Kybernetika

Luigi Chisci; Paola Falugi; Giovanni Zappa

Receding-horizon control of constrained uncertain linear systems with disturbances

Kybernetika, Vol. 38 (2002), No. 2, [169]--185

Persistent URL: http://dml.cz/dmlcz/135453

KYBERNETIKA — VOLUME 38 (2002), NUMBER 2, PAGES 169–185

REDECING–HORIZON CONTROL OF CONSTRAINED UNCERTAIN LINEAR SYSTEMS WITH DISTURBANCES

LUIGI CHISCI, PAOLA FALUGI AND GIOVANNI ZAPPA

The paper addresses receding-horizon (predictive) control for polytopic discrete-time systems subject to input/state constraints and unknown but bounded disturbances. The objective is to optimize nominal performance while guaranteeing robust stability and constraint satisfaction. The latter goal is achieved by exploiting robust invariant sets under linear and nonlinear control laws. Tradeoffs between maximizing the initial feasibility region and guaranteeing ultimate boundedness in the smallest invariant region are investigated.

1. INTRODUCTION

Predictive control represents an effective design methodology for handling hard constraints and performance issues in a joint fashion. Stability of constrained predictive control schemes has been thoroughly investigated [12] while robustness with respect to model uncertainty deserves further attention [3] though there have been recently several interesting contributions in this direction, like e. g. [1, 6, 14, 15, 16]. Typical robust constrained predictive control algorithms adopt a polytopic description of model uncertainty [6, 14, 15] and pursue minimization of a worst-case performance index [16] or of an upper bound of it [6, 14]. The present paper adheres to the polytopic description of uncertainty which seems the most natural for handling constraints, but, like [1, 15], turns aside from the objective of worst-case performance optimization for a twofold reason. First, min-max optimization may be too computationally demanding. Secondly, the paradigm of optimizing performance for the worst-case system may be unrealistic in the common situation where a nominal (most likely) model is available. For the above reasons, a more sensible approach seems to minimize the nominal performance index (i. e. the performance index for the nominal model) while *robustly* guaranteeing stability and constraint satisfaction.

Within this framework, we propose two novel predictive control algorithms for polytopic discrete-time systems with state/control constraints and subject to unknown but bounded disturbances. Both algorithms postulate a control sequence, along an infinite prediction horizon, consisting of a fixed robustly stabilizing linear state feedback $u = Fx$ plus N free control moves [18]. The receding-horizon controller selects the control at sample time t as the first element of the control sequence which minimizes the energy of control moves subject to appropriate state-dependent linear constraints.

In particular, the first algorithm imposes the constraint that after N steps the state enters a maximal polytopic set, which is feasible and robustly invariant under the feedback $u = Fx$. This approach is similar to the one adopted in [15] but uses maximal polytopic sets instead of more conservative ellipsoids.

Conversely, the second algorithm imposes the less stringent constraint that the state at the next sample time remains in a controlled robust invariant set; a similar approach is followed in [7] to cope with disturbances.

It is shown that the first algorithm guarantees asymptotic stability provided that the initial state is feasible, while the second algorithm provides a larger feasibility region but not guaranteed stability. In any case their feasibility regions turn out to be larger than the ones provided by ellipsoidal invariance constraints [15]. Moreover, the two algorithms are compared from both a computational and a performance point of view.

The rest of the paper is organized as follows: Section 2 formulates the problem of interest; Section 3 recalls background material on invariant set theory; Section 4 presents the two algorithms; Section 5 illustrates simulation results; finally Section 6 ends the paper.

2. NOTATION AND PROBLEM FORMULATION

Notation

For any subsets $\mathcal{A}, \mathcal{B}$ of $I\!R^n$, for any matrix M mapping $I\!R^n$ onto $I\!R^l$ and for any subset $\mathcal{C}$ of $I\!R^l$, the following sets are defined: $\mathcal{A}+\mathcal{B} = \{a+b | a \in \mathcal{A}, b \in \mathcal{B}\}$; $-\mathcal{A} = \{-a | a \in \mathcal{A}\}$; $\mathcal{A} - \mathcal{B} = \mathcal{A} + (-\mathcal{B})$; $M(\mathcal{A}) = \{Ma | a \in \mathcal{A}\}$; $M^{-1}(\mathcal{C}) = \{a | Ma \in \mathcal{C}\}$; $\mathcal{A} \sim \mathcal{B} = \{a | a + \mathcal{B} \subseteq \mathcal{A}\}$. Further $\mathcal{B}_1$ denotes the unit ball (in the Euclidean norm) of $I\!R^n$ and $Co\{\cdots\}$ denotes the convex hull.

Problem formulation

Consider the discrete-time uncertain LTV system

$$x(t+1) = A(t)x(t) + B(t)u(t) + Dw(t) \tag{1}$$

where

$$\begin{gathered} A(t) = A_0 + \tilde{A}(t) \quad B(t) = B_0 + \tilde{B}(t) \\ \left[\tilde{A}(t), \tilde{B}(t)\right] = \sum_{j=1}^{q} \lambda_j(t)[\tilde{A}_j, \tilde{B}_j], \quad \forall t \geq 0 \\ \sum_{j=1}^{q} \lambda_j(t) = 1, \ \lambda_j(t) \geq 0, \ j = 1, 2, \ldots, q. \end{gathered} \tag{2}$$

Here $x(t) \in I\!R^n$ is the state, $u(t) \in I\!R^m$ is the control input and $w(t) \in I\!R^p$ is the disturbance input at sample time $t = 0, 1, \ldots$. The pair (A_0, B_0) represents the *nominal* system while $(\tilde{A}(t), \tilde{B}(t))$ represents the model uncertainty which belongs to the polytopic set $\mathcal{P} = Co\{[\tilde{A}_j, \tilde{B}_j],\ j = 1, \ldots, q\}$. Notice that the coefficients λ_j are unknown and possibly time-varying. The system (1) – (2), referred to as *polytopic system*, provides a classical description of model uncertainty.

It is fundamental for the subsequent developments to make the following assumption.

Assumption 1. There exists a constant state feedback gain $F \in \mathbb{R}^{m \times n}$ which *robustly stabilizes* the polytopic system (1) – (2), i. e. makes the closed-loop system

$$x(t+1) \;=\; \Phi(t)\, x(t)\ , \quad \Phi(t) = A(t) + B(t)F \tag{3}$$

absolutely asymptotically stable (AAS) [10].

