Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 1, 229–238

Persistent URL: http://dml.cz/dmlcz/128061

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Czechoslovak Mathematical Journal, 56 (131) (2006), 229–238

ON SIGNED DISTANCE-k-DOMINATION IN GRAPHS

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(Received June 3, 2003)

Abstract. The signed distance-k-domination number of a graph is a certain variant of the signed domination number. If v is a vertex of a graph G, the open k-neighborhood of v, denoted by $N_k(v)$, is the set $N_k(v) = \{u\colon\ u \neq v \text{ and } d(u,v) \leqslant k\}$. $N_k[v] = N_k(v) \cup \{v\}$ is the closed k-neighborhood of v. A function $f\colon V \rightarrow \{-1,1\}$ is a signed distance-k-dominating function of G, if for every vertex $v \in V$, $f(N_k[v]) = \sum\limits_{u \in N_k[v]} f(u) \geqslant 1$. The signed distance-$k$-domination number, denoted by $\gamma_{k,s}(G)$, is the minimum weight of a signed distance-k-dominating function on G. The values of $\gamma_{2,s}(G)$ are found for graphs with small diameter, paths, circuits. At the end it is proved that $\gamma_{2,s}(T)$ is not bounded from below in general for any tree T.

Keywords: signed distance-k-domination number, signed distance-k-dominating function, signed domination number

MSC 2000: 05C69

1. Introduction

Let $G = (V,E)$ be a simple graph of order n and minimum degree δ. The open k-neighborhood of a vertex $v \in V$, denoted by $N_k(v)$, is the set $N_k(v) = \{u\colon\ u \neq v$ and $d(u,v) \leqslant k\}$. The closed k-neighborhood of v is the set $N_k(v) \cup \{v\}$. The k-degree of a vertex v is defined as $\deg_k(v) = |N_k(v)|$. The maximum and minimum k-degree of G are denoted by $\Delta_k(G) = \max\{\deg_k(v)\colon\ v \in V\}$, $\delta_k(G) = \min\{\deg_k(v)\colon\ v \in V\}$ respectively. If $\Delta_k(G) = \delta_k(G)$, the graph G is called distance-k-regular.

A function $f\colon V \rightarrow \{-1,1\}$ is a signed distance-k-dominating function of G, if for every vertex $v \in V$, $f(N_k[v]) = \sum\limits_{u \in N_k[v]} f(u) \geqslant 1$. The signed distance-$k$-domination number, denoted by $\gamma_{k,s}(G)$, is the minimum weight of a signed

Research supported by Natural Science Foundation of Hebei education department (2005107).

distance-k-dominating function on G. Specially, signed distance-1-dominating function and signed distance-1-domination number are called signed dominating function and signed domination number respectively. Signed domination number is denoted by $\gamma_s(G)$. It's straightforward to obtain the following result.

Theorem 1. *For any graph G,*

$$\gamma_{1,s}(G) = \gamma_s(G).$$

Theorem 2. *For any complete graph K_n $(n \geqslant 2)$,*

$$\gamma_{k,s}(K_n) = \gamma_s(K_n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Theorem 3. *Let $k \geqslant 2$. If G is a graph of order n and with diameter 2, then*

$$\gamma_{k,s}(G) = \gamma_{2,s}(G) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Corollary 1. *Let $k \geqslant 2$. For complete multipartite graph $G \cong K(m_1, m_2, \ldots, m_n)$ $(n \geqslant 2)$,*

$$\gamma_{k,s}(G) = \begin{cases} 1 & \text{if } \sum\limits_{i=1}^{n} m_i \text{ is odd,} \\ 2 & \text{if} \sum\limits_{i=1}^{n} m_i \text{ is even.} \end{cases}$$

2. Some results on signed distance-k-domination number

Theorem 4. *Let f be a signed distance-k-dominating function of G, then f is minimal if only if for each vertex v with $f(v) = 1$, there exists a vertex $u \in N_k[v]$ such that $f(N_k[v]) \in \{1, 2\}$.*

