George F. Fragulis

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# MINIMAL REALIZATIONS OF THE INVERSE OF A POLYNOMIAL MATRIX USING FINITE AND INFINITE JORDAN PAIRS

GEORGE F. FRAGULIS

A simple method is given which uses the notions of finite and infinite Jordan pairs from the theory of operators in such a way to find the minimal realization of the inverse of a given polynomial matrix. An application of the proposed method is to find the generalized state-space system which has as transfer function the inverse of the polynomial matrix.

## 1. INTRODUCTION

It is rather obvious that the connections between control theory and linear algebra are very strong. Several formulas and notions, as well as, known techniques from matrix theory and theory of operators are used efficiently in control theory. The important treatise of [4] gives a nice example of how matrix theory can be applied to the analysis and solution-finding of several difficult problems in control theory. On the other hand Gohberg and other researchers [5] presented their work on operator polynomial and general operator-valued functions, and pointed out the striking similarities among them and formulas and notions in control theory making the observation that "...from the systems theory point of view, we study here systems for which the transfer function matrix is the inverse of a polynomial matrix" [5, page 7]. In this paper we present a simple method which uses the notions of finite and infinite Jordan pairs from the operator theory in such a way to find a minimal realization of the inverse of a polynomial matrix. The notions of finite and infinite Jordan pairs were found originally in [5] and are based on the notions of finite and infinite Jordan chains [4, 5]. Our analysis is based on the theory presented in recent papers [3, 9, 10], where simple and efficient methods of finding finite and infinite Jordan chains – and as a consequence Jordan pairs – using the notions of finite and infinite elementary divisors, are given.

## 2. MAIN RESULTS

Let $A(s)$ be a polynomial matrix:

$$A(s) = A_0 + A_1 s + \cdots + A_{q_1} s^{q_1} \in \Re^{r\times r}[s] \tag{1}$$

with Smith–McMillan form at $s = \infty$ [9]:

$$S^{\infty}_{A(s)}(s) = \operatorname{diag}\left[s^{q_1}, s^{q_2} \ldots, s^{q_k}, \frac{1}{s^{\hat{q}_{k+1}}}, \ldots \frac{1}{s^{\hat{q}_r}}\right], \tag{2}$$

where $1 \leq k \leq r$ and $\hat{q}_i = -q_i$ $i = k+1, \ldots r$ such that $q_1 \geq q_2 \geq \cdots \geq q_k \geq 0$, $\hat{q}_r \geq \hat{q}_{r-1} \geq \cdots \geq \hat{q}_{k+1} \geq 0$.

Let also the finite Jordan pair $C_f \in \Re^{r\times n}$, $J_f \in \Re^{n\times n}$ with $n = \deg|A(s)|$. Let also the infinite Jordan pair $(C_\infty, J_\infty)$ of $A(s)$,with $C_\infty \in \Re^{r\times\mu}$, $J_\infty \in \Re^{\mu\times\mu}$, where $\mu$ is given by [9]:

$$\mu = (r-1)q_1 - \sum_{i=2}^{k} q_i + \sum_{j=k+1}^{r} \hat{q}_j, \tag{3}$$

where $q_i, i = 1, \ldots, k$ and $\hat{q}_j j = k+1, \ldots, r$ denote the orders of the poles and zeros at $s = \infty$ of $A(s)$ respectively. It is well known [11] that the rational matrix $A^{-1}(s)$ can be written:

$$A^{-1}(s) = C_f[sI_n - J_f]^{-1} B_f + C_\infty[sJ_\infty - I_\mu]^{-1} B_\infty, \tag{4}$$

where $B_f, B_\infty$ can be found [11]

$$\begin{bmatrix} B_f \\ B_\infty \end{bmatrix} = [I_n,\ J_\infty^{q_1-1}] \begin{bmatrix} S_{q_1-2} \\ V \end{bmatrix}^{-1} [0, 0, \ldots, 0, I_r]^T \tag{5}$$

$$V = [A_{q_1} C_f J_f^{q_1-1},\ -\sum_{i=0}^{q_1-1} A_i C_\infty J_\infty^{q_1-1-i}] \tag{6}$$

$$S_{q_1-2} = \begin{bmatrix} C_f & C_\infty J_\infty^{q_1-2} \\ C_f J_f & C_\infty J_\infty^{q_1-3} \\ \vdots & \vdots \\ C_f J_f^{q_1-2} & C_\infty \end{bmatrix}. \tag{7}$$

