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ON FINITENESS CONDITIONS FOR REES MATRIX SEMIGROUPS

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Abstract. Let $T = \mathscr{M}[S; I, J; P]$ be a Rees matrix semigroup where S is a semigroup, I and J are index sets, and P is a $J \times I$ matrix with entries from S, and let U be the ideal generated by all the entries of P. If U has finite index in S, then we prove that T is periodic (locally finite) if and only if S is periodic (locally finite). Moreover, residual finiteness and having solvable word problem are investigated.

Keywords: Rees matrix semigroup, periodicity, local finiteness, residual finiteness, word problem

MSC 2000: 20M05, 20M10

1. Introduction

After Rees matrix semigroups were introduced by Rees ([6]), they became very important family of semigroups, especially in the study of the structure theory of completely (0)-simple semigroups (see for example [3]). Although Rees matrix semigroups are defined over groups, we define them over semigroups (as in [1], [4], [5]).

Let S be a semigroup, let I and J be two index sets, and let $P = (p_{ji})_{j\in J,\, i\in I}$ be a $J \times I$ matrix with entries from S. The set

$$I \times S \times J = \{(i, s, j) \mid i \in I,\ s \in S,\ j \in J\}$$

with multiplication defined by

$$(i, s, j)(k, t, l) = (i, sp_{jk}t, l)$$

is a semigroup. This semigroup is called a *Rees matrix semigroup*, and denoted by $\mathscr{M}[S; I, J; P]$.

Finiteness conditions for semigroups (the properties of semigroups which all finite semigroups have) have been considered for certain classes of semigroup constructions (for examples, see [1], [2], [8], [7]). In this paper periodicity, local finiteness, residual finiteness of Rees matrix semigroups and solvable word problem for Rees matrix semigroups are investigated.

2. Periodicity

Recall that a semigroup S is *periodic* if, for each $s \in S$, the monogenic semigroup generated by s is finite, or equivalently there exist two distinct positive integers m, n (depending on s) such that $s^m = s^n$.

Lemma 2.1. *If S is periodic, then $\mathscr{M}[S; I, J; P]$ is periodic.*

Proof. For an arbitrary element $(i, s, j) \in \mathscr{M}[S; I, J; P]$, consider $sp_{ji} \in S$ such that there exist two positive integers $m \neq n$ such that $(sp_{ji})^m = (sp_{ji})^n$. It follows that

$$(i, s, j)^{m+1} = (i, (sp_{ji})^m s, j) = (i, (sp_{ji})^n s, j) = (i, s, j)^{n+1}.$$

Thus T is periodic as well. □

The ideal U of S generated by the set $\{p_{ji} \mid j \in J,\ i \in I\}$ of all entries of P plays a very important role in this paper as in [1].

Theorem 2.2. *The Rees matrix semigroup $\mathscr{M}[S; I, J; P]$ is periodic if and only if the ideal U of S generated by all the entries of the matrix $P = (p_{ji})_{j\in J,\, i\in I}$ is periodic.*

Proof. ($\Rightarrow$) It is clear that an arbitrary element of U can be written as $sp_{ji}t$ where $s, t \in S^1$. Consider the element $(i, tsp_{ji}ts, j)$ of $\mathscr{M}[S; I, J; P]$ such that there exist two integers $m \neq n$ such that

$$(i, tsp_{ji}ts, j)^m = (i, tsp_{ji}ts, j)^n,$$
$$(i, (tsp_{ji})^{2m-1}ts, j) = (i, (tsp_{ji})^{2n-1}ts, j).$$

It follows that $(tsp_{ji})^{2m-1}ts = (tsp_{ji})^{2n-1}ts$ or $(tsp_{ji})^{2m} = (tsp_{ji})^{2n}$, and so

$$(sp_{ji}t)^{2m+1} = sp_{ji}(tsp_{ji})^{2m}t = sp_{ji}(tsp_{ji})^{2n}t = (sp_{ji}t)^{2n+1}.$$

Thus U is periodic as well.

($\Leftarrow$) Let $(i,s,j) \in T = \mathscr{M}[S;I,J;P]$. Since U is the ideal of S generated by all the entries of the matrix $P = (p_{ji})_{j\in J,\, i\in I}$, $sp_{ji} \in U$, there exist two positive integers $p \neq q$ such that $(sp_{ji})^p = (sp_{ji})^q$. It follows that

$$(i,s,j)^{p+1} = (i,(sp_{ji})^p s,j) = (i,(sp_{ji})^q s,j) = (i,s,j)^{q+1},$$

and so T is periodic. □

Note that if the ideal U has *finite index* in S, that is $S \setminus U$ is finite, it follows from [7, Theorem 5.1] that S is periodic if and only if U is periodic. Thus we have the following corollary.

