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MEROMORPHIC FUNCTIONS SHARING TWO SETS

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Abstract. In the paper we discuss the uniqueness problem for meromorphic functions that share two sets and prove five theorems which improve and supplement some results earlier given by Yi and Yang [13], Lahiri and Banerjee [5].

Keywords: weighted sharing, shared set, meromorphic function, uniqueness

MSC 2000: 30D35

1. Introduction, Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a\in S} \{z\colon\ f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a\in S} \{z\colon\ f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$.

If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a; f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. For a set S of distinct elements of $\mathbb{C}$ we define $E_{m)}(S, f) = \bigcup_{a\in S} E_{m)}(a, f)$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a; f) = E_{\infty)}(a; g)$ we say that f, g share the value a CM.

In the paper we denote by S_1 and S_2 the sets $S_1 = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ and $S_2 = \{\infty\}$, where $\omega = \cos 2\pi/n + \mathrm{i} \sin 2\pi/n$ and n is a positive integer.

Yi [9], [11], and Song and Li [7] and other authors have investigated the problem of uniqueness of two meromorphic functions f, g for which $E_f(S_i) = E_g(S_i)$ or $\overline{E}_f(S_i) = \overline{E}_g(S_i)$, where $i = 1, 2$.

In 1997 H. X. Yi and L. Z. Yang proved the following two results.

Theorem A ([13]). *Let f and g be two nonconstant meromorphic functions such that $E_f(S_1) = E_g(S_1)$ and $\overline{E}_f(S_2) = \overline{E}_g(S_2)$. If $n \geqslant 6$ then one of the following conditions holds:*

$$f \equiv tg, \tag{1}$$

where $t^n = 1$,

$$f.g \equiv s, \tag{2}$$

where $s^n = 1$ and 0, ∞ are lacunary values of f and g.

Theorem B ([13]). *Let f and g be two nonconstant meromorphic functions such that $\overline{E}_f(S_1) = \overline{E}_g(S_1)$ and $E_f(S_2) = E_g(S_2)$. If $n \geqslant 10$ then f and g satisfy* (1) *or* (2).

Recently Lahiri and Banerjee [5] have improved Theorem A and Theorem B by relaxing the nature of sharing the sets with the idea of weighted sharing of values and sets introduced in [2], [3]. In the next definition we explain the notion.

Definition 1 ([2], [3]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $(k + 1)$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leqslant p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 2 ([3]). Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$

With the notion of weighted sharing of sets the following two results improving Theorem A and Theorem B are proved in [5].

Theorem C ([5]). *If* $E_f(S_1,2)=E_g(S_1,2)$, $E_f(S_2,0)=E_g(S_2,0)$ *and* $n\geqslant 6$ *then* f, g *satisfy one of* (1) *and* (2).

Theorem D ([5]). *If* $E_f(S_1,0)=E_g(S_1,0)$, $E_f(S_2,3)=E_g(S_2,3)$ *and* $n\geqslant 10$ *then* f, g *satisfy one of* (1) *and* (2).

Now one may ask the following questions which are the motivation of the paper:

(i) What happens in Theorem C if we relax the sharing of the set S_1 to weight one?

(ii) Can the nature of sharing the set S_2 in Theorem D be further relaxed?

(iii) Can in any way the assumption $n\geqslant 10$ in Theorem D be replaced by a weaker one?

In this paper we shall investigate the possible solutions of the above problems. We now state the following five theorems which are the main results of the paper.

Theorem 1. *If* $E_f(S_1,1)=E_g(S_1,1)$, $E_f(S_2,0)=E_g(S_2,0)$ *and* $n\geqslant 7$ *then* f, g *satisfy one of* (1) *and* (2).

Theorem 2. *If* $E_{2)}(S_1,f)=E_{2)}(S_1,g)$, $E_f(S_2,0)=E_g(S_2,0)$ *and* $n\geqslant 8$ *then* f, g *satisfy one of* (1) *and* (2).

Theorem 3. *If* $E_{3)}(S_1,f)=E_{3)}(S_1,g)$, $E_f(S_2,0)=E_g(S_2,0)$ *and* $n\geqslant 6$ *then* f, g *satisfy one of* (1) *and* (2).

Theorem 4. *If* $E_{1)}(S_1,f)=E_{1)}(S_1,g)$, $E_f(S_2,0)=E_g(S_2,0)$ *and* $n\geqslant 10$ *then* f, g *satisfy one of* (1) *and* (2).