Let us introduce:

$$A_j \triangleq A_0 + \tilde{A}_j, \quad B_j \triangleq B_0 + \tilde{B}_j, \quad \Phi_j \triangleq A_j + B_j F \quad j = 1, 2, \cdots, q.$$

Then we recall that (3) is AAS if

$$\lim_{t \to \infty} \Phi(t)\Phi(t-1) \cdots \Phi(0) = 0$$

for any sequence of matrices $\Phi(k) \in Co\{\Phi_1, \Phi_2, \cdots, \Phi_q\}$. AAS is guaranteed if there exists a norm $\|\cdot\|$ in $\mathbb{R}^n$ such that

$$\|\Phi_j\|_i \le \gamma < 1 \quad j = 1, \cdots, q \tag{4}$$

where $\|\cdot\|_i$ is the norm, induced by $\|\cdot\|$, on the algebra of $n \times n$ matrices. Particular cases of AAS are quadratic stability [2, 8] and polytopic stability [4, 5] where $\|\cdot\|$ is represented by an ellipsoidal or polytopic norm, respectively. Recent results [10] show that the polytopic system (1) – (2) is absolutely asymptotically stable if and only if it is exponentially stable. It is further assumed that the system (1) is subject to the pointwise-in-time control and state constraints

$$u(t) \in \mathcal{U},\ x(t) \in \mathcal{X} \quad \forall t \ge 0 \tag{5}$$

and the disturbance is pointwise-in-time bounded according to

$$w(t) \in \mathcal{W} \quad \forall t \ge 0 \tag{6}$$

for some appropriate sets $\mathcal{U} \subset \mathbb{R}^m$, $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{W} \subset \mathbb{R}^p$ satisfying the following assumption.

Assumption 2. $\mathcal{U}$, $\mathcal{X}$, $\mathcal{W}$ contain the origin in the interior; further $\mathcal{U}$ and $\mathcal{W}$ are compact.

The objective is to design a state-feedback regulator

$$u(t) = g(x(t)) \tag{7}$$

which possibly robustly stabilizes the system (1) subject to constraints (5) and disturbance (6). Clearly, by Assumption 1, the LTI state feedback

$$u(t) = Fx(t) \tag{8}$$

robustly stabilizes (1) but, because of constraints (5), stabilization is local in a possibly small neighborhood of the origin. The stability domain could hopefully be enlarged in a significant way by use of a nonlinear state feedback (7). The approach followed in this paper combines the theory of invariant sets with predictive control (receding-horizon) ideas.

3. INVARIANT SETS FOR POLYTOPIC SYSTEMS

This section recalls some background on invariant sets [5], revisited in the context of polytopic systems. Given a robustly stabilizing LTI feedback (8), consider the corresponding uncertain closed-loop system

$$\begin{aligned} x(t+1) &= \Phi(t)x(t) + Dw(t) \\ u(t) &= Fx(t) \\ \Phi(t) &\in Co\{\Phi_1, \Phi_2, \dots, \Phi_q\}. \end{aligned} \tag{9}$$

It is important from an analysis point of view to characterize the effect of the disturbance w on the above system. To this end, let us introduce for the system (9) the sets $\mathcal{R}_k$ of the states reachable in k steps from $x(0) = \mathbf{0}$. These sets can be computed recursively as follows

$$\begin{aligned} &\mathcal{R}_0 = \{\mathbf{0}\} \\ &\mathcal{R}_{k+1} = \varphi(1, \mathcal{R}_k) + D\mathcal{W} \end{aligned} \tag{10}$$

where, given $\mathcal{S} \subset I\!\!R^n$ and the integer $k \geq 0$, $\varphi(k, \mathcal{S})$ denotes the set of states $x(k)$ originated from $x(0) \in \mathcal{S}$ assuming that $w(\cdot) \equiv \mathbf{0}$ in (9). The following result proves that the sequence of sets $\{\mathcal{R}_k,\ k \geq 0\}$ admits a limit $\mathcal{R}_\infty$.

Theorem 1. Under assumption 1 and compactness of $\mathcal{W}$ (Assumption 2), there exists a compact set $\mathcal{R}_\infty$ such that

i) $\mathcal{R}_k \subset \mathcal{R}_\infty\ \forall\, t \geq 0$

ii) $\forall\, \epsilon > 0,\ \exists\, t \geq 0 : \mathcal{R}_\infty \subset \mathcal{R}_t + \epsilon\mathcal{B}_1$

iii) $\mathcal{R}_\infty$ is invariant under (9), i.e. $\mathcal{R}_\infty = \varphi(1, \mathcal{R}_\infty) + D\mathcal{W}$.

Proof. First we recall that with the Hausdorff metric ρ the family of compact sets of $I\!\!R^n$ is a complete metric space. Since $\mathcal{W}$ is compact, $\mathcal{R}_k$ is compact. Splitting the disturbance sequence $\{w(0), w(1), \cdots, w(k)\}$ into $\{w(0), 0, \cdots, 0\}$ and

$\{0, w(1), \cdots, w(k)\}$ and using superposition, it can be noticed that the following inclusion holds

$$\mathcal{R}_k \subset \mathcal{R}_{k+1} \subset \mathcal{R}_k + \varphi(k, D\mathcal{W})$$

In fact, any state $x \in \mathcal{R}_{k+1}$ can be written as the sum of a state in $\mathcal{R}_k$ produced by the disturbance realization $\{0, w(1), \cdots, w(k)\}$ plus a state in $\varphi(k, D\mathcal{W})$ produced by $\{w(0), 0, \cdots, 0\}$. The compactness of $\mathcal{W}$ and Assumption 1 imply the existence of $\mu > 0$ and $\lambda \in (0, 1)$ such that, for all $k \geq 0$, $\varphi(k, D\mathcal{W}) \subset \mu\lambda^k\mathcal{B}_1$. Hence, $\rho(\mathcal{R}_{k+1}, \mathcal{R}_k) \leq \mu\lambda^k$ from which $\{\mathcal{R}_k,\ k \geq 0\}$ is Cauchy and the existence of $\mathcal{R}_\infty$ is established. Letting $k \to \infty$ in (10) proves $\mathcal{R}_\infty = \varphi(1, \mathcal{R}_\infty) + D\mathcal{W}$ which, in turn, shows that $\mathcal{R}_\infty$ is invariant under (9). □

From the above theorem, it turns out that $\mathcal{R}_\infty$ is the smallest set which, for $x(0) = \mathbf{0}$, contains all possible trajectories of (9) generated from disturbances $w(\cdot)$ in $\mathcal{W}$. To ensure compatibility between the disturbance (6) and the constraints (5), the following assumption has to be made

Assumption 3. $\mathcal{R}_\infty \subset \mathcal{X}$ and $F\mathcal{R}_\infty \subset \mathcal{U}$.