Proof. Assume that f is a minimal distance-k-dominating function of G. Suppose that there exists a vertex v with $f(v) = 1$ such that for any vertex $u \in N_k[v]$, $f(N_k[u]) \geqslant 3$, then let g be a function defined by $g(v) = -1$ and $g(u) = f(u)$ for any $u \neq v$. It is obvious that g is a signed distance-k-dominating function of G, with $g < f$, which contradicts the fact that f is a minimal distance-k-dominating function of G.

Conversely, assume that for each vertex v with $f(v) = 1$ there exists a vertex $u \in N_k[v]$ such that $f(N_k[u]) \in \{1, 2\}$. Suppose that f is not minimal. Then there exists a signed distance-k-dominating function g of G such that $g < f$. Therefore, there exists a vertex $v \in V$ such that $g(v) < f(v)$ and $g(w) \leqslant f(w)$ for any vertex w $(w \neq v)$. So $g(v) = -1$ and $f(v) = 1$. Hence there exists a vertex $u \in N_k[v]$ such that $f(N_k[u]) \in \{1, 2\}$. Therefore $g(N_k[u]) \leqslant f(N_k[u]) - 2 \leqslant 0$, which contradicts the fact that g is a signed distance-k-dominating function of G. □

Theorem 5. *Let k be a positive integer. For any path P_n of order n,*

$$\gamma_{2,s}(P_n) = \begin{cases} k & \text{if } n = 5k, \\ k+1 & \text{if } n = 5k+1, \\ k+2 & \text{if } n = 5k+2 \text{ or } n = 5k+4, \\ k+3 & \text{if } n = 5k+3. \end{cases}$$

Proof. Assume that $P_n = v_1 v_2 \ldots v_n$. Let f be a minimum signed distance-2-dominating function of P_n such that $f(V(P_n)) = \gamma_{2,s}(P_n)$. Since $f(N_2[v_j]) \geqslant 1$ for every vertex v_j $(1 \leqslant j \leqslant n)$ in P_n, there are at most two vertices assigned -1 under f in every five consecutive vertices on the path P_n. We consider the following five cases.

Case 1: $n = 5k$. Since there are at most two vertices assigned -1 under f in every five consecutive vertices on P_n, we have

$$\gamma_{2,s}(P_n) = f(V(P_n)) = \sum_{j=1}^{n} f(v_j) = \sum_{i=0}^{k-1} \sum_{m=1}^{5} f(v_{5i+m}) \geqslant \sum_{i=0}^{k-1} 1 = k.$$

On the other hand, for $0 \leqslant i \leqslant k-1$, we define $g\colon V \to \{-1, 1\}$ by

$$g(v_j) = \begin{cases} -1 & \text{if } j = 5i+1,\ 5i+5, \\ 1 & \text{if } j = 5i+2,\ 5i+3,\ 5i+4. \end{cases}$$

Then g is a signed distance-2-dominating function of P_n with weight k. Therefore, we have $\gamma_{2,s}(P_n) \leqslant k$. Hence, $\gamma_{2,s}(P_n) = k$.

Case 2: $n = 5k+1$. If $f(v_1) = 1$, then $\gamma_{2,s}(P_n) = f(V(P_n)) = \sum_{j=1}^{n} f(v_j) = \sum_{i=0}^{k-1} \sum_{m=2}^{6} f(v_{5i+m}) + f(v_1) \geqslant k+1$.