First of all we shall show the following:

**Proposition 1.** Let $A(s) \in \Re^{r\times r}[s]$ be a polynomial matrix as in (1). Let also the finite Jordan pair $C_f \in \Re^{r\times n}$, $J_f \in \Re^{n\times n}$, and the matrix $B_f \in \Re^{n\times r}$ as defined in (5) with $n = \deg|A(s)|$, such that:

$$H_{\mathrm{spr}}(s) = C_f[sI_n - J_f]^{-1} B_f, \tag{8}$$

where $H_{\mathrm{spr}}(s)$ is the strictly proper part of $A^{-1}(s)$. Then the triple $(C_f, J_f, B_f)$ is a minimal realization for $H_{\mathrm{spr}}(s)$.

Proof. Consider the degree $d$ (in the finite complex plane). We have $d(H_{spr}(s)) = d(A^{-1}(s))$. The right hand side equals to the total zero multiplicity of $A(s)$ at finite points, hence is equal to $n = \deg|A(s)|$. Thus by classical realization results, any realization of dimension $n$ for $H_{spr}(s)$ is automatically minimal. □

Now we define the dual polynomial matrix $\tilde{A}(w)$ of $A(s)$ as in [5]:

$$\tilde{A}(w) := A_0 w^{q_1} + A_1 w^{q_1-1} + \cdots + A_{q_1} = w^{q_1} A(\frac{1}{w}) \in \Re^{r\times r}[w] \tag{9}$$

**Definition 2.** [5] The infinite elementary divisors (IEDs) of $A(s)$ are defined as the finite elementary divisors of $\tilde{A}(w)$ at $w = 0$ i.e. as the finite elementary divisors of $\tilde{A}(w)$ that have the form:

$$w^{\mu_j},\ \mu_j > 0. \tag{10}$$

In order to examine the structure of the IEDs of $A(s)$ we thus see that we need the zero structure at $w = 0$ of $\tilde{A}(w)$. Let $S^0_{\tilde{A}(w)}(w)$ denote the local Smith form of $\tilde{A}(w)$ at $w = 0$. Then it can be proved that:

**Proposition 3.** [10] Let $A(s) \in \Re^{r\times r}[s]$ as in (1) and let $S^0_{\tilde{A}(w)}(w)$ be the local Smith form of $\tilde{A}(w)$ at $w = 0$. Then:

$$S^0_{\tilde{A}(w)}(w) = w^{q_1} S^{\infty}_{A(s)}\left(\frac{1}{w}\right) = \mathrm{diag}[1, w^{q_1-q_2}, \ldots, w^{q_1-q_k}, w^{q_1-\widehat{q}_{k+1}}, \ldots, w^{q_1-\widehat{q}_r}] \tag{11}$$

and the infinite elementary divisors of the polynomial matrix $A(s)$ are given by:

$$w^{\mu_j}, j = 2, 3, \ldots r \tag{12}$$

$$\mu_j = q_1 - q_j > 0,\ j = 2, 3, \ldots, k \tag{13}$$

$$\hat{\mu}_j = q_1\ + \hat{q}_j\ > 0, \quad j = k+1, k+2, \ldots r. \tag{14}$$

**Remark.** We see that polynomial matrices have in general two kinds of IEDs. The first kind of IEDs that correspond to poles at $s = \infty$ of $A(s)$ with orders $q_j < q_1$, $j = 2, 3, \ldots, k$. The second kind of IEDs correspond to poles and zeros at $s = \infty$. Notice that the first kind of IEDs exist if $\mu_j = q_1 - q_j > 0$, $j = 2, 3, \ldots, k$ and that the second kind of IEDs exists only when $A(s)$ has zeros at $s = \infty$. The first kind of IEDs, i.e. the ones with degrees $\mu_j = q_1 - q_j > 0$, $j = 2, 3, \ldots, k$ we call "infinite pole IEDs". The second kind of IEDs, i.e. the ones with degrees $\hat{\mu}_j = q_1 + \hat{q}_j > 0$, $j = k+1,\ k+2, \ldots r$ we call "infinite zero IEDs".