Theorem 5. *If* $E_{1)}(S_1,f)=E_{1)}(S_1,g)$, $E_f(S_2,1)=E_g(S_2,1)$ *and* $n\geqslant 9$ *then* f, g *satisfy one of* (1) *and* (2).

Remark 1. Theorem 1, Theorem 4 and Theorem 5 provide the answer to Question (i), (ii) and (iii) respectively.

Though the standard definitions and notation of the value distribution theory are available in [1], we explain some definitions and notations which are used in the paper.

Definition 3 ([4]). For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $N(r,a;f|=1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r,a;f|\leqslant m)(N(r,a;f|\geqslant m))$ the counting function of those a-points of f whose multiplicities are not greater (less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f|\leqslant m)(\overline{N}(r,a;f|\geqslant m))$ are defined similarly, except that in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f|<m)$, $N(r,a;f|>m)$, $\overline{N}(r,a;f|<m)$ and $\overline{N}(r,a;f|>m)$ are defined analogously.

Definition 4 ([2]). We denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f|\geqslant 2)$.

Definition 5 ([13], [14], [16]). Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p=q=1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p=q\geqslant 2$, each point in these counting functions being counted only once. In the same way we can define $\overline{N}_L(r,1;g), N_E^{1)}(r,1;g), \overline{N}_E^{(2}(r,1;g)$.

Definition 6 ([2], [3]). Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

Definition 7 ([5]). Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|\,g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 8 ([5]). Let $a,b_1,b_2,\ldots,b_q\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|\,g\neq b_1,b_2,\ldots,b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not b_i-points of g for $i=1,2,\ldots,q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two nonconstant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by H and V the following two functions:

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big)$$

and

$$V=\Big(\frac{F'}{F-1}-\frac{F'}{F}\Big)-\Big(\frac{G'}{G-1}-\frac{G'}{G}\Big)=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}.$$

Lemma 1 ([13], [14]). *If F, G share $(1,0)$ and $H \not\equiv 0$ then*

$$N_E^{1)}(r,1;F) \leqslant N(r,\infty;H) + S(r,F) + S(r,G).$$

Lemma 2 ([15]). *If F, G are two nonconstant meromorphic functions such that $E_{1)}(1;F) = E_{1)}(1;G)$ and $H \not\equiv 0$ then*

$$N(r,1;F|=1) \leqslant N(r,0;H) \leqslant N(r,\infty;H) + S(r,F) + S(r,G).$$

Lemma 3 ([6]). *If $N(r,0;f^{(k)}|\, f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}|\, f \neq 0) \leqslant k\overline{N}(r,\infty;f) + N(r,0;f|<k) + k\overline{N}(r,0;f|\geqslant k) + S(r,f).$$

Lemma 4 ([5]). *Let F, G share $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} N(r,H) \leqslant{}& \overline{N}(r,0;F|\geqslant 2) + \overline{N}(r,0;G|\geqslant 2) + \overline{N}_*(r,\infty;F,G) \\ &+ \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \end{aligned}$$

as where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Lemma 5. *Let $E_{m)}(1;F) = E_{m)}(1;G)$ and let F, G share $(\infty;k)$ where $m \geqslant 1$ and $0 \leqslant k \leqslant \infty$. Also let $H \not\equiv 0$. Then*

$$\begin{aligned} N(r,\infty;H) \leqslant{}& \overline{N}(r,0;F|\geqslant 2) + \overline{N}(r,0;G|\geqslant 2) + \overline{N}_*(r,\infty;F,G) \\ &+ \overline{N}(r,1;F|\geqslant m+1) + \overline{N}(r,1;G|\geqslant m+1) \\ &+ \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'). \end{aligned}$$

Proof. We can easily verify that possible poles of H occur at (i) multiple zeros of F and G, (ii) those poles of F and G whose multiplicities are different from the multiplicities of the corresponding poles of G and F respectively, (iii) the zeros of $F-1$ and $G-1$ with multiplicities $\geqslant m+1$, (iv) zeros of F' which are not the zeros of $F(F-1)$, (v) zeros of G' which are not the zeros of $G(G-1)$.

Since all poles of H are simple, the lemma follows from the above. □

Lemma 6 ([8]). *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.*

Lemma 7 ([10]). *If $H \equiv 0$ then $T(r, G) = T(r, F) + O(1)$. Also if $H \equiv 0$ and*

$$\limsup_{\substack{r \longrightarrow \infty \\ r \in I}} \frac{\overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G)}{T(r, F)} < 1$$

where $I \subset (0, 1)$ is a set of infinite linear measure, then $F \equiv G$ or $F \cdot G \equiv 1$.