If $f(v_1) = -1$, then we divide the path $v_2 v_3 \ldots v_{5k+1}$ into k segment paths $v_{5i+2} \ldots v_{5i+6}$ $(i = 0, 1, \ldots, k-1)$. We claim that there is at least one segment path $v_{5l+2} \ldots v_{5l+6}$ $(0 \leqslant l \leqslant k-1)$ such that there is at most one vertex assigned -1

under f in the path $v_{5l+2} \dots v_{5l+6}$. Suppose to the contrary that in every segment path $v_{5i+2} \dots v_{5i+6}$ $(0 \leqslant i \leqslant k-1)$ there are two vertices assigned -1 under f. We have $f(v_{5i+2}) = f(v_{5i+3}) = f(v_{5i+4}) = 1$ and $f(v_{5i+5}) = f(v_{5i+6}) = -1$. But we have $f(N_2[v_{5k+1}]) \leqslant 0$, which is a contradiction. Therefore, $\gamma_{2,s}(P_n) = f(V(P_n)) = \sum_{j=1}^{n} f(v_j) = f(v_1) + \sum_{i=0}^{k-1} \sum_{m=2}^{6} f(v_{5i+m}) \geqslant -1 + (k-1) + \sum_{m=2}^{6} f(v_{5l+m}) \geqslant (k-2) + 3 = k+1$.

On the other hand, for $0 \leqslant i \leqslant k-1$, we define $g\colon V \to \{-1, 1\}$ by

$$g(v_j) = \begin{cases} 1 & \text{if } j = 1, \\ -1 & \text{if } j = 5i+2,\ 5i+6, \\ 1 & \text{if } j = 5i+3,\ 5i+4,\ 5i+5. \end{cases}$$

Then g is a signed distance-2-dominating function of P_n with weight $k+1$. Therefore $\gamma_{2,s}(P_n) \leqslant k+1$. Hence $\gamma_{2,s}(P_n) = k+1$.

Case 3: $n = 5k+2$. If $f(v_1) = f(v_2) = 1$, then $\gamma_{2,s}(P_n) = f(V(P_n)) = \sum_{j=1}^{n} f(v_j) = (f(v_1) + f(v_2)) + \sum_{i=0}^{k-1} \sum_{m=3}^{7} f(v_{5i+m}) \geqslant k+2$.

If either $f(v_1) = -1$ or $f(v_2) = -1$, then $f(v_1) + f(v_2) \geqslant 0$. We claim that there is at least one segment path $v_{5l+3} \dots v_{5l+7}$ $(0 \leqslant l \leqslant k-1)$ such that there is at most one vertex assigned -1 under f in the path $v_{5l+3} \dots v_{5l+7}$. Suppose to the contrary that in every segment path $v_{5i+3} \dots v_{5i+7}$ $(0 \leqslant i \leqslant k-1)$ there are two vertices assigned -1 under f. Since $f(N_2[v_2]) \geqslant 1$, we have $f(v_3) = f(v_4) = 1$. Then there are two vertices assigned -1 under f in $\{v_5, v_6, v_7\}$. Since $f(N_2[v_7]) \geqslant 1$, we have $f(v_8) = f(v_9) = 1$. Then there are two vertices assigned -1 under f in $\{v_9, v_{10}, v_{11}\}, \dots$, by a similar reason, we have that there are two vertices assigned -1 under f in $\{v_{5k}, v_{5k+1}, v_{5k+2}\}$. But $f(N_2[v_{5k+2}]) = f(v_{5k}) + f(v_{5k+1}) + f(v_{5k+2}) \leqslant -1$. This is a contradiction. Therefore, $\gamma_{2,s}(P_n) = f(V(P_n)) = \sum_{j=1}^{n} f(v_j) = (f(v_1) + f(v_2)) + \sum_{i=0}^{k-1} \sum_{m=3}^{7} f(v_{5i+m}) \geqslant 0 + (k-1) + \sum_{m=3}^{7} f(v_{5l+m}) \geqslant k+2$.