Corollary 2.3. *Let $T = \mathscr{M}[S;I,J;P]$, and let the ideal U of S generated by all the entries of the matrix $P = (p_{ji})_{j\in J,\, i\in I}$ have finite index in S. Then T is periodic if and only if S is periodic.*

3. Local finiteness

Let X be a subset of a semigroup S, then the smallest subsemigroup of S containing X is called the *subsemigroup of S generated by X*, and denoted by $\langle X\rangle$. If each finitely generated subsemigroup of a semigroup S is finite, then S is said to be *locally finite*.

First we give a technical lemma.

Lemma 3.1. *Let S be a semigroup without an identity. Then $T = \mathscr{M}[S;I,J;P]$ is locally finite if and only if $T' = \mathscr{M}[S^1;I,J;P]$ is locally finite.*

Proof. ($\Rightarrow$) Let X be a non-empty finite subset of T'. Take $Y = X \cap T$, $Z = X \setminus Y$ and $W = Y \cup YZ \cup ZY \cup ZZ$ where $YZ = \{yz \mid y \in Y,\ z \in Z\}$, etc. Then it is clear that W is a finite subset of T, and so $\langle W\rangle$ is finite. Since $\langle X\rangle = \langle W\rangle \cup Z$, T' is locally finite as well.

($\Leftarrow$) Since every subsemigroup of a locally finite semigroup is locally finite, and since T is a subsemigroup of T', the proof is complete. □

Theorem 3.2. *The Rees matrix semigroup $\mathscr{M}[S;I,J;P]$ is locally finite if and only if the ideal U of S generated by all the entries of the matrix $P = (p_{ji})_{j\in J,\, i\in I}$ is locally finite.*

Proof. ($\Rightarrow$) Let X be a finite subset of U. Since each element of U has the form $sp_{ji}t$ for some $s,t \in S^1$ and entries p_{ji} of P, we may take

$$X = \{s_k p_{j_k i_k} t_k \mid 1 \leqslant k \leqslant m\}.$$

Then define sets

$$I' = \{i_k \in I \mid 1 \leqslant k \leqslant m\},$$
$$X' = \{s_k, t_k, t_k s_k \in S^1 \mid 1 \leqslant k \leqslant m\},$$
$$J' = \{j_k \in J \mid 1 \leqslant k \leqslant m\}.$$

Since $I' \times X' \times J'$ is a finite subset of $\mathscr{M}[S^1; I, J; P]$, it follows from the above lemma that $\langle I' \times X' \times J' \rangle$ is finite. Since $I' \times \langle X \rangle \times J' \subseteq \langle I' \times X' \times J' \rangle$, the subsemigroup $\langle X \rangle$ is finite, as required.

($\Leftarrow$) Let Y be a finite subset of $\mathscr{M}[S; I, J; P]$. Define

$$I'' = \{i \in I \mid (i, s, j) \in Y\},$$
$$J'' = \{j \in J \mid (i, s, j) \in Y\},$$
$$Y'' = \{s \in S \mid (i, s, j) \in Y\},$$

and then define $X'' = \{sp_{ji}, sp_{ji}t \mid i \in I'';\ s, t \in Y'';\ j \in J''\}$. Since X'' is a finite subset of U, $\langle X'' \rangle$ is a finite subsemigroup of U.

Observe that an arbitrary element $(i, s, j) \in \langle Y \rangle \setminus Y$ can be written as a product

$$(i, s, j) = (i_1, s_1, j_1) \dots (i_k, s_k, j_k) = (i_1, s_1 p_{j_1 i_2} s_2 \dots p_{j_{k-1} i_k} s_k, j_k),$$

where $(i_1, s_1, j_1), \dots, (i_k, s_k, j_k) \in Y$ with $k \geqslant 2$. Thus $(i, s, j) \in I'' \times \langle X'' \rangle \times J''$, and so $\langle Y \rangle$ is a subset of the finite set $(I'' \times \langle X'' \rangle \times J'') \cup Y$, as required. □

If $S \setminus U$ is finite then, from the previous theorem and [7, Theorem 5.1], we have the following corollary.