In a recent paper [10] a method was introduced which showed how to find infinite Jordan chains that correspond to the "infinite zero" IEDs of the polynomial matrix

$A(s)$. Each infinite zero IED with degree $w^{\widehat{\mu}_j}$, $\widehat{\mu}_j = q_1 + \widehat{q}_j$, $j = k+1, \ldots, r$, form infinite Jordan chains of lenghts $\widehat{\mu}_j$, $j = k+1, \ldots, r$.

We define the matrix:

$$C_{\infty j} = [\widehat{x}_{j0}, \widehat{x}_{j1}, \ldots, \widehat{x}_{j,\widehat{\mu}_j - 1}] \in \Re^{r \times \widehat{\mu}_j} \quad j = k+1, \ldots, r \tag{15}$$

which consists of those infinite Jordan chains. If we now find in a similar way the infinite Jordan chain to "infinite pole" IEDs of $A(s)$ with degrees: $w^{\mu_j}$, $\mu_i = q_1 - q_i$, $i = 2, 3, \ldots, k$ we can define the matrix:

$$C_{\infty i} = [x_{j0},\ x_{j1}, \ldots, x_{j,\mu_j - 1}] \in \Re^{r \times \mu_j},\ i = 2, 3, \ldots, k. \tag{16}$$

The index $i$ has starting value $i = 2$ because $w^{\mu_1} = 1$ as we can see from (11) and no infinite Jordan chain is obtained from this infinite pole IED of $A(s)$. Now from (15) and (16) we define the following matrix:

$$C_\infty = [C_{\infty 2}, C_{\infty 3}, \ldots, C_{\infty k} \mid C_{\infty, k+1}, C_{\infty, k+2}, \ldots, C_{\infty r}] \in \Re^{r \times \mu} \tag{17}$$

with

$$\mu = (k-1) q_1 - \sum_{i=2}^{k} q_i + (r-k) q_1 + \sum_{j=k+1}^{r} \widehat{q}_j = (r-1) q_1 - \sum_{i=2}^{k} q_i + \sum_{j=k+1}^{r} \widehat{q}_j. \tag{18}$$

Now to each infinite pole IED $w^{\mu_i}$, $\mu_i = q_1 - q_i$, $i = 2, 3, \ldots, k$ corresponds a nilpotent matrix: $J_{\infty i} \in \Re^{\mu_i \times \mu_i}$. Similarly to each infinite zero IED $w^{\widehat{\mu}_j}$, $\widehat{\mu}_j = q_1 + \hat{q}_j, j = k+1, \ldots, r$ corresponds a nilpotent matrix: $J_{\infty j} \in \Re^{\widehat{\mu}_i \times \widehat{\mu}_i}$. Finally we define the matrix:

$$J_\infty = \text{blockdiag}\left[J_{\infty 2}, J_{\infty 3}, \ldots, J_{\infty k} \mid J_{\infty, k+1}, J_{\infty, k+2}, \ldots, J_{\infty r}] \in \Re^{\mu \times \mu}\right] \tag{19}$$

with $\mu$ as in (18).