Hereafter, we would like to characterize the set Σ_0 of all states $x(0)$ for the system (9) such that the constraints (5) are satisfied. Due to the stability Assumption 1 on (9), $\varphi(t, \Sigma_0) \to 0$ as $t \to \infty$ for any $x(0) \in \Sigma_0$. Let us first introduce the concept of *robust invariant set.*

Definition 1. Σ is a robust invariant set for the system (9) if for any $x(0) \in \Sigma$, $x(t)$ remains in Σ for all $t \geq 0$.

Let $X_c \triangleq \{x \in \mathcal{X} : Fx \in \mathcal{U}\}$ denote the set of states for which the constraints (5) are satisfied. Hence the desired set Σ_0 is the largest robust invariant subset of X_c, also called *maximal admissible set* [9]. Maximal admissible sets have been considered in [9] and then in [13] for systems with bounded disturbance inputs. Here the case of uncertain polytopic systems with bounded disturbances is addressed. Let us introduce the set O_i, $i = 0, 1, \ldots$, of initial states $x(0)$ from which, under the uncertain dynamics (9), the constraints (5) are satisfied for $t = 0, 1, \ldots, i$. Clearly the sets O_i can be computed recursively as follows:

$$\begin{aligned} O_0 &= X_c \\ \text{for } i &= 1, 2, \ldots \\ O_i &= X_c \cap \{x : \Phi_j x \in O_{i-1} \sim D\mathcal{W}, j = 1, 2, \ldots, q\}. \end{aligned} \tag{11}$$

Since $O_i \subseteq O_{i-1}$, O_i converges to $O_\infty \triangleq \bigcap_{i=0}^{\infty} O_i$, which, by Assumption 3, is non empty; therefore $\Sigma_0 = O_\infty$. It is important to ascertain whether the set O_∞ is finitely determined, i.e. there exists a finite i^* such that $O_{i^*} = O_\infty$. The following result holds.

Theorem 2. Under the Assumptions 1–3, if for some integer i, O_i is bounded, then O_∞ is finitely determined.

Proof. The sets O_i can also be defined recursively as follows

$$O_{i+1} = O_i \cap \{x | F\varphi(i+1,x) \in \mathcal{U}_{i+1},\ \varphi(i+1,x) \in \mathcal{X}_{i+1}\}$$

where $\mathcal{U}_i = \mathcal{U} \sim F\mathcal{R}_i$, $\mathcal{X}_i = \mathcal{X} \sim \mathcal{R}_i$. Further, by Assumptions 2 and 3, $\mathcal{U}_\infty = \mathcal{U} \sim F\mathcal{R}_\infty$ and $\mathcal{X}_\infty = \mathcal{X} \sim \mathcal{R}_\infty$ are non empty and contain the origin in the interior. If O_i is bounded there exists a ball $\mathcal{B}$ centered at the origin such that $O_i \subset \mathcal{B}$. By Assumption 1, $\varphi(t,\mathcal{B}) \to \{0\}$ as $t \to \infty$. Hence there exists an index $\ell > i$ for which $\varphi(\ell+1,\mathcal{B}) \subset \mathcal{X}_\infty$ and $F\varphi(\ell+1,\mathcal{B}) \subset \mathcal{U}_\infty$. Then $O_\ell \subset O_i \subset \mathcal{B}$ and, therefore, $\varphi(\ell+1,O_\ell) \subset \mathcal{X}_\infty$ and $F\varphi(\ell+1,O_\ell) \subset \mathcal{U}_\infty$. This, by definition of the sets O_i, implies that $O_\ell = O_{\ell+1}$ and hence $O_\infty = O_\ell$ is finitely determined. □

Remark 1. Obviously O_0 is bounded if $\mathcal{X}$ is bounded. Also, notice that the set O_n is bounded if $\mathcal{U}$ is bounded and (Φ_j, F) is observable for some j.

Remark 2. In the same way, one can define the largest set, denoted by $\bar{\Sigma}_0$, contained in X_c which is invariant under the nominal closed loop dynamics $\Phi_0 = A_0 + B_0 F$ and in absence of disturbances. $\bar{\Sigma}_0$ can be computed by the recursion (11) replacing Φ_j by Φ_0 and setting $\mathcal{W} = \{0\}$; clearly, $\Sigma_0 \subset \bar{\Sigma}_0$.

Σ_0 is the set of the initial states which are asymptotically steered to the origin under the constant linear state feedback (8) without violating the constraints. It is clearly possible to enlarge such a domain of attraction making use of a nonlinear feedback $u = g(x) \in \mathcal{U}$. This motivates the following definition of *robust controlled invariant set.*

Definition 2. Σ is a robust controlled invariant set for the polytopic system (1) if for any $x \in \Sigma$, there exists a state dependent input $u \in \mathcal{U}$ such that $A_j x + B_j u + Dw \in \Sigma$ for $j = 1, 2, \dots, q$ and all $w \in \mathcal{W}$.

Special robust controlled invariant sets are the sets Σ_N, $N \geq 0$, of all initial states $x(0)$ which can be robustly steered into Σ_0 by an N-steps feedback control sequence $\{u(0), u(1), \dots, u(N-1)\} \subset \mathcal{U}$, where each $u(i)$ is allowed to depend on the current state $x(i)$. The sets Σ_N can be computed recursively as follows

$$\begin{aligned}
\Sigma_0 &= O_\infty \\
\text{for } N &= 1, 2, \dots \\
\Sigma_N &= \mathcal{X} \cap \{x | \exists\, u \in \mathcal{U} : A_j x + B_j u \in \Sigma_{N-1} \sim D\mathcal{W}, j = 1, 2, \dots, q\}.
\end{aligned} \tag{12}$$

Notice that the sets $\Sigma_N, N = 0, 1, \dots,$ give an increasing sequence. It must be pointed out that, given $x(0) \in \Sigma_N$, it is not possible, in general, to pre-compute at time $t = 0$ the control sequence $\{u(0), u(1), \dots, u(N-1)\}$ (open-loop control) which robustly steers the state vector $x(0)$ into Σ_0 in N steps.

