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# APPROXIMATIONS FOR THE MAXIMUM OF STOCHASTIC PROCESSES WITH DRIFT[1]

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If a stochastic process can be approximated with a Wiener process with positive drift, then its maximum also can be approximated with a Wiener process with positive drift.

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## 1. INTRODUCTION AND RESULTS

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables with

$$EX_1 = \mu > 0 \text{ and } 0 < \mathrm{var} X_1 = \sigma^2 < \infty. \tag{1.1}$$

The motivation of our note is the following central limit theorem due to Teicher [6]. Let

$$S(j) = \sum_{1 \le i \le j} X_i$$

and

$$0 \le \alpha < 1. \tag{1.2}$$

**Theorem 1.1.** If (1.1) and (1.2) hold, then

$$\frac{1}{\sigma n^{1/2-\alpha}} \left\{ \max_{1 \le j \le n} \frac{S(j)}{j^\alpha} - \mu n^{1-\alpha} \right\} \xrightarrow{\mathcal{D}} N(0,1),$$

where $N(0,1)$ denotes a standard normal random variable.

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Since

$$\frac{1}{\sigma n^{1/2-\alpha}}\left\{\frac{S(n)}{n^\alpha} - \mu n^{1-\alpha}\right\} \xrightarrow{\mathcal{D}} N(0,1),$$

Theorem 1.1 strongly suggests that

$$\max_{1\le j\le n} \frac{S(j)}{j^\alpha} - \frac{S(n)}{n^\alpha} = o_P(n^{1/2-\alpha}),$$

i.e. $S(j)/j^\alpha$ reaches its largest value on $[1,n]$ nearly at $j=n$. Indeed, Chow and Hsiung [1] proved the following result:

**Theorem 1.2.** If (1.1) and (1.2) hold, then

$$\max_{1\le j\le n} \frac{S(j)}{j^\alpha} - \frac{S(n)}{n^\alpha} = o(n^{1/2-\alpha}) \quad \text{a.s.} \tag{1.3}$$

For generalizations of (1.3) we refer to Chow, Hsiung and Yu [2].

We show that (1.3) holds not only for partial sums of independent identically distributed random variables, but for any process if they can be approximated with a Wiener process with drift. Let $\Gamma(t)$ be a stochastic process on $\mathcal{D}[1,\infty)$.

**Theorem 1.3.** We assume that there exist a Wiener process $\{W(t), 1\le t<\infty\}$ and constants $\tau>0$, $\gamma>0$ such that

$$\Gamma(t) - (\tau W(t) + \gamma t) = o(t^{1/\nu}) \quad \text{a.s. } (t\to\infty) \tag{1.4}$$

with some $\nu>2$. If (1.2) holds, then

$$\sup_{1\le t\le T} \frac{\Gamma(t)}{t^\alpha} - \frac{\Gamma(T)}{T^\alpha} = o(T^{1/\nu-\alpha}) \quad \text{a.s. } (T\to\infty) \tag{1.5}$$

and

$$\sup_{1\le t\le T} \frac{\Gamma(t)}{t^\alpha} - \frac{\tau W(T)+\gamma T}{T^\alpha} = o(T^{1/\nu-\alpha}) \quad \text{a.s. } (T\to\infty). \tag{1.6}$$

Theorem 1.3 implies immediately an improvement of the rate in (1.3) under stronger moment conditions on $X_1$.

**Theorem 1.4.** If (1.1), (1.2) hold and

$$E|X_1|^\nu < \infty \text{ with some } \nu>2, \tag{1.7}$$

then

$$\max_{1\le j\le n} \frac{S(j)}{j^\alpha} - \frac{S(n)}{n^\alpha} = o(n^{1/\nu-\alpha}) \quad \text{a.s. } (n\to\infty). \tag{1.8}$$

Theorems 1.3 and 1.4 will be proven in the next section. The following two corollaries are immediate consequences of (1.6) and the properties of the Wiener process. Let $[\cdot]$ denote the integer part function.