In the sequel we shall present a method which shows how to reduce the degree of $\mu$ and make it minimal. If we take the $\sigma$th power of $J_\infty$ in (19), because of its block diagonal form, we shall also take the same powers of $J_{\infty i}$, $i = 2, 3, \ldots, r$. But the matrices $J_{\infty i}$, $i = 2, 3, \ldots, r$ are nilpotent, hence they shall have $\sigma$-zero rows in the end.Clearly the index of nilpotency of $J_\infty$ is equal to the index of nilpotency of its maximum Jordan block i.e. equal to $q_1 + \widehat{q}_r$. We consider again the form of $B_\infty$ as given in (5):

$$\begin{bmatrix} 0_n \\ B_\infty \end{bmatrix} = [0_n,\ J_\infty^{q_1 - 1}] \begin{bmatrix} S_{q_1 - 2} \\ V \end{bmatrix}^{-1} [0, 0, \ldots, 0, I_r]^T. \tag{20}$$

First of all we remark that $J_\infty^{q_1 - 1} \neq 0$ because its index of nilpotency is $q_1 + \widehat{q}_r > q_1 - 1$. Because of the block diagonal form of $J_\infty$ we must take also the $(q_1 - 1)$-power of $J_{\infty i}$, $i = 2, 3, \ldots, k$ and $(q_1 - 1)$-power of the $J_{\infty j}$, $j = k+1, \ldots, r$. The index of nilpotency of the Jordan blocks $J_{\infty i}$, $i = 2, 3, \ldots, k$ is clearly $q_1 - q_i$, $i = 2, 3, \ldots, k$, i.e. $J_{\infty i}^{q_1 - q_i} \equiv 0$ for $i = 2, 3, \ldots, k$. But $q_1 - 1 \geq q_1 - q_i$, $i = 1, 2, \ldots k$ because $q_1 \geq q_2 \geq \ldots \geq q_k > 0$. Hence

$$J_{\infty i}^{q_1 - 1} = 0, \quad i = 1, 2, \ldots, k. \tag{21}$$

If we take the $(q_1-1)$-power of the Jordan blocks $J_{\infty j}$, $j=k+1,\dots,r$ then each one shall have $(q_1-1)$-zero rows in the end. From the above analysis we have that if we take the $(q_1-1)$-power of $J_\infty$, then $J_\infty$ has:

$$m_1=\sum_{i=2}^{k}(q_1-q_i)=(k-1)\,q_1-\sum_{i=2}^{k}q_i \tag{22}$$

zero-rows from the beginning because of the Jordan blocks $J_{\infty i}$, $i=2,3,\dots,k$ (see (21)) and

$$m_2=(r-k)\,(q_1-1) \tag{23}$$

zero-rows in the end of each block $J_{\infty j}$, $j=k+1,\dots,r$ ($(r-k)$ is the number of the Jordan blocks $J_{\infty j}, j=k+1,\dots,r$). From (22) and (23) it is clear that the $(q_1-1)$-power of $J_\infty$ has $(m_1+m_2)$ zero-rows. Now from the definition of $B_\infty$ in (20) we have that the matrix $[0_n,\, J_{\infty i}^{q_1-1}]$ is multiplied with $\begin{bmatrix} S_{q_1-2} \\ V \end{bmatrix}^{-1}$ $[0,0,\dots,0,I_r]$ – which is nonsingular by definition – hence the product $[0_n,\, J_{\infty}^{q_1-1}]\begin{bmatrix} S_{q_1-2} \\ V \end{bmatrix}^{-1}[0,0,\dots,0,I_r]^T$ has also $(m_1+m_2)$ zero-rows. Then from the above it is clear that the matrix $B_\infty$ has always $(m_1+m_2)$ zero-rows which are not useful and we can eliminate them. If from the matrix $B_\infty\in\Re^{\mu\times r}$ with $\mu$ as in (18) we eliminate the appropriate zero-rows( the $m_1$ zero-rows from the start and the $m_2$ zero-rows which correspond to the last zero rows of each Jordan block $J_{\infty j}$, $j=k+1,\dots,r$) we obtain a new matrix $\widetilde{B}\in\Re^{\widetilde{\mu}\times r}$ with:

$$\begin{aligned}\widetilde{\mu}=\mu-(m_1+m_2)&=(r-1)\,q_1-\sum_{i=2}^{k}q_i+\sum_{j=k+1}^{r}\widehat{q}_j-\left[(k-1)\,q_1-\sum_{i=2}^{k}q_i+(r-k)\,(q_1-1)\right]\\ &=(r-k)+\sum_{j=k+1}^{r}\widehat{q}_j\end{aligned} \tag{24}$$

rows. As we can easily see the value of $\widetilde{\mu}$ (which is the value of $\mu$ after the elimination of the zero-rows) is equal to the order of a minimal realization of $H_{pol}(s)$ [9].