Remark 2. Let $F = f^n$ and $G = g^n$, where n ($\geqslant 5$) is an integer. If $H \equiv 0$ then Lemma 7 implies that f and g satisfy one of (1) and (2).

Lemma 8 ([12], [13]). *If F, G share $(\infty, 0)$ and $V \equiv 0$ then $F \equiv G$.*

Lemma 9. *Let $F = f^n$, $G = g^n$ and $V \not\equiv 0$. If f, g share (∞, k), where $0 \leqslant k < \infty$, and $E_{m)}(1; F) = E_{m)}(1; G)$, then*

$$\begin{aligned}(nk + n - 1)\overline{N}(r, \infty; f| \geqslant k + 1) &= (nk + n - 1)\overline{N}(r, \infty; F| \geqslant nk + n) \\ &\leqslant \frac{m+1}{m}[\overline{N}(r, 0; f) + \overline{N}(r, 0; g)] + \frac{2}{m}\overline{N}(r, \infty; f) + S(r, f) + S(r, g).\end{aligned}$$

Proof. Since f, g share $(\infty; k)$, it follows that F, G share $(\infty; nk)$ and so a pole of F with multiplicity p ($\geqslant nk+1$) is a pole of G with multiplicity r ($\geqslant nk+1$) and vice versa. We note that F and G have no pole of multiplicity q where $nk < q < nk + n$. So using Lemma 3 and Lemma 6 we get from the definition of V

$$\begin{aligned}(nk + n - 1)\overline{N}(r, \infty; f| \geqslant k + 1) &\leqslant N(r, 0; V) \leqslant N(r, \infty; V) + S(r, f) + S(r, g) \\ &\leqslant \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, 1; F| \geqslant m + 1) \\ &\quad + \overline{N}(r, 1; G| \geqslant m + 1) + S(r, f) + S(r, g) \\ &\leqslant \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \frac{1}{m}N(r, 0; F'| F = 1) \\ &\quad + \frac{1}{m}N(r, 0; G'| G = 1) + S(r, f) + S(r, g) \\ &\leqslant \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \frac{1}{m}[N(r, 0; F'| F \neq 0) \\ &\quad - N_0(r, 0; F') + N(r, 0; G'| G \neq 0) - N_0(r, 0; G')] + S(r, f) + S(r, g) \\ &\leqslant \frac{m+1}{m}[\overline{N}(r, 0; f) + \overline{N}(r, 0; g)] + \frac{2}{m}\overline{N}(r, \infty; f) + S(r, f) + S(r, g).\end{aligned}$$

This proves the lemma. □

Lemma 10. *Let F, G share $(1,1)$. Then*

$$\overline{N}_{F>2}(r,1;G) \leqslant \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) - \frac{1}{2}N_0(r,0;F') + S(r,F).$$

P r o o f. Using Lemma 3 we get

$$\begin{aligned}\overline{N}_{F>2}(r,1;G) &\leqslant \overline{N}(r,1;F|\geqslant 3)\\ &\leqslant \frac{1}{2}N(r,0;F'|\,F=1)\\ &\leqslant \frac{1}{2}N(r,0;F'|\,F\neq 0) - \frac{1}{2}N_0(r,0;F')\\ &\leqslant \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) - \frac{1}{2}N_0(r,0;F') + S(r,F).\end{aligned}$$

□

Lemma 11 ([2])**.** *If F, G share $(1,2)$ then*

$$\begin{aligned}&\overline{N}_0(r,0;G') + \overline{N}(r,1;G|\geqslant 2) + \overline{N}_*(r,1;F,G)\\ &\quad\leqslant \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G).\end{aligned}$$

Lemma 12 ([14])**.** *If $H \equiv 0$ and F, G share $(\infty,0)$ then F, G share $(1,\infty)$, (∞,∞).*

Lemma 13. *If F, G share $(1,2)$ and (∞,k), where $0 \leqslant k \leqslant \infty$, then one of the following cases occurs:*

(i) $$\begin{aligned}T(r,F)+T(r,G) \leqslant 2\{&N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)\\ &+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)\}+S(r,F)+S(r,G),\end{aligned}$$

(ii) $F \equiv G$,

(iii) $FG \equiv 1$.

