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Ján Ohriska

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Ján Ohriska, Košice

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Abstract. In this paper two sequences of oscillation criteria for the self-adjoint second order differential equation $(r(t)u'(t))' + p(t)u(t) = 0$ are derived. One of them deals with the case $\int^{\infty} \mathrm{d}t/r(t) = \infty$, and the other with the case $\int^{\infty} \mathrm{d}t/r(t) < \infty$.

Keywords: oscillation theory

MSC 2000: 34C10

1. Introduction

We consider the second order self-adjoint differential equation

$$(r(t)u'(t))' + p(t)u(t) = 0, \quad t \geqslant t_0, \tag{1}$$

where

(i) $r \in C[t_0, \infty), \quad r(t) > 0$ for $t \geqslant t_0$,

(ii) $p \in C[t_0, \infty)$.

We call a function u a solution of the equation (1) for $t \geqslant t_0$ if $u(t) \in C^1[t_0, \infty)$, $r(t)u'(t) \in C^1[t_0, \infty)$ and it satisfies the equation (1) for $t \geqslant t_0$.

In the sequel we shall restrict our attention to non-trivial solutions of the equations considered. Such a solution is called *oscillatory* if it has arbitrarily large zeros and *non-oscillatory* otherwise. An equation is said to be *oscillatory* if one, and thereby each of its solutions is oscillatory, otherwise it is said to be *non-oscillatory.*

We say that the equation (1) is in the *canonical* form if $\int^{\infty} \mathrm{d}t/r(t) = \infty$. In this case we denote by $R(t) = \int^{t} \mathrm{d}s/r(s)$ a primitive function of the function $1/r(t)$.

We say that the equation (1) is in the *non-canonical* form if $\int^{\infty} \mathrm{d}t/r(t) < \infty$ and in this case we put $\varrho(t) = \int_t^{\infty} \mathrm{d}s/r(s)$.

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Among many authors who discussed the oscillatory nature of equation (1) we make a mention of D. Willett. In his excellent survey [8], to the date of its publication, we find the following sentence: "Although there are many results concerning the classification of equations of the form (1) with respect to these properties (you understand 'with respect to oscillation') no completely satisfactory answer has yet been obtained." It is well known that second order differential equations are most important in applications. Numerous phenomena in physical, biological, and engineering sciences can be described by a second order differential equation. Moreover, as we can see e.g. in [1, Theorems 2.19 and 2.20] and in [3, Theorems 2 and 4], results on second order equations play very important role in the study of higher order differential equations.

First we consider a special case of the equation (1), namely the equation

$$u''(t) + p(t)u(t) = 0. \tag{2}$$

Regarding the earlier results on the oscillatory and non-oscillatory character of solutions of the equation (2) one can find e.g. in [4] or in [7] the following criteria due to A. Kneser and E. Hille.

Theorem A. *The equation* (2) *is non-oscillatory if*

$$\limsup_{t\to\infty} t^2 p(t) < \frac{1}{4}$$

and it is oscillatory if

$$\liminf_{t\to\infty} t^2 p(t) > \frac{1}{4}.$$

Theorem B. *The equation* (2) *is non-oscillatory if*

$$\limsup_{t\to\infty} (t\ln t)^2 \Big(p(t) - \frac{1}{4t^2}\Big) < \frac{1}{4}$$

and it is oscillatory if

$$\liminf_{t\to\infty} (t\ln t)^2 \Big(p(t) - \frac{1}{4t^2}\Big) > \frac{1}{4}.$$

It is clear that Theorem A is noneffective e.g. if $\lim\limits_{t\to\infty} t^2 p(t) = 1/4$. In such a case Theorem B can be useful but we see that Theorem B is noneffective e.g. if $\lim\limits_{t\to\infty} (t\ln t)^2 (p(t) - 1/(4t^2)) = 1/4$.