On the other hand, for $0 \leqslant i \leqslant k-1$, we define a function g by

$$g(v_j) = \begin{cases} 1 & \text{if } j = 1,\ 2, \\ -1 & \text{if } j = 5i+3,\ 5i+7, \\ 1 & \text{if } j = 5i+4,\ 5i+5,\ 5i+6. \end{cases}$$

Then g is a signed distance-2-dominating function of P_n with weight $k+2$. Therefore $\gamma_{2,s}(P_n) \leqslant k+2$. Hence $\gamma_{2,s}(P_n) = k+2$.

Case 4: $n = 5k+3$. Since $f(N_2[v_1]) \geqslant 1$, there is at most one vertex assigned -1 under f in $\{v_1, v_2, v_3\}$.

If $f(v_1) = f(v_2) = f(v_3) = 1$, then $\gamma_{2,s}(P_n) = f(V(P_n)) = \sum\limits_{j=1}^{n} f(v_j) = (f(v_1) + f(v_2) + f(v_3)) + \sum\limits_{i=0}^{k-1} \sum\limits_{m=4}^{8} f(v_{5i+m}) \geqslant k+3$.

If there is exactly one vertex assigned -1 under f in $\{v_1, v_2, v_3\}$, then we claim that there is at least one segment path $v_{5l+4} \ldots v_{5l+8}$ $(0 \leqslant l \leqslant k-1)$ such that there is at most one vertex in the path $v_{5l+4} \ldots v_{5l+8}$ assigned -1 under f. Suppose to the contrary that in every segment path $v_{5i+4} \ldots v_{5i+8}$ $(0 \leqslant i \leqslant k-1)$ there are two vertices assigned -1 under f. Since $f(N_2[v_2]) \geqslant 1$, we have $f(v_4) = 1$. Then there are two vertices assigned -1 under f in $\{v_5, v_6, v_7, v_8\}$. Since $f(N_2[v_7]) \geqslant 1$, we have $f(v_9) = 1$. Then there are two vertices assigned -1 under f in $\{v_{10}, v_{11}, v_{12}, v_{13}\}, \ldots$, by similar reason, we have that there are two vertices assigned -1 under f in $\{v_{5k}, v_{5k+1}, v_{5k+2}, v_{5k+3}\}$. But $f(N_2[v_{5k+2}]) = \sum\limits_{m=0}^{3} f(v_{5k+m}) \leqslant 0$. This is a contradiction. Therefore, $\gamma_{2,s}(P_n) = f(V(P_n)) = \sum\limits_{j=1}^{n} f(v_j) = (f(v_1) + f(v_2) + f(v_3)) + \sum\limits_{i=0}^{k-1} \sum\limits_{m=4}^{8} f(v_{5i+m}) \geqslant 1 + (k-1) + \sum\limits_{m=4}^{8} f(v_{5l+m}) \geqslant k+3$.

On the other hand, for $0 \leqslant i \leqslant k-1$, we define a function g by

$$g(v_j) = \begin{cases} 1 & \text{if } j = 1,\ 2,\ 3, \\ -1 & \text{if } j = 5i+4,\ 5i+5, \\ 1 & \text{if } j = 5i+6,\ 5i+7,\ 5i+8. \end{cases}$$

Then g is a signed distance-2-dominating function of P_n with weight $k+3$. Therefore $\gamma_{2,s}(P_n) \leqslant k+3$. Hence $\gamma_{2,s}(P_n) = k+3$.

Case 5: $n = 5k+4$. Since $f(N_2[v_2]) \geqslant 1$, there is at most one vertex assigned -1 under f in $\{v_1, v_2, v_3, v_4\}$. Furthermore, there are at most two vertices assigned -1 under f in every segment path $v_{5i+5} \ldots v_{5i+9}$ $(0 \leqslant i \leqslant k-1)$. Therefore, we have $\gamma_{2,s}(P_n) = f(V(P_n)) = \sum\limits_{j=1}^{n} f(v_j) = \sum\limits_{j=1}^{4} f(v_j) + \sum\limits_{i=0}^{k-1} \sum\limits_{m=5}^{9} f(v_{5i+m}) \geqslant k+2$.