Hence if we eliminate now from $C_\infty$ the columns which correspond to the $(m_1+m_2)$ zero-rows in $B_\infty$ we shall obtain a matrix $\tilde{C}_\infty\in\Re^{r\times\widetilde{\mu}}$. Similarly if we eliminate from $J_\infty$ the columns and the rows which correspond to the $(m_1+m_2)$ zero-rows in $B_\infty$ we shall obtain a matrix $\tilde{J}_\infty\in\Re^{r\times\widetilde{\mu}}$. From the above it is clear that there exists a triple of matrices $\tilde{C}_\infty\in\Re^{r\times\widetilde{\mu}}$, $\tilde{J}_\infty\in\Re^{\widetilde{\mu}\times\widetilde{\mu}}$ and $\tilde{B}_\infty\in\Re^{\widetilde{\mu}\times r}$ such that $H_{pol}(s)=\tilde{C}_\infty[s\tilde{J}_\infty-I_{\widetilde{\mu}}]^{-1}\tilde{B}_\infty$ and $\tilde{\mu}=(r-k)+\sum_{j=k+1}^{r}\widehat{q}_j$, is the least order among all realizations of $H_{pol}(s)$. Hence $\tilde{\mu}$ is the order of a minimal realization i.e. $(\tilde{C}_\infty$, $\tilde{J}_\infty$, $\tilde{B}_\infty)$ is a minimal realization of $H_{pol}(s)$. We can now state the following:

**Proposition 5.** Let $A(s) \in \Re^{r\times r}[s]$ be a polynomial matrix as in (1) with Smith–McMillan form at $s=\infty$ as in (2). Let also the infinite Jordan pair $(C_\infty, J_\infty)$ of $A(s)$ with $C_\infty \in \Re^{r\times\mu}$, $J_\infty \in \Re^{\mu\times\mu}$, where $\mu$ is given by (see (18)):

$$\mu = (r-1)q_1 - \sum_{i=2}^{k} q_i + \sum_{j=k+1}^{r} \widehat{q}_j \tag{25}$$

and $B_\infty \in \Re^{\mu\times\mu}$ as this defined in (5)-(7). Let also $A^{-1}(s) = H_{spr}(s) + H_{pol}(s)$. Then:

(i) The triple of matrices $(C_\infty, J_\infty, B_\infty)$ is a realization for the polynomial part $H_{pol}(s)$ of $A^{-1}(s)$.

(ii) From $(\underset{\sim}{C_\infty}, \underset{\sim}{J_\infty}, \underset{\sim}{B_\infty})$ we can find a triple of matrices $(\tilde{C}_\infty, \tilde{J}_\infty, \tilde{B}_\infty)$, with $\tilde{C}_\infty \in \Re^{r\times\tilde{\mu}}$, $\tilde{J}_\infty \in \Re^{\tilde{\mu}\times\tilde{\mu}}$ and $\tilde{B}_\infty \in \Re^{\tilde{\mu}\times r}$ where $\tilde{\mu}$ is given by (see (24))

$$\tilde{\mu} = (r-k) + \sum_{j=k+1}^{r} \widehat{q}_j. \tag{26}$$

Clearly $\tilde{\mu} \leq \mu$ and the triple $(\tilde{C}_\infty, \tilde{J}_\infty, \tilde{B}_\infty)$ constitutes a minimal realization of the polynomial part $H_{pol}(s)$ of $A^{-1}(s)$, i.e.: $H_{pol}(s) = \tilde{C}_\infty[s\tilde{J}_\infty - I_{\tilde{\mu}}]^{-1}\tilde{B}_\infty$ and $\tilde{\mu} = \delta_M[1/w\ A(1/w)]$.