In spite of this problem with one quarter, it is natural to ask whether there is some analogue of Theorem A or of Theorem B for the equation (1) in the canonical or in the non-canonical form. We note that such analogue of Theorem A is known provided the equation (1) is in the canonical form, and we recall it here.

Theorem C (Theorem 2.3 in [6]). *Let* (i) *and* (ii) *be satisfied. Let the equation* (1) *be in the canonical form. Then the equation* (1) *is non-oscillatory if*

$$\limsup_{t\to\infty} R^2(t)r(t)p(t) < \frac{1}{4}$$

and it is oscillatory if

$$\liminf_{t\to\infty} R^2(t)r(t)p(t) > \frac{1}{4}.$$

Other questions will be positively answered in such a way that we will derive two sequences of assertions on oscillation and non-oscillation of equation (1) in both, the canonical and non-canonical forms, which will contain the above mentioned analogues.

2. Preliminaries

In the sequel we will use such notions like a v-derivative of a function, a v-transformation of a differential equation, and certain generalized Euler equations. Thus we introduce here these notions and give some necessary information about them.

Definition 1 (Definition 1.1 in [5]). Let functions f and v be defined on a neighborhood $O(t)$ of a point $t \in \mathbb{R}$ and let the conditions $x \in O(t)$, $x \neq t$ imply $v(x) \neq v(t)$. If the limit

$$\lim_{x\to t} \frac{f(x) - f(t)}{v(x) - v(t)}$$

is finite, then it is called the *v-derivative* of the function f at the point t and is denoted by $f'_v(t)$ or $\mathrm{d}f(t)/\mathrm{d}v$.

Theorem D (Theorem 1.2 in [5]). *Let there exist $v'(t) \neq 0$ on an interval $I \subset \mathbb{R}$. Then for $t \in I$ the v-derivative $f'_v(t)$ exists if and only if the derivative $f'(t)$ exists. Moreover,*

$$f'_v(t) = \frac{f'(t)}{v'(t)}.$$

Definition 2 (Definition 1.2 in [6]). Let functions f and v be as in Definition 1. Let the function f'_v be defined on some neighborhood $O(t)$ of a point $t \in \mathbb{R}$. If the limit

$$\lim_{x\to t} \frac{f'_v(x) - f'_v(t)}{v(x) - v(t)}$$

is finite, then it is called the *second v-derivative* of the function f at the point t and denoted by

$$f''_{v^2}(t) \quad \text{or} \quad \frac{\mathrm{d}^2 f(t)}{\mathrm{d}v^2}.$$

For the purposes of this paper we simplify the notion of a v-transformation of a differential equation, presented in [5] and done also in [6]. Thus suppose that the following conditions are satisfied:

a) I and I_1 are intervals in $\mathbb{R}$,
b) $v \in C(I_1)$, v is a strictly monotone function, $v\colon\ I_1 \to I$,
c) φ is the inverse function to v,
d) $p\colon\ I \to \mathbb{R}$.

Consider the differential equation (2) for $t \in I$. If the independent variable t is replaced by the function $v(s)$ in the coefficient p of equation (2) and $u''(t)$ is replaced by $y''_{v^2}(s)$ in the sense that $v(s)$ replaces even the independent variable as the argument of the function with respect to which the derivatives of the unknown function are calculated ($y'(s) = y'_w(s)$, where $w(s) \equiv s$), then equation (2) is transformed into the equation

$$y''_{v^2}(s) + p(v(s))y(s) = 0, \quad s \in I_1. \tag{3}$$

The above mentioned process of obtaining equation (3) from equation (2) is called the *v-transformation* of equation (2).

It is useful to note that a φ-transformation of (3) leads again to (2).

Now we can introduce the following result which is a special case of Theorem 2.1 proved in [5].

Theorem E. *Let the conditions* a)–d) *be satisfied. A function $u(t)$ is a solution of equation* (2) *on I if and only if the function $y(s) = u(v(s))$ is a solution of equation* (3) *on I_1.*

Now we present the so called generalized Euler equations and introduce their solutions.