On the other hand, for $0 \leqslant i \leqslant k-1$, we define a function g by

$$g(v_j) = \begin{cases} -1 & \text{if } j = 1, \\ 1 & \text{if } j = 2,\ 3,\ 4, \\ -1 & \text{if } j = 5i+5,\ 5i+6, \\ 1 & \text{if } j = 5i+7,\ 5i+8,\ 5i+9. \end{cases}$$

Then g is a signed distance-2-dominating function of P_n with weight $k+2$. Therefore, $\gamma_{2,s}(P_n) \leqslant k+2$. Hence $\gamma_{2,s}(P_n) = k+2$. $\square$

Theorem 6. *Let k be a positive integer. For any circuit C_n of order n,*

$$\gamma_{2,s}(C_n) = \begin{cases} k & n = 5k, \\ k+1 & n = 5k+1 \text{ or } n = 5k+3, \\ k+2 & n = 5k+2 \text{ or } n = 5k+4. \end{cases}$$

Proof. Assume that $C_n = v_1 v_2 \dots v_n v_1$. Let f be a minimum signed distance-2-dominating function of C_n such that $f(V(C_n)) = \gamma_{2,s}(C_n)$.

Case 1: $n = 5k$. Since there are at most two vertices assigned -1 under f in every five consecutive vertices on C_n, we have $\gamma_{2,s}(C_n) = f(V(C_n)) = \sum\limits_{j=1}^{n} f(v_j) = \sum\limits_{i=0}^{k-1} \sum\limits_{m=1}^{5} f(v_{5i+m}) \geqslant k$.

On the other hand, for $0 \leqslant i \leqslant k-1$, we define a function g by

$$g(v_j) = \begin{cases} -1 & \text{if } j = 5i+1,\ 5i+5, \\ 1 & \text{if } j = 5i+2,\ 5i+3,\ 5i+4. \end{cases}$$

Then g is a signed distance-2-dominating function of C_n with weight k. Therefore, $\gamma_{2,s}(C_n) \leqslant k$. It follows that $\gamma_{2,s}(C_n) = k$.

Case 2: $n = 5k+1$. Without loss of generality, we assume that $f(v_1) = 1$. Since there are at most two vertices assigned -1 under f in every five consecutive vertices on C_n, we have $\gamma_{2,s}(C_n) = f(V(C_n)) = \sum\limits_{j=1}^{n} f(v_j) = f(v_1) + \sum\limits_{i=0}^{k-1} \sum\limits_{m=2}^{6} f(v_{5i+m}) \geqslant k+1$.

On the other hand, for $0 \leqslant i \leqslant k-1$, we define a function g by

$$g(v_j) = \begin{cases} 1 & \text{if } j = 1, \\ -1 & \text{if } j = 5i+2,\ 5i+6, \\ 1 & \text{if } j = 5i+3,\ 5i+4,\ 5i+5. \end{cases}$$

Then g is a signed distance-2-dominating function of C_n with weight $k+1$. Therefore, $\gamma_{2,s}(C_n) \leqslant k+1$. It follows that $\gamma_{2,s}(C_n) = k+1$.

Case 3: $n = 5k+2$. It is easy to see that there exists two consecutive vertices of C_n, say v_1, v_2, such that $f(v_1) = f(v_2) = 1$. Since there are at most two vertices assigned -1 under f in every five consecutive vertices on C_n, we have $\gamma_{2,s}(C_n) = f(V(C_n)) = \sum\limits_{j=1}^{n} f(v_j) = (f(v_1) + f(v_2)) + \sum\limits_{i=0}^{k-1} \sum\limits_{m=3}^{7} f(v_{5i+m}) \geqslant k+2$.