**Corollary 1.1.** We assume that the conditions of Theorem 1.3 are satisfied.

(i) If $0 \le \alpha < 1/2$, then

$$n^{\alpha-1/2}\left\{\sup_{1\le t\le [nu]+1}\frac{\Gamma(t)}{t^\alpha}-\gamma([nu]+1)^{1-\alpha}\right\}\xrightarrow{\mathcal{D}[0,1]}\frac{\tau W(u)}{u^\alpha}. \tag{1.9}$$

(ii) If $1/2 < \alpha < 1$, then

$$n^{\alpha-1/2}\left\{\sup_{1\le t\le [nu]+1}\frac{\Gamma(t)}{t^\alpha}-\gamma([nu]+1)^{1-\alpha}\right\}\xrightarrow{\mathcal{D}[1,\infty]}\frac{\tau W(u)}{u^\alpha}. \tag{1.10}$$

(iii) For any $0 < c_1 < c_2 < \infty$

$$n^{\alpha-1/2}\left\{\sup_{1\le t\le [nu]+1}\frac{\Gamma(t)}{t^\alpha}-\gamma([nu]+1)^{1-\alpha}\right\}\xrightarrow{\mathcal{D}[c_1,c_2]}\frac{\tau W(u)}{u^\alpha}. \tag{1.11}$$

**Corollary 1.2.** If the conditions of Theorem 1.3 are satisfied, then

$$\limsup_{T\to\infty}\frac{T^\alpha}{(2T\log\log T)^{1/2}}\left|\sup_{1\le t\le T}\frac{\Gamma(t)}{t^\alpha}-\gamma T^{1-\alpha}\right|=\tau \quad \text{a.s.}$$

## 2. PROOFS

The first two lemmas show that $\Gamma(t)/t^\alpha$ and $(\tau W(t)+\gamma t)/t^\alpha$ will reach their largest value on $[1,T]$ on the second half of this interval.

**Lemma 2.1.** If (1.2) holds and $\gamma > 0$, then there is a random variable $T_1$ such that

$$\sup_{1\le t\le T}\frac{\tau W(t)+\gamma t}{t^\alpha}=\sup_{T/2\le t\le T}\frac{\tau W(t)+\gamma t}{t^\alpha}, \text{ if } T\ge T_1. \tag{2.1}$$

Proof. By the law of iterated logarithm for $W$ we have

$$\frac{1}{T^{1-\alpha}}\sup_{1\le t\le T}\frac{\tau W(t)+\gamma t}{t^\alpha}\longrightarrow\gamma \quad \text{a.s. } (T\to\infty) \tag{2.2}$$

and

$$\frac{1}{T^{1-\alpha}}\sup_{1\le t\le T/2}\frac{\tau W(t)+\gamma t}{t^\alpha}\longrightarrow\left(\frac{1}{2}\right)^{1-\alpha}\gamma \quad \text{a.s. } (T\to\infty), \tag{2.3}$$

implying the statement of Lemma 2.1. □

**Lemma 2.2.** If the conditions of Theorem 1.3 are satisfied, then there is a random variable $T_2$ such that

$$\sup_{1 \le t \le T} \frac{\Gamma(t)}{t^\alpha} = \sup_{T/2 \le t \le T} \frac{\Gamma(t)}{t^\alpha}, \quad \text{if } t \ge T_2.$$

Proof. The approximation in (1.4) implies that

$$\sup_{1 \le t \le T} \frac{|\Gamma(t) - (\tau W(t) + \gamma t)|}{t^\alpha} = O(\max(1, T^{1/\nu - \alpha})) \quad \text{a.s.}$$

and therefore (2.2) and (2.3) yield

$$\frac{1}{T^{1-\alpha}} \sup_{1 \le t \le T} \frac{\Gamma(t)}{t^\alpha} \longrightarrow \gamma \quad \text{a.s. } (t \to \infty) \tag{2.4}$$

and

$$\frac{1}{T^{1-\alpha}} \sup_{1 \le t \le T/2} \frac{\Gamma(t)}{t^\alpha} \longrightarrow \left(\frac{1}{2}\right)^{1-\alpha} \gamma \quad \text{a.s. } (T \to \infty). \tag{2.5}$$