P r o o f. First we suppose that $H \not\equiv 0$. By the second fundamental theorem we obtain

$$T(r,F) \leqslant \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - N_0(r,0;F') + S(r,F). \tag{3}$$

Since F, G share $(1,2)$ we note that

$$\begin{aligned}\overline{N}(r,1;F) &= N(r,1;F|=1) + \overline{N}(r,1;F|\geqslant 2)\\ &= N(r,1;F|=1) + \overline{N}(r,1;G|\geqslant 2).\end{aligned}$$

Hence by Lemmas 1, 4 and 11 we get from (3)

$$
\begin{aligned}
T(r,F) &\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)\\
&\quad+\overline{N}_*(r,\infty;F,G)+\overline{N}(r,1;G|\geqslant 2)+\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
&\leqslant N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
&\quad+\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G).
\end{aligned}
\tag{4}
$$

Similarly we obtain

$$
\begin{aligned}
T(r,G) &\leqslant N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
&\quad+\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G).
\end{aligned}
\tag{5}
$$

Adding (4) and (5) we get (i).

Next we suppose that $H\equiv 0$. Then by integration we get

$$
F\equiv\frac{AG+B}{CG+D},
\tag{6}
$$

where A, B, C, D are constants and $AD-BC\neq 0$. Also

$$
T(r,F)=T(r,G)+O(1).
\tag{7}
$$

We now consider the following cases.

Case 1. Let $AC\neq 0$. Since F, G share (∞,k), it follows from Lemma 12 that F, G share (∞,∞). So from (6) we obtain that F and G have no pole. Again since $F-A/C\equiv(BC-AD)/(C(CG+D))$, it follows that $F-A/C$ has no zero. So by the second fundamental theorem we get

$$
\begin{aligned}
T(r,F) &\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}\Bigl(r,\frac{A}{C};F\Bigr)+S(r,F)\\
&=\overline{N}(r,0;F)+S(r,F),
\end{aligned}
$$

which implies (i) in view of (7).

Case 2. Let $AC=0$. Since F is nonconstant it follows that A and C are not simultaneously zero.

Subcase 2.1. $A\neq 0$ and $C=0$. Then $F=\alpha G+\beta$, where $\alpha=A/D$ and $\beta=B/D$.

If F has no 1-point, by the second fundamental theorem we get

$$
T(r,F)\leqslant\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+S(r,F),
$$

which implies (i) in view of (7).

If F and G have some 1-points then $\alpha+\beta=1$ and so $F\equiv\alpha G+1-\alpha$.

If $\alpha\neq 1$ then by the second fundamental theorem we get

$$\begin{aligned}T(r,F)&\leqslant \overline{N}(r,0;F)+\overline{N}(r,1-\alpha;F)+\overline{N}(r,\infty;F)+S(r,F)\\&=\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,F),\end{aligned}$$

which implies (i) in view of (7).

If $\alpha=1$ then $F\equiv G$, which is (ii).

Subcase 2.2. Let $A=0$ and $C\neq 0$. Then $F=1/(\gamma G+\delta)$, where $\gamma=C/B$ and $\delta=D/B$.

If F has no 1-point then as in Subcase 2.1 we obtain (i).

If F and G have some 1-points then $\gamma+\delta=1$ and so $F\equiv 1/(\gamma G+1-\gamma)$.

If $\gamma\neq 1$ then by the second fundamental theorem we get

$$\begin{aligned}T(r,F)&\leqslant \overline{N}(r,0;F)+\overline{N}\Big(r,\frac{1}{1-\gamma};F\Big)+\overline{N}(r,\infty;F)+S(r,F)\\&=\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,F),\end{aligned}$$

which implies (i) in view of (7).

If $\gamma=1$ then $FG\equiv 1$, which is (iii). This proves the lemma. □

Lemma 14. *If $E_{3)}(1;F)=E_{3)}(1;G)$ and F, G share (∞,k) then the conclusion of Lemma* 13 *holds.*

Proof. If $H\equiv 0$, then by Lemma 12 F, G share $(1,\infty)$, (∞,k). Hence the result follows from Lemma 13.

