$.

In a similar way we can prove (7.15). □

Example 7.5. It is worthy to recall that if $\mathcal{C}$ is not Boolean then in (7.14) and (7.15) is not necessarily the equality. Indeed, take the MV-algebra M_{10} from Example 5.1, and set $\mathcal{A} = \{0.3, 0.7\}$, $\mathcal{B} = \{0.4, 0.6\}$, and $\mathcal{C} = \{0.2, 0.8\}$. Then $H_*^{\mathcal{R}}(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) = 0.255581$, $H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{C}) = 0.130903$, and $H_*^{\mathcal{R}}(\mathcal{B}|\mathcal{A} \vee \mathcal{C}) = 0$.
$H^*_{\mathcal{R}}(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) = 0.519130$, $H^*_{\mathcal{R}}(\mathcal{A}|\mathcal{C}) = 0.266581$, $H^*_{\mathcal{R}}(\mathcal{B}|\mathcal{A} \vee \mathcal{C}) = 0.289279$.[3]

As a corollary of (7.14) – (7.15) and (7.10) – (7.11)

$$H^*_{\mathcal{R}}(\bigvee_{i=1}^{n} \mathcal{A}_i | \bigvee_{j=1}^{m} \mathcal{B}_j) \leq \sum_{i=1}^{n} H^*_{\mathcal{R}}\left(\mathcal{A}_i | \bigvee_{j=1}^{m} \mathcal{B}_j\right), \tag{7.16}$$

and if $\mathcal{B}_1, \ldots, \mathcal{B}_m$ are Boolean decompositions, then

$$H_*^{\mathcal{R}}(\bigvee_{i=1}^{n} \mathcal{A}_i | \bigvee_{j=1}^{m} \mathcal{B}_j) \leq \sum_{i=1}^{n} H_*^{\mathcal{R}}\left(\mathcal{A}_i | \bigvee_{j=1}^{m} \mathcal{B}_j\right). \tag{7.17}$$

[3]The equalities in inequalities of formulas between (7.7) and (7.8) imply equalities in formulas (7.14) – (7.15).

We recall that that if (E,s,T) is a dynamical system, then

$$H_*^{\mathcal{R}}(T\mathcal{A}|T\mathcal{B}) \le H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}), \quad H_{\mathcal{R}}^*(T\mathcal{A}|T\mathcal{B}) \ge H_{\mathcal{R}}^*(\mathcal{A}|\mathcal{B}), \tag{7.18}$$

and if $\mathcal{A}$ or $\mathcal{B}$ is Boolean and if T preserves central elements, then in (7.18) we have the equality.

Proposition 7.6. Let (E,s,T) be a dynamical system, where E is with (RDP) and let T preserve central elements. Then, for any partition $\mathcal{A}$ and any Boolean partition $\mathcal{B}$, we have

$$h_*^{\mathcal{R}}(\mathcal{A},T) \le h_*^{\mathcal{R}}(\mathcal{B},T) + H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}) \tag{7.19}$$

and

$$h_{\mathcal{R}}^*(\mathcal{A},T) \le h_{\mathcal{R}}^*(\mathcal{B},T) + H_{\mathcal{R}}^*(\mathcal{A}|\mathcal{B}). \tag{7.20}$$

Proof. By Proposition 3.4, (7.14) - (7.15), (7.17), and (7.18), we have

$$\begin{aligned}
H_*^{\mathcal{R}}\Big(\bigvee_{i=0}^{n-1} T^i\mathcal{A}\Big) &\le H_*^{\mathcal{R}}\left(\bigvee_{i=0}^{n-1} T^i\mathcal{A} \vee \bigvee_{j=0}^{n-1} T^j\mathcal{B}\right) \\
&= H_*^{\mathcal{R}}\left(\bigvee_{i=0}^{n-1} T^i\mathcal{B}\right) + H_*^{\mathcal{R}}\left(\bigvee_{i=0}^{n-1} T^i\mathcal{A} \Big| \bigvee_{j=0}^{n-1} T^j\mathcal{B}\right) \\
&\le H_*^{\mathcal{R}}\left(\bigvee_{i=0}^{n-1} T^i\mathcal{B}\right) + \sum_{i=0}^{n-1} H_*^{\mathcal{R}}\left(T^i\mathcal{A} \Big| \bigvee_{j=0}^{n-1} T^j\mathcal{B}\right) \\
&\le H_*^{\mathcal{R}}\left(\bigvee_{i=0}^{n-1} T^i\mathcal{B}\right) + \sum_{i=0}^{n-1} H_*^{\mathcal{R}}(T^i\mathcal{A}|T^i\mathcal{B}) \\
&= H_*^{\mathcal{R}}\left(\bigvee_{i=0}^{n-1} T^i\mathcal{B}\right) + nH_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}).
\end{aligned}$$

Hence: $h_*^{\mathcal{R}}(\mathcal{A},T) = \lim_n \frac{1}{n} H_*^{\mathcal{R}}(\bigvee_{i=0}^{n-1} T^i\mathcal{A}) \le \frac{1}{n} H_*^{\mathcal{R}}(\bigvee_{i=0}^{n-1} T^i\mathcal{B}) + H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}) = h_*^{\mathcal{R}}(\mathcal{B},T) + H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B})$.

The second inequality follows from (7.16). □

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