Therefore for receding-horizon operation it is convenient to introduce the smaller class of robust controlled invariant sets $\underline{\Sigma}_N$ for which a sequence of N control moves around the linear feedback (8), driving the initial state $x(0)$ into Σ_0, can be pre-computed. Let us consider the feedback transformation $u(t) = Fx(t) + c(t)$ where $c(t)$ denotes the new control variable and, accordingly, re-express the system (1) – (2) as

$$\begin{aligned} x(t+1) &= \Phi(t)x(t) + B(t)c(t) + Dw(t) \\ u(t) &= Fx(t) + c(t) \\ [\Phi(t), B(t)] &\in Co\{[\Phi_1, B_1], [\Phi_2, B_2], \ldots, [\Phi_q, B_q]\}. \end{aligned} \tag{13}$$

With reference to the above system, let us denote by $\underline{\Sigma}_N$ the set of all the initial states which can be robustly steered to Σ_0, while satisfying the constraints, by choice of an N-steps sequence $\{c(0), c(1), \ldots, c(N-1)\}$ depending on $x(0)$ only. Notice that the actual control sequence $\{u(0), u(1), \ldots, u(N-1)\}$ turns out to be the sum of the linear feedback $Fx(t)$ plus the pre-computed sequence $\{c(0), c(1), \ldots, c(N-1)\}$. The set $\underline{\Sigma}_N$ can be computed by the following recursion

$$\begin{aligned} S_0 &= \Sigma_0 = \underline{\Sigma}_0 = O_\infty \\ \text{for } N &= 1, 2, \ldots \\ S_N &= \left\{ \begin{bmatrix} x \\ c \\ z \end{bmatrix} : \begin{bmatrix} \Phi_j x + B_j c + Dw \\ z \end{bmatrix} \in S_{N-1} \text{ for } j = 1, 2, \ldots, q \text{ and } \forall\, w \in \mathcal{W}; \right. \\ &\qquad \left. Fx + c \in \mathcal{U}; x \in \mathcal{X} \right\} \\ \underline{\Sigma}_N &= \left\{ x : \begin{bmatrix} x \\ c \\ z \end{bmatrix} \in S_N \text{ for some } \begin{bmatrix} c \\ z \end{bmatrix} \right\} \end{aligned} \tag{14}$$

where, at stage N, $c = c(0)$ denotes the first move and $z = \left[c^T(1), c^T(2), \ldots, c^T(N-1)\right]^T$ the subsequent $N-1$ moves. Notice that the above recursion employs sets S_N belonging to spaces whose dimension increases with N; each vector of S_N represents a state x (first n components) and a control sequence $[c^T(0), \ldots, c^T(N-1)]^T$ which robustly steers x in Σ_0. Hence $\underline{\Sigma}_N$ is just the projection of S_N onto the state space $\mathbb{R}^n$. In particular, from the construction (14), it turns out that

$$\begin{bmatrix} x(0) \\ c(0) \\ \vdots \\ c(N-1) \end{bmatrix} \in S_N \Rightarrow \begin{bmatrix} x(1) \\ c(1) \\ \vdots \\ c(N-1) \end{bmatrix} \in S_{N-1} \Rightarrow \begin{bmatrix} x(N-1) \\ c(N-1) \end{bmatrix} \in S_1 \Rightarrow x(N) \in \Sigma_0. \tag{15}$$

for any realization of the system dynamics $(A(t), B(t))$ and of the disturbance $w(t),\ t = 0, \ldots, N-1$.

Remark 3. Clearly, since Σ_0 is invariant for $c(t) = 0$, the sets $\underline{\Sigma}_N$, $N = 0, 1, \ldots$ form an increasing sequence; moreover $\underline{\Sigma}_N \subset \Sigma_N$.

4. PREDICTIVE CONTROL ALGORITHMS

Predictive control algorithms are based on the optimization, along the control horizon, of a given cost-functional, subject to suitable constraints. These constraints must ensure the feasibility, at each step, of the corresponding system behavior and, possibly, asymptotic convergence to the origin, or, in case of persistent disturbances, to an ultimate boundedness set. When the dynamics is uncertain, the cost optimization can be carried out either in a min-max or in a nominal sense. Hereafter we shall consider only optimization with respect to the nominal model. This choice is due both to the need of limiting the on-line computational complexity of predictive control algorithms and to the belief that, in practice, attention to the worst case performance does not pay too much. Therefore we assume that the feedback gain matrix F, which robustly stabilizes the system, is the optimal LQ feedback gain for the nominal model (this can always be ensured by a suitable choice of the LQ cost functional), so that the optimal control policy is to reduce the gap between the actual input $u(t)$ and the feedback control signal $Fx(t)$. Feasibility and stability constraints, conversely, are provided by the robust invariant sets $\underline{\Sigma}_N$ and Σ_N, previously introduced.

Given $x(0)$, the control sequence $\underline{c} \triangleq [c^T(0), \ldots, c^T(N-1)]^T$ steers $x(0)$ to Σ_0 iff $[x(0)^T, \underline{c}^T]^T \in S_N$. This suggests the following on-line receding-horizon scheme, provided that S_N is determined off-line.

Robust Predictive Control. (RPC(N)) At each sample time t, find

$$\begin{bmatrix} \hat{c}(t|t) \\ \hat{c}(t+1|t) \\ \vdots \\ \hat{c}(t+N-1|t) \end{bmatrix} = \arg \min_{c(t+k|t), 0 \le k \le N-1} \sum_{k=0}^{N-1} \|c(t+k|t)\|^2, \tag{16}$$

subject to

$$\begin{bmatrix} x(t) \\ c(t|t) \\ c(t+1|t) \\ \vdots \\ c(t+N-1|t) \end{bmatrix} \in S_N. \tag{17}$$

Then apply to the system the control signal

$$u(t) = Fx(t) + \hat{c}(t|t) \tag{18}$$

The above algorithm selects, at time t, among all sequences $\{c(t|t), \ldots, c(t+N-1|t)\}$ which robustly enforce $x(t+N) \in \Sigma_0$, the one with minimum l_2 norm. Notice that if $\mathcal{U}$ and $\mathcal{X}$ are polyhedral sets, S_N turns out to be a convex polytope and hence (16)-(17) amounts to a *Quadratic Programming (QP)* problem. As far as stability is concerned, the following result holds.