Next suppose that $H\not\equiv 0$. Then by the second fundamental theorem, Lemma 2 and Lemma 5 we get for $m=3$

$$\begin{aligned}T(r,F)&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-N_0(r,0;F')+S(r,F)\\&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)\\&\quad+\overline{N}_*(r,\infty;F,G)+\overline{N}(r,1;F|\geqslant 4)+\overline{N}(r,1;G|\geqslant 4)\\&\quad+\overline{N}(r,1;F|\geqslant 2)+\overline{N}_0(r,0;G')+S(r,F)+S(r,G).\end{aligned}\tag{8}$$

Again by the second fundamental theorem we get

$$T(r,G)\leqslant \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,1;G)-N_0(r,0;G')+S(r,G).\tag{9}$$

We note that

$$\begin{aligned}(10)\qquad &\overline{N}(r,1;F|\geqslant 2)+\overline{N}(r,1;F|\geqslant 4)+\overline{N}(r,1;G)+\overline{N}(r,1;G|\geqslant 4)\\ &=\frac{1}{2}N(r,1;F|=1)+\frac{1}{2}N(r,1;G|=1)+\overline{N}(r,1;F|\geqslant 2)\\ &\quad+\overline{N}(r,1;G|\geqslant 2)+\overline{N}(r,1;F|\geqslant 4)+\overline{N}(r,1;G|\geqslant 4)\\ &\leqslant\frac{1}{2}N(r,1;F)+\frac{1}{2}N(r,1;G)\\ &\leqslant\frac{1}{2}T(r,F)+\frac{1}{2}T(r,G).\end{aligned}$$

Adding (8) and (9) we get by using (10)

$$\begin{aligned}T(r,F)+T(r,G)\leqslant 2\{&N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\ &+\overline{N}_*(r,\infty;F,G)\}+S(r,F)+S(r,G).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 15 ([16]). *If* F, G *share* $(1,1)$, *then*

$$\begin{aligned}&2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)-\overline{N}_{F>2}(r,1;G)\\ &\qquad\leqslant N(r,1;G)-\overline{N}(r,1;G).\end{aligned}$$

Lemma 16. *Let* F, G *be two nonconstant meromorphic functions such that they share* $(1,1)$, $(\infty,0)$ *and* $H\not\equiv 0$. *Then*

$$\begin{aligned}T(r,F)\leqslant\ &N_2(r,0;F)+N_2(r,0;G)+\frac{1}{2}\overline{N}(r,0;F)+\frac{7}{2}\overline{N}(r,\infty;F)\\ &+S(r,F)+S(r,G).\end{aligned}$$

P r o o f. By the second fundamental theorem we get

$$\begin{aligned}(11)\quad T(r,F)+T(r,G)\leqslant\ &\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\\ &+\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_0(r,0;F')-N_0(r,0;G')\\ &+S(r,F)+S(r,G).\end{aligned}$$

Since F, G share $(1,1)$ and $(\infty;0)$ we note that $N_E^{1)}(r,1;F)=N(r,1;F|=1)$ and

$\overline{N}_*(r,\infty;F,G) \leqslant \overline{N}(r,\infty;F)$. So using Lemmas 1, 4, 10 and 15 we get

$$
\begin{aligned}
\overline{N}(r,1;F) + \overline{N}(r,1;G) \qquad\qquad & \\
\leqslant N(r,1;F|=1) &+ \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) \\
&+ \overline{N}_E^{(2}(r,1;F) + \overline{N}(r,1;G) \\
\leqslant N(r,1;F|=1) &+ N(r,1;G) - \overline{N}_L(r,1;F) \\
&- \overline{N}_L(r,1;G) + \overline{N}_{F>2}(r,1;G) \\
\leqslant \overline{N}(r,0;F|\geqslant 2) &+ \overline{N}(r,0;G|\geqslant 2) + \overline{N}_*(r,\infty;F,G) \\
&+ \overline{N}_*(r,1;F,G) + T(r,G) - m(r,1;G) + O(1) \\
&- \overline{N}_L(r,1;F) - \overline{N}_L(r,1;G) + \frac{1}{2}\overline{N}(r,0;F) \\
&+ \frac{1}{2}\overline{N}(r,\infty;F) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') \\
&+ S(r,F) + S(r,G) \\
\leqslant \overline{N}(r,0;F|\geqslant 2) &+ \overline{N}(r,0;G|\geqslant 2) + \overline{N}(r,\infty;F) \\
&+ T(r,G) + \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) \\
&+ \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G).
\end{aligned}
\tag{12}
$$

From (11) and (12) we obtain

$$
\begin{aligned}
T(r,F) \leqslant N_2(r,0;F) + N_2(r,0;G) + \frac{1}{2}\overline{N}(r,0;F) + \frac{7}{2}\overline{N}(r,\infty;F) \\
+ S(r,F) + S(r,G).
\end{aligned}
$$