Lemma 2.2 follows from (2.4) and (2.5). □

Let $F_0(t)$ be the uniform distribution function on $[0,1]$. For any $0 < \alpha < 1$, $F_\alpha(t)$ denotes the uniform distribution function on $[1, 1/\alpha]$.

**Lemma 2.3.** Let $0 \le \alpha < 1$ and $Y_1, Y_2, \dots$ be independent, identically distributed random variables with distribution function $F_\alpha(t)$. Then

$$\max_{1 \le j \le n} \frac{1}{j^\alpha} \sum_{1 \le i \le j} Y_i = \frac{1}{n^\alpha} \sum_{1 \le i \le n} Y_i.$$

Proof. It is enough to show that

$$\left(1 + \frac{1}{j}\right)^\alpha \sum_{1 \le i \le j} Y_i \le \sum_{1 \le i \le j+1} Y_i \quad \text{for all } 1 \le j < \infty. \tag{2.6}$$

Since $Y_i \ge 0$, (2.6) holds if $\alpha = 0$. If $0 < \alpha < 1$, we observe that $1 \le Y_i \le 1/\alpha$ and

$$\left(1 + \frac{1}{j}\right)^\alpha - 1 \le \frac{\alpha}{j}.$$

Hence

$$\left\{\left(1 + \frac{1}{j}\right)^\alpha - 1\right\} \sum_{1 \le i \le j} Y_i \le \frac{\alpha}{j} \sum_{1 \le i \le j} Y_i \le 1 \le Y_{j+1},$$

completing the proof of (2.6). □

**Lemma 2.4.** If (1.2) holds and $\tau > 0$, $\gamma > 0$, then

$$\sup_{1\le t\le T} \frac{\tau W(t)+\gamma t}{t^\alpha} - \frac{\tau W(T)+\gamma T}{T^\alpha} = O\left(\frac{\log T}{T^\alpha}\right) \quad \text{a.s.}$$

Proof. Let $\mu_* = \mu_*(\alpha)$ and $\sigma_* = \sigma_*(\alpha)$ be the mean and standard deviation of a random variable with distribution function $F_\alpha(t)$. Next we define

$$c = \left(\frac{\mu_*}{\gamma}\frac{\tau}{\sigma_*}\right)^2. \tag{2.7}$$

Obviously,

$$\begin{aligned}\sup_{1\le t\le T} \frac{\tau W(t)+\gamma t}{t^\alpha} &= \tau \sup_{1/c\le s\le T/c} \frac{W(cs)+\frac{\gamma}{\tau}cs}{(cs)^\alpha} \\ &= \tau c^{1/2-\alpha} \sup_{1/c\le s\le T/c} \frac{W_1(s)+\frac{\gamma}{\tau}c^{1/2}s}{s^\alpha},\end{aligned} \tag{2.8}$$

where

$$W_1(s) = c^{-1/2}W(cs), \ \ 0\le s<\infty \tag{2.9}$$

is a Wiener process. By (2.7) and (2.8) we have

$$\sup_{1\le t\le T} \frac{\tau W(t)+\gamma t}{t^\alpha} = \frac{\tau}{\sigma_*}c^{1/2-\alpha} \sup_{1/c\le t\le T/c} \frac{\sigma_* W_1(t)+\mu_* t}{t^\alpha}. \tag{2.10}$$