Lemma 1. *Let the Euler equation*

$$(r(t)y'(t))' + \frac{c}{r(t)R^2(t)}y(t) = 0, \quad t > t_0,\ c \in \mathbb{R} \tag{4}$$

be in the canonical form. The linearly independent solutions of (4) *are the functions*

$$y_1(t) = [R(t)]^{\frac{1+\sqrt{1-4c}}{2}}, \quad y_2(t) = [R(t)]^{\frac{1-\sqrt{1-4c}}{2}} \quad \text{if } c < \frac{1}{4},$$
$$y_1(t) = \sqrt{R(t)}, \qquad y_2(t) = \sqrt{R(t)}\ln R(t) \quad \text{if } c = \frac{1}{4},$$

and

$$y_1(t) = \sqrt{R(t)}\cos\Big(\frac{\sqrt{4c-1}}{2}\ln R(t)\Big),$$
$$y_2(t) = \sqrt{R(t)}\sin\Big(\frac{\sqrt{4c-1}}{2}\ln R(t)\Big) \quad \textit{if } c > \frac{1}{4}.$$

P r o o f. One can verify the validity of Lemma 1 directly but we show what is the way to obtain this result. Using the notion of the v-derivative of a function, the definition of the function $R(t)$ and Theorem D we can write equation (4) in the form

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}R^2} + \frac{c}{R^2(t)} y(t) = 0. \tag{4_1}$$

Denote by Φ the inverse function to R. Then the v-transformation of equation (4_1) with $v(s) = \Phi(s)$ yields the Euler equation $u''(s) + \frac{c}{s^2}u(s) = 0$, $s > 0$. Now by solving this equation and using Theorem E we have the result of Lemma 1. □

Similarly to the above the following assertion can be proved.

Lemma 2. *Let the Euler equation*

$$(r(t)y'(t))' + \frac{c}{r(t)\varrho^2(t)} y(t) = 0, \quad t > t_0, \ c \in \mathbb{R} \tag{5}$$

be in the non-canonical form. The linearly independent solutions of (5) *are the functions*

$$y_1(t) = [\varrho(t)]^{\frac{1+\sqrt{1-4c}}{2}}, \quad y_2(t) = [\varrho(t)]^{\frac{1-\sqrt{1-4c}}{2}} \quad \textit{if } c < \frac{1}{4},$$
$$y_1(t) = \sqrt{\varrho(t)}, \qquad y_2(t) = \sqrt{\varrho(t)}\ln\varrho(t) \quad \textit{if } c = \frac{1}{4},$$

and

$$y_1(t) = \sqrt{\varrho(t)}\cos\Big(\frac{\sqrt{4c-1}}{2}\ln\varrho(t)\Big),$$
$$y_2(t) = \sqrt{\varrho(t)}\sin\Big(\frac{\sqrt{4c-1}}{2}\ln\varrho(t)\Big) \quad \textit{if } c > \frac{1}{4}.$$

3. The canonical case

Now we study the equation (1) under the assumption $\int^\infty \mathrm{d}t/r(t) = \infty$ and use the following notation. For $t \geqslant t_0$ we put

$$r_0(t) = r(t), \quad p_0(t) = p(t) \tag{6}$$

and for $i = 0, 1, 2, \ldots$

$$R_i(t) = \int^t \frac{\mathrm{d}s}{r_i(s)}, \quad r_{i+1}(t) = R_i(t) r_i(t), \quad p_{i+1}(t) = R_i(t) p_i(t) - \frac{1}{4R_i(t) r_i(t)}. \tag{7}$$

Lemma 3. *Let $i \in \{0, 1, 2, \ldots\}$. If the equation*

$$(r_i(t) u'(t))' + p_i(t) u(t) = 0, \quad t > t_0 \tag{E_i}$$

is in the canonical form then also the equation

$$(r_{i+1}(t) u'(t))' + p_{i+1}(t) u(t) = 0, \quad t > t_0 \tag{E_{i+1}}$$

is such and moreover, a function $y(t)$ is a solution of $(\mathrm{E_i})$ *if and only if the function $w(t) = R_i^{-1/2}(t) y(t)$ is a solution of* $(\mathrm{E_{i+1}})$.