Theorem 3. Provided that $x(0) \in \underline{\Sigma}_N$, the receding-horizon control (16)-(18) guarantees that: (i) the constraints (5) are satisfied; (ii) $x(t) \to \mathcal{R}_\infty$ as $t \to \infty$.

Proof. The hypothesis that $x(0) \in \underline{\Sigma}_N$ along with (15) imply that $x(1) \in \underline{\Sigma}_{N-1} \subset \underline{\Sigma}_N$. Hence, by induction, $x(t) \in \underline{\Sigma}_N$ for all $t \geq 0$ and satisfaction of the constraints (5) is guaranteed. Next, consider the Bellman function

$$V_t = V(x(t)) \triangleq \sum_{k=0}^{N-1} \|\hat{c}(t+k|t)\|^2$$

Since, by (15), $\{\hat{c}(t+1|t), \ldots, \hat{c}(t+N-1|t), \mathbf{0}\}$ is feasible at time $t+1$,

$$V_t - V_{t+1} \geq \|c(t)\|^2 \geq 0 \tag{19}$$

where $c(t) \triangleq \hat{c}(t|t)$. Hence $\{V_t\}_{t\geq 0}$ is a nonnegative monotonic non-increasing scalar sequence and, as $t \to \infty$, must converge to $V_\infty < \infty$. Summing the $V_t - V_{t+1}$ of (19), for t from 0 to ∞, we have

$$\infty > V_0 - V_\infty \geq \sum_{t=0}^{\infty} \|c(t)\|^2 \geq 0 \Rightarrow \lim_{t\to\infty} \|c(t)\|^2 = 0$$

which proves that $\lim_{t\to\infty} c(t) = \mathbf{0}$. Since

$$x(t) = \Phi(t,0)x(0) + \sum_{k=0}^{t-1} \Phi(t,k+1)B(k)c(k) + \sum_{k=0}^{t-1} \Phi(t,k+1)Dw(k)$$

$$\Phi(t,k) = \Phi(t-1)\cdots\Phi(k+1)\Phi(k)$$

and, by Assumption 1, $\Phi(t,k)$ exponentially converges to $\mathbf{0}$ as $t \to \infty$, it follows that $x(t) \to \mathcal{R}_\infty$ as $t \to \infty$. □

Algorithm RPC(N) ensures, therefore, asymptotic stability with domain of attraction $\underline{\Sigma}_N \subset \Sigma_N$. Notice that $\underline{\Sigma}_N$ is actually a conservative region; any state in Σ_N could in fact be steered into Σ_0 by an N-steps feedback sequence. Therefore an alternative algorithm called IC–PC(N) is introduced hereafter.

Invariance Constraint Predictive Control. (IC–PC(N)) Let Σ be a robust controlled invariant set. At each sample time t, find

$$\begin{bmatrix} \hat{c}(t|t) \\ \hat{c}(t+1|t) \\ \vdots \\ \hat{c}(t+N-1|t) \end{bmatrix} = \arg \min_{c(t+k|t), 0\leq k\leq N-1} \sum_{k=0}^{N-1} \|c(t+k|t)\|^2, \tag{20}$$

subject to the robust *invariance constraint*

$$\Phi_j x(t) + B_j c(t|t) \in \Sigma_N \sim D\mathcal{W} \quad j = 1, 2, \ldots, q \tag{21}$$

and to the *nominal constraints*

$$\begin{aligned} &u(t+k|t) \in \mathcal{U} \text{ and } x(t+k|t) \in \mathcal{X}, \quad && k \geq 0 \\ &c(t+k|t) = \mathbf{0}, && k \geq N \end{aligned}$$

where $u(t+k|t)$ and $x(t+k|t)$ denote the disturbance free predictions with respect to the pre-specified nominal LTI system $[A_0, B_0]$. Then apply $u(t) = Fx(t) + \hat{c}(t|t)$ to the system.

Remark 4. In this algorithm, constraints on the future input and state values are imposed for the nominal model only. They should ensure a satisfactory performance for the real plant. Notice that the constraints $u(t+k|t) \in \mathcal{U}$ and $x(t+k|t) \in \mathcal{X}$ for $k \geq N$ are equivalent to impose that $x(t+N|t)$ belongs to $\overline{\Sigma}_0$, the maximal admissible set under the linear closed-loop dynamics of the nominal model. Conversely stability and feasibility are ensured by the robust constraint (21), which guarantees that the state $x(t)$ will never leave Σ and, hence, that the optimization problem will remain feasible at future time instants.

Theorem 4. Let $\Sigma = \Sigma_L$ where $L \geq N$ is the integer such that

$$\Sigma_{L-1} \subset \overline{\Sigma}_{N-1}\ , \quad \Sigma_L \not\subset \overline{\Sigma}_{N-1}. \tag{22}$$

Then, if $x(0) \in \Sigma$, the IC–PC(N) algorithm guarantees that $x(t) \in \Sigma$ for all $t \geq 0$ and that the constraints (5) are satisfied.