On the other hand, for $0 \leqslant i \leqslant k-1$, we define a function g by

$$g(v_j) = \begin{cases} 1 & \text{if } j = 1,\ 2, \\ -1 & \text{if } j = 5i+3,\ 5i+7, \\ 1 & \text{if } j = 5i+4,\ 5i+5,\ 5i+6. \end{cases}$$

Then g is a signed distance-2-dominating function of C_n with weight $k+2$. Therefore, $\gamma_{2,s}(C_n) \leqslant k+2$. It follows that $\gamma_{2,s}(C_n) = k+2$.

Case 4: $n = 5k+3$. If there does not exist any vertex assigned -1 under f, then $f(V) \geqslant k+1$. Otherwise, we can assume that $f(v_1) = -1$. Since $f(N_2[v_1]) \geqslant 1$, either $f(v_2) = f(v_3) = 1$ or $f(v_{n-1}) = f(v_n) = 1$. Without loss of generality, we assume that $f(v_2) = f(v_3) = 1$. Since there are at most two vertices assigned -1 under f in every five consecutive vertices on C_n, we have $\gamma_{2,s}(C_n) = f(V(C_n)) = \sum\limits_{j=1}^{n} f(v_j) = (f(v_1) + f(v_2) + f(v_3)) + \sum\limits_{i=0}^{k-1} \sum\limits_{m=4}^{8} f(v_{5i+m}) \geqslant k+1$.

On the other hand, for $0 \leqslant i \leqslant k-1$, we define a function g by

$$g(v_j) = \begin{cases} -1 & \text{if } j = 1, \\ 1 & \text{if } j = 2,\, 3, \\ -1 & \text{if } j = 5i+4,\, 5i+6, \\ 1 & \text{if } j = 5i+5,\, 5i+7,\, 5i+8. \end{cases}$$

Then g is a signed distance-2-dominating function of C_n with weight $k+1$. Therefore, $\gamma_{2,s}(C_n) \leqslant k+1$. It follows that $\gamma_{2,s}(C_n) = k+1$.

Case 5: $n = 5k+4$. It is easy to see that there are at most two vertices assigned -1 under f in $\{v_1, v_2, v_3, v_4\}$.

If there is at most one vertex assigned -1 under f in $\{v_1, v_2, v_3, v_4\}$, then $\gamma_{2,s}(C_{5k+4}) = f(V(C_{5k+4})) = \sum\limits_{j=1}^{n} f(v_j) = \sum\limits_{j=1}^{4} f(v_j) + \sum\limits_{i=0}^{k-1} \sum\limits_{m=5}^{9} f(v_{5i+m}) \geqslant k+2$.

If there are exactly two vertices assigned -1 under f in $\{v_1, v_2, v_3, v_4\}$, then we have the following claim.

Claim 1. *If there are exactly two vertices assigned* -1 *under* f *in* $\{v_1, v_2, v_3, v_4\}$*, then there exists a segment path* $v_{5l+5} \ldots v_{5l+9}$ $(0 \leqslant l \leqslant k-1)$ *on* C_{5k+4} *in which there is at most one vertex assigned* -1 *under* f*.*

Proof. Suppose to the contrary that in every segment path $v_{5i+5} \ldots v_{5i+9}$ $(0 \leqslant i \leqslant k-1)$ on C_{5k+4} there are two vertices assigned -1 under f. Then we have $f(V(C_{5k+4})) \leqslant k$. There are six cases to be considered.

Case a: $f(v_1) = f(v_2) = -1$ and $f(v_3) = f(v_4) = 1$. In this case, we have $f(v_{5k+4}) = f(v_{5k+3}) = f(v_{5k+2}) = 1$. Since there are at most two vertices assigned -1 under f in every five consecutive vertices on the C_{5k+4}, we have $f(V(C_{5k+4})) = \sum\limits_{j=1}^{n} f(v_j) = (f(v_{5k+2}) + f(v_{5k+3}) + f(v_{5k+4}) + f(v_1)) + \sum\limits_{i=0}^{k-1} \sum\limits_{m=2}^{6} f(v_{5i+m}) \geqslant k+2$, which contradicts the fact that $f(V(C_{5k+4})) \leqslant k$.