Using the K–M–T approximation (cf. Komlós, Major and Tusnády [3, 4] and Major [5]) we can define $Y_1^*, Y_1^*, \ldots$, a sequence of independent, identically distributed random variables with distribution function $F_\alpha(t)$ such that

$$\sum_{1\le i\le t} Y_i^* - (\sigma_* W_1(t)+\mu_* t) = O(\log t) \quad \text{a.s. } (t\to\infty). \tag{2.11}$$

By Lemmas 2.1, 2.2 and (2.10) there is random variable $T_0$ such that

$$\sup_{1/c\le t\le T/c} \frac{\sigma_* W_1(t)+\mu_* t}{t^\alpha} = \sup_{T/(2c)\le t\le T/c} \frac{\sigma_* W_1(t)+\mu_* t}{t^\alpha}$$

and

$$\sup_{1/c\le t\le T/c} \frac{1}{t^\alpha}\sum_{1\le i\le t} Y_i^* = \sup_{T/(2c)\le t\le T/c} \frac{1}{t^\alpha}\sum_{1\le i\le t} Y_i^*,$$

if $T\ge T_0$. Hence (2.11) yields, as $T\to\infty$,

$$\sup_{1/c\le t\le T/c} \frac{\sigma_* W_1(t)+\mu_* t}{t^\alpha} - \sup_{1/c\le t\le T/c} \frac{1}{t^\alpha}\sum_{1\le i\le t} Y_i^* = O(T^{-\alpha}\log T) \quad \text{a.s.} \tag{2.12}$$

Putting together Lemma 2.3 and (2.11) we conclude

$$
\begin{aligned}
\sup_{1/c\le t\le T/c} \frac{1}{t^\alpha}\sum_{1\le i\le t} Y_i^* &= \left(\frac{T}{c}\right)^{-\alpha} \sum_{1\le i\le T/c} Y_i^* \\
&= \left(\frac{T}{c}\right)^{-\alpha}\left\{\sigma_* W_1\left(\frac{T}{c}\right)+\mu_*\frac{T}{c}\right\} + O(T^{-\alpha}\log T) \quad \text{a.s.}
\end{aligned} \tag{2.13}
$$

$(T\to\infty)$. Next we use (2.7), (2.9) and (2.10) to obtain

$$
\begin{aligned}
&\left(\frac{T}{c}\right)^{-\alpha}\left\{\sigma_* W_1\left(\frac{T}{c}\right)+\mu_*\frac{T}{c}\right\} \\
&\qquad = \left(\frac{T}{c}\right)^{-\alpha}\left\{\sigma_* c^{-1/2} W(T)+\mu_*\frac{T}{c}\right\} \\
&\qquad = \frac{1}{T^\alpha}c^{\alpha-1/2}\sigma_*\left\{W(T)+\frac{\mu_*}{\sigma_*}c^{-1/2}T\right\} \\
&\qquad = \frac{1}{T^\alpha}c^{\alpha-1/2}\frac{\sigma_*}{\tau}\left\{\tau W(T)+\gamma T\right\}.
\end{aligned} \tag{2.14}
$$

Lemma 2.4 now follows from (2.8) and (2.12) – (2.14). □

P r o o f o f T h e o r e m 1.3. Using (1.4) and Lemmas 2.1 and 2.2 we get that

$$
\sup_{1\le t\le T}\frac{\Gamma(t)}{t^\alpha} - \sup_{1\le t\le T}\frac{\tau W(t)+\gamma t}{t^\alpha} = o(T^{1/\nu-\alpha}) \quad \text{a.s.}
$$

Hence Theorem 1.3 follows from Lemma 2.4. □

P r o o f o f T h e o r e m 1.4. By the K–M–T approximation there is a Wiener process $\{W(t), 0\le t<\infty\}$ such that

$$
S(t) - (\sigma W(t)+\mu t) = o(t^{1/\nu}) \quad \text{a.s.} \ (t\to\infty).
$$