Proof. For any $x \in \Sigma_L$ there exists $u \in \mathcal{U}$ such that $Ax + Bu \in \Sigma_{L-1} \sim D\mathcal{W} \subset \overline{\Sigma}_{N-1} \cap (\overline{\Sigma}_L \sim D\mathcal{W})$, for all $[A, B] \in [A_0, B_0] + \mathcal{P}$. Hence if $x(0) \in \Sigma_L$, IC–PC(N) can find $u(0) \in \mathcal{U}$ such that $x(1) \in \Sigma_L$ and, by induction, guarantees that $x(k) \in \Sigma_L \subset \mathcal{X}$ and $u(k) \in \mathcal{U}$ for all $k \geq 0$. □

Compared to RPC(N), IC–PC(N) has a larger feasibility domain $\Sigma_L \supset \underline{\Sigma}_N$. However, robust asymptotic stability cannot be guaranteed. A counterexample exhibiting a limit cycle will be shown in the next section. From a computational point of view the IC–PC algorithm is cheaper than the RPC algorithm. In fact, the latter requires the off-line computation of the invariant set S_N in the higher dimensional space $\mathcal{R}^{n+mN}$ which is harder than the computation of Σ_N in $\mathcal{R}^n$. Details on the computation of these sets are reported in the Appendix. Nevertheless, the most crucial issue is on-line computation which, for both algorithms, amounts to solving at each sampling interval, a QP problem in mN variables. Compared to traditional predictive control, the algorithms RPC and IC–PC may involve a considerably larger number of linear inequalities in the QP problem. However this implies a negligible extra computational load if *interior point* algorithms [17] are used for the solution of QP.

5. A NUMERICAL EXAMPLE

Consider the system (1) with $q = 2$,

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1.8 \end{bmatrix}, \quad B = B_1 = B_2 = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, \quad D = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \tag{23}$$

subject to input saturation constraints $|u(t)| \le 2$ and disturbance $|w(t)| \le 0.3$. We assumed $A_0 = (A_1 + A_2)/2$ and $B_0 = B$ as the nominal model. Also consider the feedback gain $F = [-11.80 - 18.70]$ which quadratically stabilizes the system (23). With reference to the above feedback gain, Figure 1 compares the robust domain of attraction Σ_0 with $\overline{\Sigma}_0$, the domain of attraction that we should have for the nominal system $[A_0, B_0]$ with no uncertainty and without disturbances. For this example it turns out that, in Theorem 4, $L = 21$ much larger than $N = 3$. Figure 2 compares, for a control horizon $N = 3$, the regions $\underline{\Sigma}_3$ and Σ_{21} where the two algorithms RPC(N) and IC–PC(N) guarantee feasibility (and for RPC(N) also asymptotic stability). Figure 2 also displays the ellipsoidal domain of attraction $\widetilde{\Sigma}_3$ obtained with the approach in [15] for $N = 3$; notice that $\widetilde{\Sigma}_3$ is significantly smaller than $\underline{\Sigma}_3$ and Σ_{21}. This clearly demonstrates the conservatism of using ellipsoids instead of polytopes as robust invariant sets. Figure 3 reports the sets Σ_N for $0 \le N \le 10$ and shows how the feasibility region increases for a larger control horizon. Table 1 reports the number of inequalities describing Σ_N and, respectively, S_N for $0 \le N \le 10$. It can be seen that the complexity of S_N is increasing with N more rapidly than for Σ_N and, hence, the RPC algorithm turns out to be more expensive in terms of computation and memory requirements than IC–PC.

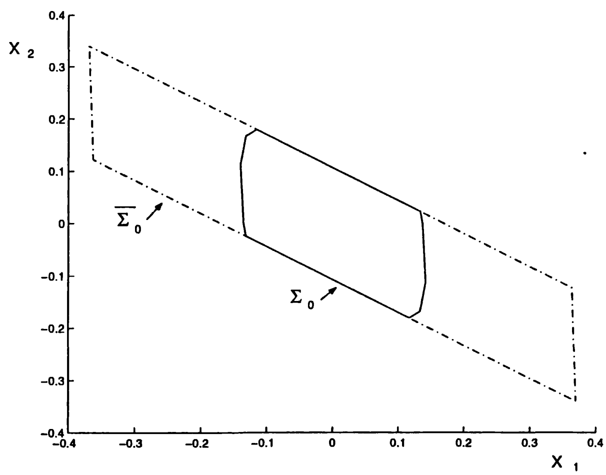

Fig. 1. Σ_0 (solid) and $\overline{\Sigma}_0$ (dashed).

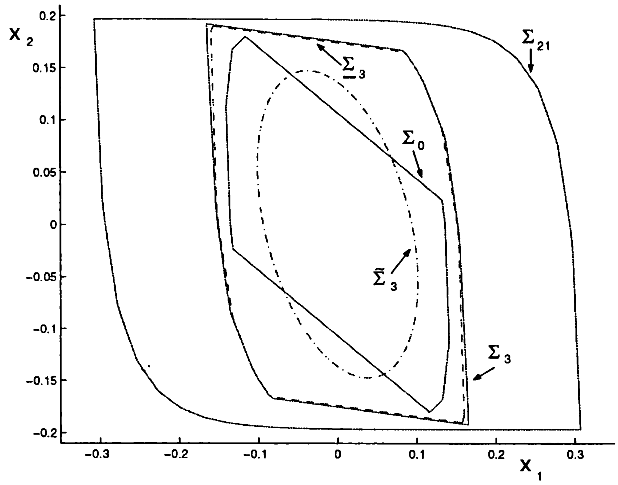

Fig. 2. $\Sigma_0 \subset \Sigma_3 \subset \Sigma_{21}$ (solid), $\underline{\Sigma}_3$ (dashed) and $\widetilde{\Sigma}_3$ (dash-dotted).

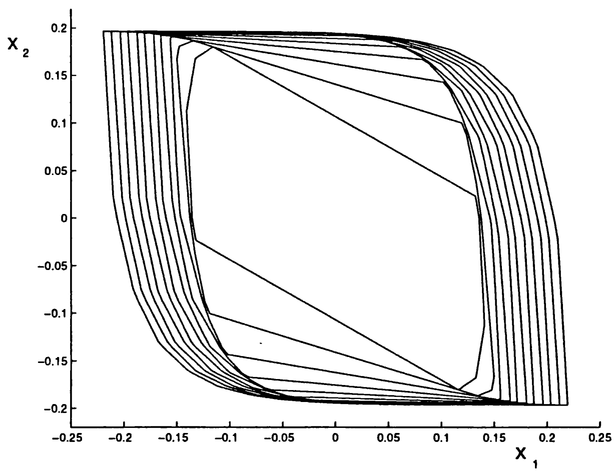

Fig. 3. Σ_N, $0 \leq N \leq 10$.