Case b: $f(v_1) = f(v_3) = -1$ and $f(v_2) = f(v_4) = 1$. In this case, we have $f(v_{5k+4}) = f(v_{5k+3}) = 1$. Then we have $f(V(C_{5k+4})) = \sum_{j=1}^{n} f(v_j) = (f(v_1)+f(v_2)+f(v_{5k+3}) + f(v_{5k+4})) + \sum_{i=0}^{k-1}\sum_{m=3}^{7} f(v_{5i+m}) \geqslant k+2$, which contradicts the fact that $f(V(C_{5k+4})) \leqslant k$.

Case c: $f(v_1) = f(v_4) = -1$ and $f(v_2) = f(v_3) = 1$. In this case, we have $f(v_5) = 1$. Then we have $f(V(C_{5k+4})) = \sum_{j=1}^{n} f(v_j) \geqslant k + \sum_{j=2}^{5} f(v_j) = k+2$, which contradicts the fact that $f(V(C_{5k+4})) \leqslant k$.

Case d: $f(v_1) = f(v_2) = 1$ and $f(v_3) = f(v_4) = -1$. In this case, we have $f(v_5) = f(v_6) = 1$ and argue as in Case a.

Case e: $f(v_1) = f(v_4) = 1$ and $f(v_2) = f(v_3) = -1$. In this case, we have $f(v_5) = 1$ and argue as in Case a.

Case f: $f(v_1) = f(v_3) = 1$ and $f(v_2) = f(v_4) = -1$. In this case, we have $f(v_5) = 1$ and argue as in Case b. □

By Claim 1, we have that $\gamma_{2,s}(C_{5k+4}) = f(V(C_{5k+4})) = \sum_{j=1}^{n} f(v_j) = \sum_{j=1}^{4} f(v_j) + \sum_{i=0}^{k-1}\sum_{m=5}^{9} f(v_{5i+m}) \geqslant 0 + (k-1) + \sum_{m=5}^{9} f(v_{5l+m}) \geqslant k+2$.

On the other hand, for $0 \leqslant i \leqslant k-1$, we define $g\colon V \to \{-1, 1\}$ by

$$g(v_j) = \begin{cases} -1 & \text{if } j = 1, \\ 1 & \text{if } j = 2,\ 3,\ 4, \\ -1 & \text{if } j = 5i+5,\ 5i+9, \\ 1 & \text{if } j = 5i+6,\ 5i+7,\ 5i+8. \end{cases}$$

Then g is a signed distance-2-dominating function of C_{5k+4} with weight $k+2$. Therefore, we have $\gamma_{2,s}(V(C_{5k+4})) \leqslant k+2$. It follows that $\gamma_{2,s}(C_{5k+4}) = k+2$. □

Theorem 7. *If G is distance-k-regular and $\Delta_k(G) = \delta_k(G) = r$, then*

$$\gamma_{k,s}(G) \geqslant \frac{n}{r+1},$$

and this bound is sharp.

Proof. Let f be a minimum signed distance-k-dominating function of G such that $f(V(G)) = \gamma_{k,s}(G)$. Let $N = \sum_{v\in V} f(N_k[v])$. Then

$$N = \sum_{v\in V} f(N_k[v]) = (r+1)\sum_{v\in V} f(v) = (r+1)f(V) \tag{1}$$

$$N = \sum_{v\in V} f(N_k[v]) \geqslant \sum_{v\in V} 1 = |V| = n. \tag{2}$$

Combining (1) and (2), we have

$$\gamma_{k,s}(G) \geqslant \frac{n}{r+1}.$$

Now we show that the bound is sharp. Let n be odd and $n \geqslant 3$. We consider the complete graph K_n. We have $r = \Delta_k(G) = \delta_k(G) = n-1$. By Theorem 2, $\gamma_{k,s}(K_n) = 1 = n/(r+1)$. This completes the proof. $\square$

Let $T_{s,t,k}$ be a graph obtained from a path P_k by adding s paths of length 1 and t paths of length 2 to each vertex u of P_k and identifying one end vertex of each of them with u. $T_{3,4,4}$ is illustrated in Fig. 1. Then we have the following theorem.