Corollary 3.3. *Let $T = \mathscr{M}[S; I, J; P]$, and let the ideal U of S generated by all the entries of the matrix $P = (p_{ji})_{j \in J, i \in I}$ have finite index in S. Then T is locally finite if and only if S is locally finite.*

4. Residual finiteness

We call a semigroup S residually finite if, for each pair $s \neq t \in S$, there exists a homomorphism Φ from S onto a finite semigroup such that $\Phi(s) \neq \Phi(t)$, or equivalently, there exists a congruence ϱ with finite index (that is, ϱ has finitely many equivalence classes) such that $(s, t) \notin \varrho$. (Residual finiteness of completely (0)-simple semigroups, which are Rees matrix semigroups $\mathscr{M}[G; I, J; P]$ over groups, was investigated in [2].)

Let K be a subset of I. If, for each $i \in I$, there exist $s_i \in S^1$ and $k_i \in K$ such that

$$(1) \qquad p_{ji} = p_{jk_i} s_i \quad \text{for all } j \in J,$$

then we call K a (*left*) *co-index* of I. Let L be a subset of J. If, for each $j \in J$, there exist $t_j \in S^1$ and $l_j \in L$ such that

$$(2) \qquad p_{ji} = t_j p_{l_j i} \quad \text{for all } i \in I,$$

then we call L a (*right*) *co-index* of J. Given left and right co-indices K and L respectively, we fix all s_i, k_i $(i \in I)$ and t_j, l_j $(j \in J)$ and moreover, we take $s_i = 1$ if $i \in K$ and $t_j = 1$ if $j \in L$. If, for all fixed s_i and t_j, $s_i s t_j = s_i t t_j$ implies $s = t$, then we call K and L *normal co-indices.* Notice that if S is a group then all co-indices are normal. Notice also that if both I and J have finite normal co-indices, then there are finitely many rows and columns of P such that each row (column) of P is a right (left) multiple of one of these finitely many rows (columns).

Theorem 4.1. *If S is residually finite, and if both I and J have finite normal co-indices, then the Rees matrix semigroup $T = \mathscr{M}[S; I, J; P]$ is residually finite.*

Proof. Let (i_1, s_1, j_1) and (i_2, s_2, j_2) be arbitrary different elements of T. If $i_1 \neq i_2$, then we consider the left zero semigroup $L_2 = \{a_1, a_2\}$ $(ab = a)$ of order 2 and the mapping $\varphi\colon T \longrightarrow L_2$, defined by

$$\varphi(i, s, j) = \begin{cases} a_1 & \text{if } i = i_1, \\ a_2 & \text{if } i \neq i_1. \end{cases}$$

It is clear that φ is an onto homomorphism such that $\varphi(i_1, s_1, j_1) \neq \varphi(i_2, s_2, j_2)$. If $j_1 \neq j_2$, then this is shown similarly. If $i_1 = i_2$ and $j_1 = j_2$ then $s_1 \neq s_2$. Let K and L be finite normal co-indices. Then $s_{i_1} s_1 t_{j_1} \neq s_{i_1} s_2 t_{j_1}$. Moreover, since S is residually finite, there exist a finite semigroup S' and an onto homomorphism Φ from S onto S' such that $\Phi(s_{i_1} s_1 t_{j_1}) \neq \Phi(s_{i_1} s_2 t_{j_1})$.

Now define a submatrix $P' = (p_{kl})_{k \in K, l \in L}$ of P where p_{kl} is the corresponding entry of P and consider the finite Rees matrix semigroup $T' = \mathscr{M}[S'; K, L; P'']$ where $P'' = (\Phi(p_{kl}))_{k \in K,\, l \in L}$, and the map $\psi\colon T \longrightarrow T'$ defined by

$$\psi\colon (i, s, j) \mapsto (k_i, \Phi(s_i s t_j), l_j)$$

where k_i, s_i, t_j and l_j are defined as in (1) and (2). Since k_i and l_j are unique, and since s_i and t_j are fixed, the map ψ is well-defined, and clearly onto. For