This proves the lemma. □

Lemma 17 ([5]). *Let $F = f^n$, $G = g^n$ and $V \not\equiv 0$. If f, g share $(\infty, 0)$ and F, G share $(1,k)$, where $1 \leqslant k \leqslant \infty$, then*

$$
\begin{aligned}
&\Big(n - 1 - \frac{1}{k}\Big)\overline{N}(r,\infty;f) \\
&\qquad \leqslant \frac{k+1}{k}\overline{N}(r,0;f) + \overline{N}(r,0;g) - \frac{1}{k}N(r,0;f'|\,f \neq 0,1,\omega,\dots,\omega^{n-1}) \\
&\qquad\quad + S(r,f) + S(r,g).
\end{aligned}
$$

3. Proofs of the theorems

P r o o f of Theorem 1. Assume that $F = f^n$, $G = g^n$ and f, g do not satisfy (1). Since $E_f(S_1, 1) = E_g(S_1, 1)$ and $E_f(S_2, 0) = E_g(S_2, 0)$, it follows that F, G share $(1,1)$ and $(\infty, 0)$. If possible, we suppose that $H \not\equiv 0$. Then by the second fundamental theorem and Lemma 16 we obtain

$$
\begin{aligned}
(13)\qquad T(r,F) &\leqslant N_2(r,0;F) + N_2(r,0;G) + \frac{1}{2}\overline{N}(r,0;F) + \frac{7}{2}\overline{N}(r,\infty;F) \\
&\quad + S(r,F) + S(r,G) \\
&\leqslant \frac{5}{2}\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + \frac{7}{2}\overline{N}(r,\infty;f) + S(r,f) + S(r,g).
\end{aligned}
$$

Since $F \not\equiv G$ we get by Lemma 8 that $V \not\equiv 0$. So by Lemma 17 for $k = 1$ we get from (13)

$$
\begin{aligned}
(14)\qquad nT(r,f) &\leqslant \frac{5}{2}\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + \frac{7}{2(n-2)}\{2\overline{N}(r,0;f) + \overline{N}(r,0;g)\} \\
&\quad + S(r,f) + S(r,g) \\
&\leqslant \Big\{\frac{5}{2} + \frac{7}{n-2}\Big\}T(r,f) + \Big\{2 + \frac{7}{2(n-2)}\Big\}T(r,g) + S(r,f) + S(r,g).
\end{aligned}
$$

Similarly we obtain

$$
(15)\qquad nT(r,g) \leqslant \Big\{2 + \frac{7}{2(n-2)}\Big\}T(r,f) + \Big\{\frac{5}{2} + \frac{7}{n-2}\Big\}T(r,g) + S(r,f) + S(r,g).
$$

Adding (14) and (15) we get

$$
\Big\{n - \frac{9}{2} - \frac{21}{2(n-2)}\Big\}\{T(r,f) + T(r,g)\} \leqslant S(r,f) + S(r,g),
$$

which is a contradiction for any integer $n \geqslant 7$. Hence $H \equiv 0$ and so the theorem follows from Lemma 7 and Remark 2. $\square$

P r o o f of Theorem 2. Assume that $F = f^n$, $G = g^n$ and f, g do not satisfy (1). Since $E_{2)}(S_1, f) = E_{2)}(S_1, g)$ and $E_f(S_2, 0) = E_g(S_2, 0)$, it follows that $E_{2)}(1, F) = E_{2)}(1, G)$ and F, G share $(\infty, 0)$. If possible, we suppose that $H \not\equiv 0$. Then by the second fundamental theorem and Lemma 5 for $m = 2$ and $k = 0$ we obtain

$$
\begin{aligned}
(16)\qquad T(r,F) &\leqslant \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + N(r,1;F|=1) + \overline{N}(r,1;F|\geqslant 2) \\
&\quad - N_0(r,0;F') + S(r,F) + S(r,G) \\
&\leqslant N_2(r,0;F) + \overline{N}(r,0;G|\geqslant 2) + 2\overline{N}(r,\infty;F) + \overline{N}(r,1;F|\geqslant 3) \\
&\quad + \overline{N}(r,1;G|\geqslant 3) + \overline{N}(r,1;F|\geqslant 2) + \overline{N}_0(r,0;F') \\
&\quad + \overline{N}_0(r,0;G') + S(r,F) + S(r,G).
\end{aligned}
$$