## 3. AN APPLICATION

The proposed method can be applied to the so-called realization theory of transfer function matrices of Linear Multivariable Systems [6], i.e. physical systems of the form $(\sum)$ :

$$\begin{aligned} A(\rho)\beta(t) &= B(\rho)\,u(t) \\ y(t) &= C(\rho)\,\beta(t), \end{aligned} \tag{27}$$

where $\rho := \mathrm{d}/\mathrm{d}t$ is the differential operator, $A(\rho)$, $B(\rho)$, $C(\rho)$ are polynomial matrices and $\beta(t)$, $y(t)$, $u(t)$ are respectively the pseudostate, the output and the input vectors of the system $(\sum)$. The transfer function matrix of $(\sum)$ is (in frequency-domain): $G(s) = C(s)A^{-1}(s)B(s)$ which is a rational matrix(not necessarily proper) in general. It would be interesting to find certain singular systems in generalized state-space form [1], i.e. physical systems of the form $(\sum_1)$ :

$$\begin{aligned} E\rho x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{28}$$

where $E$, $A$, $B$, $C$ are constant matrices with appropriate dimensions and $x(t)$, $y(t)$, $u(t)$ are respectively the generalized state,the output and the input vectors of the system $(\sum_1)$, which give rise to the transfer function matrix $G(s)$. In other words the transfer function matrix of system $(\sum_1)$ which is given by: $G_1(s) = C[sE-A]^{-1}B$ satisfies the following condition: $G_1(s) = C[sE-A]^{-1}B = C(s)A^{-1}(s)\,B(s) = G(s)$.

**Definition 6.** [2] Assume that $G_1(s) \in \Re^{r\times r}(s)$ is a rational matrix. If there exists a quadruple of matrices $(E, A, B, C)$ such that: $G_1(s) = C[sE - A]^{-1}B$ – where $E, A \in \Re^{\widehat{n}\times\widehat{n}}$, $B \in \Re^{\widehat{n}\times r}$, $C \in \Re^{r\times\widehat{n}}$ are constant matrices with $\widehat{n} \in N - \{0\}$, then the generalized state-space system described by $(\sum_1)$ will be called a singular system realization of $G_1(s)$, or simply a realization of $G_1(s)$. Furthermore the system $(\sum_1)$ is called a minimal realization of $G_1(s)$ iff any other realization of $G_1(s)$ has order greater than $\widehat{n}$, or equivalently iff the generalized state-space system $(\sum_1)$ has the least number of generalized states $(x(t))$.

Any rational matrix $G(s)$ (not necessarily proper) may be represented as the sum of its strictly proper part $H_{spr}(s)$ and its polynomial part $H_{pol}(s)$, i.e. $G(s) = H_{spr}(s) + H_{pol}(s)$. We know that the inverse of a polynomial matrix $F(s) \in \Re^{r\times r}[s]$ is a rational matrix in general. If we now consider the case where $F^{-1}(s) := G(s)$ then the proposed method finds a minimal realization – as this defined in Definition 6 – of a transfer function matrix $(G(s))$ of a system $(\sum)$ which has the property its inverse to be a polynomial matrix.To be more precise let a system $(\sum)$ which give rise to a transfer function matrix $G(s) = C(s)\,A^{-1}(s)\,B(s)$, $(s) \in \Re^{r\times r}$, and assume that $G(s)$ has the following property $G(s)^{-1} := F(s) \in \Re^{r\times r}[s]$. Now Proposition 1 states that we can find a triple of matrices $C_f \in \Re^{r\times n}$, $J_f \in \Re^{n\times n}$, $B_f \in \Re^{n\times r}$ with $n = \deg|F(s)|$, such that: $H_{spr}(s) = C_f[sI_n - J_f]^{-1}B_f$, where $H_{spr}(s)$ is the strictly proper part of $G(s) = F^{-1}(s)$ and the triple $(C_f, J_f, B_f)$ is a minimal realization of $H_{spr}(s)$. Also Proposition 5 states that we can find a triple of matrices $(\tilde{C}_\infty, \tilde{J}_\infty, \tilde{B}_\infty)$, with $\tilde{C}_\infty \in \Re^{r\times\tilde{\mu}}$, $\tilde{J}_\infty \in \Re^{\tilde{\mu}\times\tilde{\mu}}$, $\tilde{B}_\infty \in \Re^{\tilde{\mu}\times r}$ with $\tilde{\mu} = (r-k) + \sum_{j=k+1}^{r} \widehat{q}_j$ such that $H_{pol}(s) = \tilde{C}_\infty[s\tilde{J}_\infty - I_{\tilde{\mu}}]^{-1}\tilde{B}_\infty$, where $H_{pol}(s)$ is the polynomial part of $G(s) = F^{-1}(s)$ and the triple $(\tilde{C}_\infty, \tilde{J}_\infty, \tilde{B}_\infty)$ is a minimal realization of $H_{pol}(s)$. Let now define:

$$E := \begin{bmatrix} I_n & 0 \\ 0 & \tilde{J}_\infty \end{bmatrix} \in \Re^{(n+\tilde{\mu})\times(n+\tilde{\mu})} \tag{29}$$

$$A := \begin{bmatrix} J_f & 0 \\ 0 & I_{\tilde{\mu}} \end{bmatrix} \in \Re^{(n+\tilde{\mu})\times(n+\tilde{\mu})} \tag{30}$$

$$B := \begin{bmatrix} B_f \\ \tilde{B}_\infty \end{bmatrix} \in \Re^{(n+\tilde{\mu})\times r} \tag{31}$$

$$C := \begin{bmatrix} C_f, & \tilde{C}_\infty \end{bmatrix} \in \Re^{r\times(n+\tilde{\mu})}. \tag{32}$$

We can now define the following *generalized state-space* system:

$$\begin{aligned} E\rho x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned} \tag{33}$$

It is easy to verify that: $G(s) = H_{spr}(s) + H_{pol}(s) = C[sE - A]^{-1}B$. Hence the system (33) determined by the matrices (29)–(32) is a realization of $G(s)$. Furthermore this realization is also a minimal one.

**Definition 7.** The order $\hat{n}$ of the minimal realization of the transfer function matrix $G(s)$ defined by (29)–(32) is called the minimum generalized order of $G(s)$. Furthermore $\hat{n}$ is the dimension of the generalized state-space system (33) and is equal to:

$$\hat{n} = n + \tilde{\mu} = \deg|F(s)| + (r-k) + \sum_{j=k+1}^{r} \hat{q}_j, \tag{34}$$

where $\hat{q}_j$, $j = k+1, \dots, r$ denote the orders of the zeros at $s = \infty$ of the polynomial matrix $F(s)$ which can be found using the Smith–McMillan form at $s = \infty$ [8]).

We can now state the following:

**Theorem 8.** Let a linear multivariable system of the form (27) which give rise to a rational transfer function matrix $G(s) \in \Re^{r\times r}(s)$ and has the property to have a polynomial inverse $F(s) \in \Re^{r\times r}[s]$. Then we can find a generalized state-space system of the form (33) with $[E, A, B, C]$ as in (29)–(32) and minimum generalized order $\hat{n}$ as in (34), such that the system (33) to be a minimal realization of the rational matrix $G(s)$ (according to Definition 6). Furthermore since the two systems (27) and (33)give rise to the same transfer function matrix $G(s)$ they have the same sets of finite and infinite transmission poles and zeros ([6, 7]).