$(i_1, s_1, j_1), (i_2, s_2, j_2) \in T$, it follows from (2) and (1) that

$$\begin{aligned}
\psi(i_1, s_1, j_1)\psi(i_2, s_2, j_2) &= (k_{i_1}, \Phi(s_{i_1} s_1 t_{j_1}), l_{j_1})(k_{i_2}, \Phi(s_{i_2} s_2 t_{j_2}), l_{j_2}) \\
&= (k_{i_1}, \Phi(s_{i_1} s_1 t_{j_1})\Phi(p_{l_{j_1} k_{i_2}})\Phi(s_{i_2} s_2 t_{j_2}), l_{j_2}) \\
&= (k_{i_1}, \Phi(s_{i_1} s_1 (t_{j_1} p_{l_{j_1} k_{i_2}}) s_{i_2} s_2 t_{j_2}), l_{j_2}) \\
&= (k_{i_1}, \Phi(s_{i_1} s_1 (p_{j_1 k_{i_2}} s_{i_2}) s_2 t_{j_2}), l_{j_2}) \\
&= (k_{i_1}, \Phi(s_{i_1} s_1 p_{j_1 i_2} s_2 t_{j_2}), l_{j_2}) \\
&= \psi(i_1, s_1 p_{j_1 i_2} s_2, j_2) = \psi((i_1, s_1, j_1)(i_2, s_2, j_2))
\end{aligned}$$

so that ψ is a homomorphism. Moreover, it is clear that $\psi(i_1, s_1, j_1) \neq \psi(i_2, s_2, j_2)$, as required. □

Notice that if S is residually finite, and if both I and J are finite, then the Rees matrix semigroup $T = \mathscr{M}[S; I, J; P]$ is residually finite. Now consider the cyclic group $C_2 = \{1, a\}$ of order 2, the matrix $P_1 = (p_{ji})_{j \in \mathbb{N},\, i \in \mathbb{N}}$ where $\mathbb{N}$ is the set of natural numbers and

$$p_{ji} = \begin{cases} 1 & \text{if } j \leqslant i, \\ a & \text{if } j > i, \end{cases}$$

and the Rees matrix semigroup $T_1 = \mathscr{M}[C_2; \mathbb{N}, \mathbb{N}; P_1]$. Clearly C_2 is residually finite but we will show that T_1 is not residually finite.

For $(1, 1, 1)$ and $(1, a, 1)$ in T_1, assume that there exists a congruence ϱ on T_1 with finite index such that $((1, 1, 1), (1, a, 1)) \notin \varrho$. Let $(i, a, j) \in T_1$ be arbitrary, and let $j < l$. Then, since ϱ has finite index, we may assume either $((i, a, j), (k, a, l)) \in \varrho$ or $((i, a, j), (k, 1, l)) \in \varrho$ for some $k, l \in \mathbb{N}$.

If $((i, a, j), (k, a, l)) \in \varrho$, then we have

$$\begin{aligned}
(1, 1, 1)(i, a, j)(j, 1, 1) &= (1, a p_{jj}, 1) = (1, a, 1), \\
(1, 1, 1)(k, a, l)(j, 1, 1) &= (1, a p_{lj}, 1) = (1, 1, 1),
\end{aligned}$$

and so $((1, 1, 1), (1, a, 1)) \in \varrho$, which is a contradiction.

If $((i, a, j), (k, 1, l)) \in \varrho$, then we have

$$\begin{aligned}
(1, 1, 1)(i, a, j)(l, 1, 1) &= (1, a p_{jl}, 1) = (1, a, 1), \\
(1, 1, 1)(k, 1, l)(l, 1, 1) &= (1, p_{ll}, 1) = (1, 1, 1),
\end{aligned}$$

and so $((1, 1, 1), (1, a, 1)) \in \varrho$, which is again a contradiction. Thus T_1 cannot be a residually finite semigroup.

This example shows that the residual finiteness of S is not sufficient for the residual finiteness of $\mathscr{M}[S; I, J; P]$. Moreover, consider the Rees matrix semigroup

$T_2 = \mathscr{M}[S; I, J; P_2]$ where S is a non-residually finite semigroup with a zero 0, and the matrix $P_2 = (p_{ji})_{j\in J,\, i\in I}$ with $p_{ji} = 0$. (Note that since adding a zero into a non-residually finite semigroup gives a non-residually finite semigroup with a zero, examples of non-residually finite semigroups with a zero exist.) It is easy to show that T_2 is residually finite. This last example shows that the converse of the above theorem is not true in general.

5. Word problem

A semigroup S is said to have a *solvable word problem with respect to a generating set* A if there exists an algorithm which, for any two words $u, v \in A^+$, decides whether the relation $u = v$ holds in S or not. It is a well-known fact that, for a finitely generated semigroup S, the solvability of the word problem does not depend on the choice of the finite generating set for S. Thus we say that a finitely generated semigroup S has a *solvable word problem* if S has a solvable word problem with respect to any finite generating set.