Simulations have been run to compare the two proposed algorithms RPC(N) and IC–PC(N), taking into account the uncertainty, and a traditional constrained predictive control algorithm referred to as NPC (Nominal Predictive Control) designed for the nominal model ignoring uncertainty and disturbances. A control horizon $N = 3$ has been selected. Figure 4 shows the state trajectories of NPC, RPC(3), IC–PC(3) for an initial state $x(0) \in \underline{\Sigma}_3$, $A(t) = A_2$ and $w(t) = -0.3$, $\forall t \geq 0$;

Table 1. Number of linear inequalities describing S_N and Σ_N vs. N.

N	S_N(RPC)	Σ_N(IC–PC)
0	10	10
1	20	14
2	42	18
3	86	22
4	174	28
5	350	32
6	702	36
7	1342	40
8	2686	44
9	5246	48
10	10238	52

Figure 5 shows the corresponding input responses $u(t)$. Notice that both RPC and IC–PC ensure feasibility and asymptotic stability, while NPC exhibits an infeasible and unstable behavior. Figure 6 shows the state trajectories of RPC and IC–PC for the same $A(t)$ as above, an initial state $x(0) \in \Sigma_{21} \backslash \Sigma_3$ and $w(t) = 0.3\cos(1000\pi t)$. Notice that in this case RPC becomes infeasible and unstable. Although IC–PC(N) ensures feasibility and boundedness of the state in Σ_N, in general it does not guarantee ultimate boundedness in $\mathcal{R}_\infty$. To this end, Figure 7 shows a limit cycle behaviour of IC–PC(3) for the system under consideration with

$$A_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1.9 \end{bmatrix}, \quad F = [-49, -16.5],$$

a particular choice of the initial state, $A(t) = A_2$ and $w(t) = 0$, $\forall t \geq 0$. This example, however, is quite "pathological" since the selected gain F is highly detuned for the true system (A_2, B), i.e. the eigenvalues of $A_2 + BF$ are very close to the unit circle.

6. CONCLUSION

The paper has faced the control of polytopic uncertain systems subject to control/state constraints and unknown but bounded disturbances. Two predictive control algorithms have been proposed. Both combine nominal performance optimization with robust feasibility and stability. The two algorithms differ on the constraints imposed on future inputs and states. When robust constraints are imposed over the whole prediction horizon, we get a smaller feasibility region but asymptotic stability can be proved. When robust constraints are imposed only on the one-step-ahead prediction while the nominal constraints are considered for the subsequent steps, we

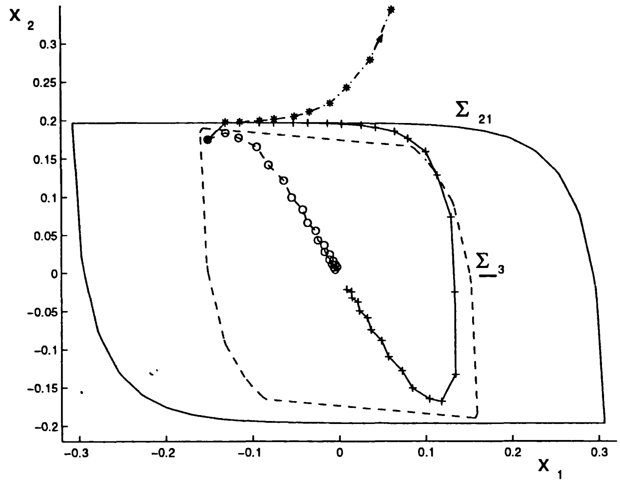

Fig. 4. NPC (dash-dotted), RPC (dashed) and IC–PC (solid).

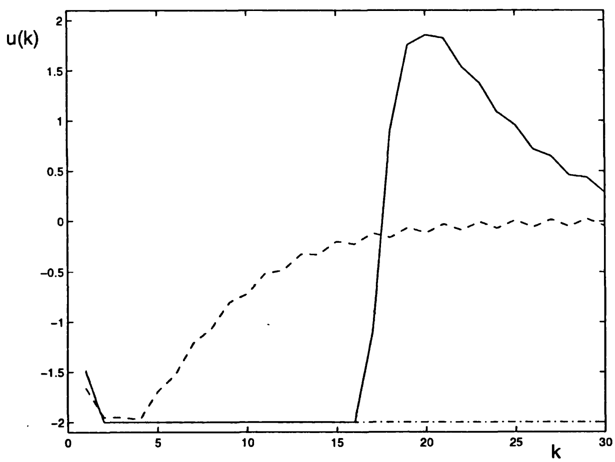

Fig. 5. Input responses of NPC (dash-dotted), RPC (dashed) and IC–PC (solid).

get a larger feasibility region but asymptotic stability cannot be guaranteed. Computational and robust stability requirements have been discussed and performance illustrated by simulation examples.

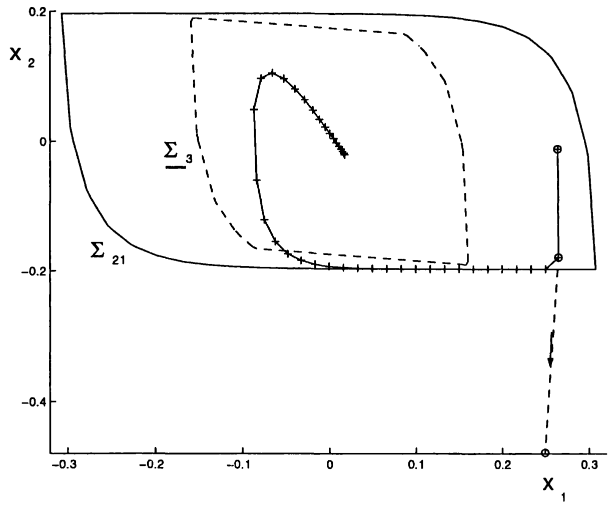

Fig. 6. RPC (dashed) and IC–PC (solid).

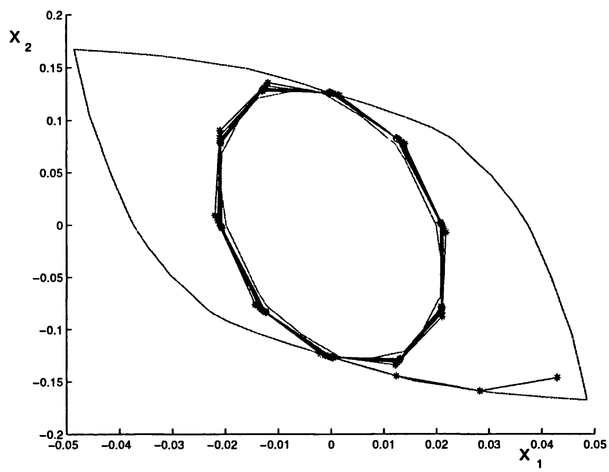

Fig. 7. Limit cycle behavior of IC–PC.