Since $\overline{N}(r,1;F|\geqslant 2)+\overline{N}(r,1;F|\geqslant 3)+\overline{N}_0(r,0;F')\leqslant N(r,0;F'|\,F\neq 0)$ and $\overline{N}(r,1;G|\geqslant 3)+\overline{N}_0(r,0;G')\leqslant N(r,0;G'|\,G\neq 0)$, it follows from Lemma 3, Lemma 6 and from (16)

$$nT(r,f)\leqslant 3\overline{N}(r,0;f)+2\overline{N}(r,0;g)+4\overline{N}(r,\infty;f)+S(r,f)+S(r,g). \tag{17}$$

Since $F\not\equiv G$ we get by Lemma 8 that $V\not\equiv 0$. Now by Lemma 9 for $m=2$ and $k=0$ we get from (17)

$$\begin{aligned}nT(r,f)&\leqslant 3\overline{N}(r,0;f)+2\overline{N}(r,0;g)+\frac{6}{n-2}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\&\quad+S(r,f)+S(r,g)\\&\leqslant\Big\{3+\frac{6}{n-2}\Big\}T(r,f)+\Big\{2+\frac{6}{n-2}\Big\}T(r,g)+S(r,f)+S(r,g).\end{aligned}\tag{18}$$

Similarly we obtain

$$nT(r,g)\leqslant\Big\{2+\frac{6}{n-2}\Big\}T(r,f)+\Big\{3+\frac{6}{n-2}\Big\}T(r,g)+S(r,f)+S(r,g).\tag{19}$$

Adding (18) and (19) we get

$$\Big\{n-5-\frac{12}{n-2}\Big\}\{T(r,f)+T(r,g)\}\leqslant S(r,f)+S(r,g),$$

which is a contradiction for any integer $n\geqslant 8$. Hence $H\equiv 0$ and so the theorem follows from Lemma 7 and Remark 2. □

P r o o f of Theorem 3. Assume that $F=f^n$, $G=g^n$ and f, g do not satisfy (1). Since $E_{3)}(S_1,f)=E_{3)}(S_1,g)$ and $E_f(S_3,0)=E_g(S_3,0)$, it follows that $E_{3)}(1,F)=E_{3)}(1,G)$ and F, G share $(\infty,0)$. Since $F\not\equiv G$, by Lemma 8 we get $V\not\equiv 0$. Now using Lemma 6 and Lemma 9 we get

$$\begin{aligned}nT(r,f)+nT(r,g)&\leqslant 2N_2(r,0;F)+2N_2(r,0;G)+6\overline{N}(r,\infty;F)+S(r,F)+S(r,G)\\&\leqslant 4\overline{N}(r,0;f)+4\overline{N}(r,0;g)+3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)\\&\quad+S(r,f)+S(r,g)\\&\leqslant\Big\{4+\frac{24}{3n-5}\Big\}T(r,f)+\Big\{4+\frac{24}{3n-5}\Big\}T(r,g)\\&\quad+S(r,f)+S(r,g),\end{aligned}$$

i.e.

$$\Big(n-4-\frac{24}{3n-5}\Big)T(r,f)+\Big(n-4-\frac{24}{3n-5}\Big)T(r,g)\leqslant S(r,f)+S(r,g),$$

which is a contradiction for any integer $n\geqslant 6$ and so condition (i) of Lemma 13 does not hold. Hence we must have $FG\equiv 1$. So f, g must satisfy one of (1) and (2). This proves the theorem. □

P r o o f of Theorem 4. Assume that $F=f^n$, $G=g^n$ and f, g do not satisfy (1). Since $E_{1)}(S_1,f)=E_{1)}(S_1,g)$ and $E_f(S_2,0)=E_g(S_2,0)$, it follows that $E_{1)}(1,F)=E_{1)}(1,G)$ and F, G share $(\infty,0)$. If possible, we suppose that $H\not\equiv 0$. Then by Lemma 8 we have $V\not\equiv 0$. So proceeding in the same way as in Theorem 2 we obtain by the second fundamental theorem, Lemma 6 and Lemma 9 for $m=1$ and $k=0$

$$
\begin{aligned}
(20)\quad nT(r,f) &\leqslant 4\overline{N}(r,0;f)+2\overline{N}(r,0;g)+5\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
&\leqslant \Big(4+\frac{10}{n-3}\Big)T(r,f)+\Big(2+\frac{10}{n-3}\Big)T(r,g)+S(r,f)+S(r,g).
\end{aligned}
$$

Similarly we obtain

$$
(21)\qquad nT(r,g)\leqslant\Big\{2+\frac{10}{n-3}\Big\}T(r,f)+\Big(4+\frac{10}{n-3}\Big)T(r,g)+S(r,f)+S(r,g).
$$