Since finite generation is important in this section, we recall the main result of [1]:

Theorem 5.1. *Let S be a semigroup, let I and J be index sets, let $P = (p_{ji})_{j\in J,\, i\in I}$ be a $J \times I$ matrix with entries from S, and let U be the ideal of S generated by the set $\{p_{ji} \mid j \in J,\ i \in I\}$ of all entries of P. Then the Rees matrix semigroup $\mathscr{M}[S; I, J; P]$ is finitely generated (finitely presented) if and only if the following three conditions are satisfied:*

(i) *both I and J are finite;*
(ii) *S is finitely generated (respectively, finitely presented); and*
(iii) *the set $S \setminus U$ is finite.*

In this section we assume $T = \mathscr{M}[S; I, J; P]$ is finitely generated, and so the sets I, J and $S \setminus U$ are finite and S is finitely generated.

Let $T = \mathscr{M}[S; I, J; P]$ have a solvable word problem. Since I and J are finite, $T' = \mathscr{M}[S^1; I, J; P]$ is a small extension of T, that is $T' \setminus T = I \times \{1\} \times J$ is finite, T' has a solvable word problem (see [7, Theorem 5.1 (i)]. Let Z be a finite generating set for the ideal U. First note that each $z \in Z$ has the form $s_z p_{j_z i_z} t_z$ where $s_z, t_z \in S^1$, then consider the set

$$X = I \times \{1, s, s_z, t_z, s_z t_z, t_z s_z \mid s \in S \setminus U,\ z \in Z\} \times J$$

which is a finite generating set for T' (see [1]).

Let $u \equiv (s_{z_1} p_{j_{z_1} i_{z_1}} t_{z_1}) \ldots (s_{z_m} p_{j_{z_m} i_{z_m}} t_{z_m})$ and $v \equiv (s_{z'_1} p_{j_{z'_1} i_{z'_1}} t_{z'_1}) \ldots (s_{z'_n} p_{j_{z'_n} i_{z'_n}} t_{z'_n})$ be arbitrary words in Z^+. Then, for any $i \in I$ and $j \in J$, consider the elements

$$(i, u, j) = (i, s_{z_1}, j_{z_1})(i_{z_1}, t_{z_1} s_{z_2}, j_{z_2}) \ldots (i_{z_m}, t_{z_m}, j),$$

and

$$(i, v, j) = (i, s_{z'_1}, j_{z'_1})(i_{z'_1}, t_{z'_1} s_{z'_2}, j_{z'_2}) \ldots (i_{z'_n}, t_{z'_n}, j)$$

in T'. Since T' has a solvable word problem, the relation $(i, u, j) = (i, v, j)$ is decidable, and so $u = v$ is decidable. Therefore we have

Proposition 5.2. *If $\mathscr{M}[S; I, J; P]$ has a solvable word problem, then the ideal U of S generated by the entries of P has a solvable word problem.*

Let the semigroup S have a solvable word problem. Let X be a finite generating set for $T = \mathscr{M}[S; I, J; P]$. Then the set

$$Y = \{s \in S \mid (i, s, j) \in X \text{ for some } i \in I,\ j \in J\} \cup \{p_{ji} \mid i \in I,\ j \in J\}$$

is a finite generating set for S (see [1]).

Let $u \equiv (i_1, s_1, j_1) \ldots (i_m, s_m, j_m)$, $v \equiv (k_1, t_1, l_1) \ldots (k_n, t_n, l_n)$ be arbitrary elements in X. Since the relation $u = v$ is decidable in T if and only if $i_1 = k_1$, $j_m = l_n$ and the relation $s_1 p_{j_1 i_2} s_2 \ldots s_{m-1} p_{j_{m-1} i_m} s_m = t_1 p_{l_1 k_2} t_2 \ldots t_{n-1} p_{l_{n-1} k_n} t_n$ is decidable in S, and since S has a solvable word problem, $u = v$ can be decidable in T. Therefore we have

Proposition 5.3. *Let $T = \mathscr{M}[S; I, J; P]$ be a finitely generated Rees matrix semigroup over a semigroup S. If S has a solvable word problem, T has a solvable word problem as well.*

Finally, we have the following theorem:

Theorem 5.4. *Let $T = \mathscr{M}[S; I, J; P]$ be a finitely generated Rees matrix semigroup over a semigroup S. Then T has a solvable word problem if and only if S has a solvable word problem.*

Proof. Since $S \setminus U$ is finite, it follows from [7, Theorem 5.1 (i)] that U has a solvable word problem if and only if S has a solvable word problem. Thus the result follows from Propositions 5.2 and 5.3. □

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