## 4. EXAMPLE

Let the following PMD:

$$\begin{bmatrix} \rho+1 & 0 \\ 0 & \rho+1 \end{bmatrix} \beta(t) = \begin{bmatrix} 1 & \rho^3 \\ 0 & \rho+1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & -\rho^2 \\ 0 & 1 \end{bmatrix} \beta(t).$$

The transfer function matrix of the above PMD is given by:
$G(s) = C(s)A^{-1}(s)B(s) \Rightarrow$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{-s^2}{s+1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{-1}{s+1} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -s+1 \\ 0 & 1 \end{bmatrix} = H_{spr}(s) + H_{pol}(s)$$

$G(s)$ is a rational matrix which has the property $G^{-1}(s) = F(s) \in \Re^{2\times 2}[s]$ where: $F(s) = \begin{bmatrix} s+1 & s^2 \\ 0 & 1 \end{bmatrix}$, with $S^{\infty}_{F(s)}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & \frac{1}{s} \end{bmatrix}$. A finite Jordan pair for $F(s)$ can be found,with $n = 1$ : $C_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $J_f = [-1]$ and an infinite Jordan pair for $F(s)$, $\mu = 3$, $C_\infty = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $J_\infty = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Then we can find $B_f$ and $B_\infty$ as follows (according to equations (5) – (7)).

$$\begin{bmatrix} B_f \\ B_\infty \end{bmatrix} = [I_1, J_\infty] \begin{bmatrix} S_0 \\ V \end{bmatrix}^{-1} [0_2, I_2]^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -- & -- \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_f \\ -- \\ B_\infty \end{bmatrix}.$$

Hence $B_f = \begin{bmatrix} 1 & -1 \end{bmatrix}$ and $B_\infty = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$ and therefore $(C_f, J_f, B_f)$ is a minimal realization of $H_{spr}(s)$ of $G(s)$. The triple $(C_\infty, J_\infty, B_\infty)$ is a realization of $H_{pol}(s)$ but not minimal. Applying the proposed method we have that there is only one block $J_{\infty 2}$ in $J_\infty$ and therefore $m_1 = 0$ and $m_2 = 1$. Hence if we eliminate the last row of $B_\infty$, the last column of $C_\infty$, and the last row and column of $J_\infty$ we shall obtain: $\tilde{C}_\infty = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $\tilde{J}_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\tilde{B}_\infty = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ (with $\tilde{\mu} = 2$) which is a minimal realization of $H_{pol}(s)$ of $G(s)$. Let now define:

$$E := \begin{bmatrix} I_1 & 0 \\ 0 & \tilde{J}_\infty \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A := \begin{bmatrix} J_f & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B := \begin{bmatrix} B_f \\ \tilde{B}_\infty \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad C := \begin{bmatrix} C_f, \ \tilde{C}_\infty \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The quadruple of matrices $[E, A, B, C]$ give rise to the following generalized state-space system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

which is a minimum realization of the matrix $G(s)$ with minimum generalized order $\hat{n} = n + \tilde{\mu} = 1 + 2 = 3$, which represents also the dimension of the generalized state $x(t)$.

## 5. CONCLUSIONS

In the first part of the present paper we investigated the problem of finding the minimal realization of the inverse of a given polynomial matrix by adopting the notions of finite and infinite Jordan pairs. In the sequel we applied the proposed method in order to find the generalized state-space system which has as transfer function matrix the inverse of a polynomial matrix. We remark here that the problem of transforming a linear multivariable system of the form (27) to a generalized state space system of the form (33) is called linearization and has been considered by many researchers. In our paper we study a special case of linearization ; that is linearization for the class of transfer function matrices $G(s)$ of systems (27) with the property of having a polynomial inverse, i. e.:

$$\mathcal{V} = \left\{ G(s) \in \Re^{r \times r}(s) /\ G^{-1}(s) \in \Re^{r\times r}[s] \right\}.$$

The structural properties of the elements of the class $\mathcal{V}$ as defined above, as well as, the forms of the matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ of the system $(\sum)$ in (27) which give rise to a transfer function matrix $G(s) \epsilon\ \mathcal{V}$ is a topic of further research.

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*Dr. George F. Fragulis, Department of Mathematics, Faculty of Science, Aristotle University of Thessaloniki 54006. Greece.*