Hence (1.4) holds and the result follows from Theorem 1.3. □

P r o o f o f C o r o l l a r y 1.1. Assume that $0\le\alpha<1/2$. By Theorem 1.3 there is a Wiener process $\{W(t), 0\le t<\infty\}$ such that

$$
n^{\alpha-1/2}\sup_{0\le u\le 1}\left|\sup_{1\le t\le [nu]+1}\frac{\Gamma(t)}{t^\alpha} - \frac{\tau W([nu]+1)+\gamma([nu]+1)}{([nu]+1)^\alpha}\right| = o(n^{1/\nu-1/2}) \quad \text{a.s.}
$$

Hence (1.9) is proven if

$$
n^{\alpha-1/2}\frac{W([nu]+1)}{([nu]+1)^\alpha} \xrightarrow{\mathcal{D}[0,1]} \frac{W(u)}{u^\alpha}. \tag{2.15}
$$

Obviously,

$$\sup_{0\le u\le \epsilon} \frac{|W([nu]+1)|}{([nu]+1)^\alpha} \le \sup_{0\le u\le [n\epsilon]+1} \frac{|W(u)|}{u^\alpha}$$

and by the scale transformation of $W$ we have

$$n^{\alpha-1/2} \sup_{0\le u\le [n\epsilon]+1} \frac{|W(u)|}{u^\alpha} \overset{\mathcal{D}}{=} \sup_{0\le u\le [n\epsilon]+1} \frac{|W(u/n)|}{(u/n)^\alpha} = \sup_{0\le u\le ([n\epsilon]+1)/n} \frac{|W(u)|}{u^\alpha}.$$

By the law of the iterated logarithm for $W$ at 0 we have

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} P\left\{ \sup_{0\le u\le ([n\epsilon]+1)/n} \frac{|W(u)|}{u^\alpha} > \delta \right\} = 0 \text{ for all } \delta > 0. \tag{2.16}$$

The scale transformation of $W$ and the almost sure continuity of $W(u)/u^\alpha$ on $[c_1, c_2]$, $0 < c_1 \le c_2$ yield

$$n^{\alpha-1/2} \frac{W([nu]+1)}{([nu]+1)^\alpha} \overset{\mathcal{D}[c_1,c_2]}{\longrightarrow} \frac{W(u)}{u^\alpha}. \tag{2.17}$$

Clearly, (2.15) follows from (2.16) and (2.17).

Assume that $1/2 < \alpha < 1$. Using again Theorem 1.3 there is a Wiener process $\{W(t), 0 \le t < \infty\}$ such that

$$n^{\alpha-1/2} \sup_{1\le u<\infty} \left| \sup_{1\le t\le [nu]+1} \frac{\Gamma(t)}{t^\alpha} - \frac{\tau W([nu]+1) + \gamma([nu]+1)}{([nu]+1)^\alpha} \right| = o(1) \text{ a.s.}$$

Hence (1.10) is proven if we show that

$$n^{\alpha-1/2} \frac{W([nu]+1)}{([nu]+1)^\alpha} \overset{\mathcal{D}[1,\infty]}{\longrightarrow} \frac{W(u)}{u^\alpha}. \tag{2.18}$$

For any $T > 0$ we have that

$$\sup_{T\le u<\infty} \frac{|W([nu]+1)|}{([nu]+1)^\alpha} \le \sup_{[nT]\le u<\infty} \frac{|W(u)|}{u^\alpha}$$

and by the scale transformation of $W$ we have

$$n^{\alpha-1/2} \sup_{[nT]\le u<\infty} \frac{|W(u)|}{u^\alpha} \overset{\mathcal{D}}{=} \sup_{[nT]/n\le u<\infty} \frac{|W(u)|}{u^\alpha}.$$

The law of the iterated logarithm for $W$ at $\infty$ yields that

$$\sup_{T\le u\le\infty} \frac{|W(u)|}{u^\alpha} \to 0 \text{ a.s. } (T\to\infty). \tag{2.19}$$

Now (2.18) follows from (2.17) and (2.19).

Theorem 1.3 and (2.17) imply immediately (1.11). □

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