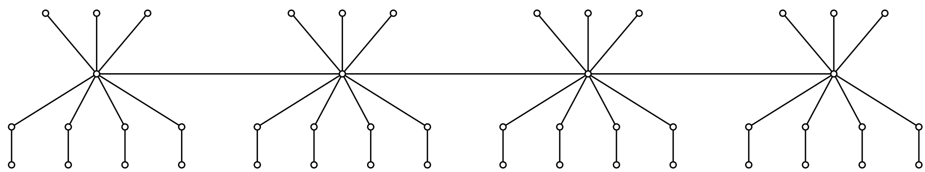

Fig. 1. $T_{3,4,4}$

Theorem 8. *For any positive integer k, there exists a tree T such that*

$$\gamma_{2,s}(T) \leqslant -k.$$

Proof. For $T_{s,t,k}$ where $t \geqslant 2s-2$, $s \geqslant 2$ and $k \geqslant 5$, let c_i $(1 \leqslant i \leqslant k)$ denote the vertex on path P_k of $T_{s,t,k}$. Denote by M_i the set of vertices adjacent to c_i whose degree is one. Denote by T_i the set of vertices adjacent to c_i whose degree is two and L_i the set of vertices adjacent to a vertex in T_i and different from c_i.

We define a function on $T_{s,t,k}$ as follows:

$$f(u) = \begin{cases} -1 & u \in M_i \cup L_i, \\ 1 & u \in T_i \cup \{c_i\}. \end{cases}$$

In the following, we prove that f is a signed distance-2-dominating function on $T_{s,t,k}$. We consider the following three cases.

If $i = 1$ or $i = k$, then for any vertex $u \in M_i$, we have $f(N_2[u]) = t+2-s \geqslant 1$. If $u = c_i$, then $f(N_2[u]) = t+3-2s \geqslant 1$. For any vertex $u \in T_i$, we have $f(N_2[u]) = t+1-s \geqslant 1$. For any vertex $u \in L_i$, we have $f(N_2[u]) = 1$.

If $i = 2$ or $i = k-1$, then for any vertex $u \in M_i$, we have $f(N_2[u]) = t+3-s \geqslant 1$. If $u = c_i$, we have $f(N_2[u]) = 2t+4-3s \geqslant 1$. For any vertex $u \in T_i$, we have $f(N_2[u]) = t+2-s \geqslant 1$. For any vertex $u \in L_i$, we have $f(N_2[u]) = 1$.

If $3 \leqslant i \leqslant k-2$, then for any vertex $u \in M_i$, we have $f(N_2[u]) = t+3-s \geqslant 1$. If $u = c_i$, we have $f(N_2[u]) = 2t+5-3s \geqslant 1$. For any vertex $u \in T_i$, we have $f(N_2[u]) = t+2-s \geqslant 1$. For any vertex $u \in L_i$, we have $f(N_2[u]) = 1$.

Therefore, f is a signed distance-2-dominating function of $T_{s,t,k}$. Let $T = T_{s,t,k}$. Then $\gamma_{2,s}(T) \leqslant w(f) = -(s-1)k \leqslant -k$.

For $1 \leqslant k \leqslant 5$, let $T = T_{s,t,5}$. Then $\gamma_{2,s}(T) \leqslant -5 \leqslant -k$. For $k \geqslant 6$, let $T = T_{s,t,k}$. Then $\gamma_{2,s}(T) \leqslant -k$. This completes the proof. $\square$

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