Adding (20) and (21) we get

$$
\Big\{n-6-\frac{20}{n-3}\Big\}\{T(r,f)+T(r,g)\}\leqslant S(r,f)+S(r,g),
$$

which is a contradiction for any integer $n\geqslant 10$. Hence $H\equiv 0$ and so the theorem follows from Lemma 7 and Remark 2. $\square$

P r o o f of Theorem 5. Assume that $F=f^n$, $G=g^n$ and f, g do not satisfy (1). Since $E_{1)}(S_1,f)=E_{1)}(S_1,g)$ and $E_f(S_2,1)=E_g(S_2,1)$, it follows that $E_{1)}(1,F)=E_{1)}(1,G)$ and F, G share (∞,n). We note that $\overline{N}(r,\infty;F|\geqslant n+1)=\overline{N}(r,\infty;F|\geqslant 2n)=\overline{N}(r,\infty;f|\geqslant 2)$. If possible, we suppose $H\not\equiv 0$. Then by Lemma 8 we have $V\not\equiv 0$. So by the second fundamental theorem, Lemma 3 and Lemma 5 for $m=1$ and $k=n$ we obtain

$$
\begin{aligned}
(22)\quad T(r,F) &\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;F|\geqslant 2)+\overline{N}(r,0;G|\geqslant 2)\\
&\quad+\overline{N}_*(r,\infty;F,G)+2\overline{N}(r,1;F|\geqslant 2)+\overline{N}(r,1;G|\geqslant 2)\\
&\quad+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
&\leqslant N_2(r,0;F)+\overline{N}(r,0;G|\geqslant 2)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;f|\geqslant 2)\\
&\quad+2N(r,0;F'|\,F\neq 0)+N(r,0;G'|\,G\neq 0)+S(r,F)+S(r,G)\\
&\leqslant N_2(r,0;F)+N_2(r,0;G)+4\overline{N}(r,\infty;F)+2\overline{N}(r,0;F)\\
&\quad+\overline{N}(r,\infty;f|\geqslant 2)+S(r,F)+S(r,G).
\end{aligned}
$$

So using Lemma 6 and Lemma 9 for $m = 1$ and $k = 1$ we obtain from (22)

$$
\begin{aligned}
nT(r,f) &\leqslant 4\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + 4\overline{N}(r,\infty;f) \\
&\quad + \frac{2}{2n-1}[\overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f)] \\
&\quad + S(r,f) + S(r,g) \\
&\leqslant \left[4 + \frac{2}{2n-1}\right] T(r,f) + \left[2 + \frac{2}{2n-1}\right] T(r,g) \\
&\quad + \left\{4 + \frac{2}{2n-1}\right\} \overline{N}(r,\infty;f) + S(r,f) + S(r,g).
\end{aligned}
\tag{23}
$$

Now again using Lemma 9 for $m = 1$ and $k = 0$ we get from (23)

$$
\begin{aligned}
nT(r,f) &\leqslant \left[4 + \frac{2}{2n-1}\right] T(r,f) + \left[2 + \frac{2}{2n-1}\right] T(r,g) \\
&\quad + \frac{2}{n-3}\left\{4 + \frac{2}{2n-1}\right\}[\overline{N}(r,0;f) + \overline{N}(r,0;g)] \\
&\quad + S(r,f) + S(r,g) \\
&\leqslant \left[4 + \frac{18n-10}{(n-3)(2n-1)}\right] T(r,f) + \left[2 + \frac{18n-10}{(n-3)(2n-1)}\right] T(r,g) \\
&\quad + S(r,f) + S(r,g).
\end{aligned}
\tag{24}
$$

Similarly we obtain

$$
\begin{aligned}
nT(r,g) &\leqslant \left[2 + \frac{18n-10}{(n-3)(2n-1)}\right] T(r,f) + \left[4 + \frac{18n-10}{(n-3)(2n-1)}\right] T(r,g) \\
&\quad + S(r,f) + S(r,g).
\end{aligned}
\tag{25}
$$

Adding (24) and (25) we get

$$
\left\{n - 6 - \frac{36n-20}{(2n-1)(n-3)}\right\}\{T(r,f) + T(r,g)\} \leqslant S(r,f) + S(r,g).
$$

This is a contradiction for any integer $n \geqslant 9$. Hence $H \equiv 0$ and so the theorem follows from Lemma 7 and Remark 2. $\square$

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