APPENDIX – COMPUTATION OF INVARIANT SETS

Let $\mathcal{X}$, $\mathcal{U}$, $\mathcal{W}$ be polyhedra described by

$$\mathcal{X} = \{x : M_x x \leq v_x\}, \quad \mathcal{U} = \{u : M_u u \leq v_u\}, \quad \mathcal{W} = Co\{w_j; j = 1, 2, \ldots \ell\}.$$

Computation of sets Σ_i

Let $\Sigma_i = \{x : Mx \leq v\}$. Then:

1. Compute $\delta = \max\limits_{1\leq j\leq \ell} MDw_j$ and

$$\mathcal{Z} = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : \begin{bmatrix} MA_1 & MB_1 \\ \vdots & \vdots \\ MA_q & MB_q \\ \mathbf{0} & M_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} v-\delta \\ \vdots \\ v-\delta \\ v_u \end{bmatrix} \right\}.$$

2. Compute, by the Fourier–Motzkin elimination algorithm [11] together with an LP subroutine in order to eliminate redundant inequalities, the projection $\overline{\mathcal{Z}} = \{x : \overline{M}x \leq \overline{v}\}$ of $\mathcal{Z}$ onto $\mathbb{R}^n$.

3. Compute
$$\Sigma_{i+1} = \left\{ x : \begin{bmatrix} \overline{M} \\ M_x \end{bmatrix} x \leq \begin{bmatrix} \overline{v} \\ v_x \end{bmatrix} \right\}.$$

Computation of sets S_i

Let $S_i = \{s : Ms \leq v\} \subset \mathbb{R}^{n+mi}$ and partition M as $M = [M_1, M_2]$ where M_1 and M_2 denote the first n and, respectively, last mi columns. Then, setting $\delta = \max\limits_{1\leq j\leq \ell} M_1 D w_j$, we get $S_{i+1} = \{s : \overline{M}s \leq \overline{v}\} \subset \mathbb{R}^{n+m(i+1)}$ where

$$\overline{M} = \begin{bmatrix} M_1\Phi_1 & M_1B_1 & M_2 \\ \vdots & \vdots & \vdots \\ M_1\Phi_q & M_1B_q & M_2 \\ M_uF & M_u & \mathbf{0} \\ M_x & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \overline{v} = \begin{bmatrix} v-\delta \\ \vdots \\ v-\delta \\ v_u \\ v_x \end{bmatrix}.$$

(Received January 18, 2001.)

REFERENCES

[1] T. A. Badgwell: Robust model predictive control. Internat. J. Control *68* (1997), 797–818.
[2] R. R. Barmisch: Necessary and sufficient conditions for quadratic stabilizability of an uncertain system. J. Optim. Theory Appl. *46* (1985), 399–408.
[3] A. Bemporad and M. Morari: Robust model predictive control: a survey. In: Robustness in Identification and Control (A. Garulli, A. Tesi and A. Vicino, eds., Lecture Notes in Control and Information Sciences 245), Springer–Verlag, Berlin 1999, pp. 207–226.
[4] F. Blanchini: Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov function. IEEE Trans. Automat. Control *39* (1994), 428–433.
[5] F. Blanchini: Set invariance in control. Automatica *35* (1999), 1747–1768.
[6] A. Casavola, M. Giannelli, and E. Mosca: Min-max predictive control strategies for input-saturated polytopic uncertain systems. Automatica *36* (2000), 125–133.

[7] L. Chisci and G. Zappa: Robustifying a predictive controller against persistent disturbances. In: Proc. European Control Conference'99 (CD ROM), Karlsruhe 1999.
[8] J. C. Geromel, P. L. D. Peres, and J. Bernussou: On a convex parameter space method for linear control design of uncertain systems. SIAM J. Control Optim. *29* (1991), 381–402.
[9] E. G. Gilbert and K. Tin Tan: Linear systems with state and control constraints: the theory and application of maximal output admissible sets. IEEE Trans. Automat. Control *36* (1991), 1008–1021.
[10] L. Gurvits: Stability of discrete linear inclusion. Linear Algebra Appl. *231* (1995), 47–85.
[11] S. S. Keerthi and E. G. Gilbert: Computation of minimum-time feedback control laws for discrete-time systems with state-control constraints. IEEE Trans. Automat. Control *32* (1987), 432–435.
[12] S. S. Keerthi and E. G. Gilbert: Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: stability and moving-horizon approximations. J. Optim. Theory Appl. *57* (1988), 265–293.
[13] I. Kolmanovsky and E. G. Gilbert: Maximal output admissible sets for discrete-time systems with disturbance inputs. In: Proc. 1995 Amer. Control Conference, Seattle 1995, pp. 1995–1999.
[14] M. V. Kothare, V. Balakrishnan, and M. Morari: Robust constrained model predictive control using linear matrix inequalities. Automatica *32* (1996), 1361–1379.
[15] B. Kouvaritakis, J. A. Rossiter, and J. Schuurmans: Efficient robust predictive control. IEEE Trans. Automat. Control *4* (2000), 1545–1549.
[16] J. H. Lee and Z. Yu: Worst case formulations of model predictive control for systems with bounded parameters. Automatica *33* (1997), 763–781.
[17] C. Rao, J. B. Rawlings, and S. Wright: Application of interior point methods to model predictive control. J. Optim. Theory Appl. *99* (1998), 723–757.
[18] J. A. Rossiter, B. Kouvaritakis, and M. J. Rice: A numerically robust state-space approach to stable predictive control strategies. Automatica *34* (1998), 65–73.

Prof. Dr. Luigi Chisci, Dr. Paola Falugi and Prof. Dr. Giovanni Zappa, Dipartimento di Sistemi e Informatica – Università di Firenze, via Santa Marta 3, 50139 Firenze. Italy.
e-mails: chisci,falugi,zappa@dsi.unifi.it