Proof. The canonical form of (E_i) means that $R_i(t) \to \infty$ as $t \to \infty$ and $i \in \{0, 1, 2, \ldots\}$. But then $R_{i+1}(t) = \int^t (R_i(s) r_i(s))^{-1} \mathrm{d}s = \ln R_i(t) \to \infty$ as $t \to \infty$. Moreover, one can verify directly that the relation between the solutions of (E_i) and (E_{i+1}) is as introduced in Lemma 3 but again, similarly to the proof of Lemma 1 we show the way how to obtain this result.

Using the notion of the v-derivative of a function we can write the equation (E_i) in the form

$$\frac{\mathrm{d}^2 u(t)}{\mathrm{d}R_i^2} + r_i(t) p_i(t) u(t) = 0$$

and by Theorem E the v-transformation of this equation with $v(s) = \Phi_i(s)$, where Φ_i is the inverse function to R_i, yields the equation

$$y''(s) + r_i(\Phi_i(s)) p_i(\Phi_i(s)) y(s) = 0, \quad s > 0 \tag{8}$$

and we know that $u(t) = y(R_i(t))$, or $y(s) = u(\Phi_i(s))$. Now the change of both the independent and the dependent variable of the form

$$s = \mathrm{e}^x, \quad y(s) = s^{1/2} z(x)$$

transforms the equation (8) to the equation

$$z''(x) + \left[e^{2x} r_i(\Phi_i(e^x)) p_i(\Phi_i(e^x)) - \frac{1}{4}\right] z(x) = 0, \quad x \in \mathbb{R}. \tag{9}$$

Besides, we see that $z(x) = s^{-1/2} u(\Phi_i(s))$, where $s = e^x$.

The v-transformation of (9) with $v(\xi) = \ln R_i(\xi)$ gives the equation

$$\frac{d^2 w(\xi)}{d(\ln R_i)^2} + \left[R_i^2(\xi) r_i(\xi) p_i(\xi) - \frac{1}{4}\right] w(\xi) = 0, \quad \xi > t_0$$

or what is the same, the equation

$$(R_i(\xi) r_i(\xi) w'(\xi))' + \left[R_i(\xi) p_i(\xi) - \frac{1}{4R_i(\xi) r_i(\xi)}\right] w(\xi) = 0, \quad \xi > t_0$$

which is equation (E_{i+1}), and also $w(\xi) = z(\ln R_i(\xi)) = R_i^{-1/2}(\xi) u(\Phi_i(R_i(\xi))) = R_i^{-1/2}(\xi) u(\xi)$, which completes the proof. □

Now we can state the following result.

Theorem 1. *Let* (i) *and* (ii) *be satisfied. Let the equation* (1) *be in the canonical form. Then equation* (1) *is nonoscillatory if for some* $i \in \{0, 1, 2, \ldots\}$

$$\limsup_{t\to\infty} R_i^2(t) r_i(t) p_i(t) < \frac{1}{4}, \tag{10}$$

and equation (1) *is oscillatory if for some* $i \in \{0, 1, 2, \ldots\}$

$$\liminf_{t\to\infty} R_i^2(t) r_i(t) p_i(t) > \frac{1}{4}. \tag{11}$$

Proof. With regard to Lemma 3 we know that the (non-)oscillation of the equation (E_i) for some $i \in \{0, 1, 2, \ldots\}$ implies the same property of the equation (E_i) for every $i \in \{0, 1, 2, \ldots\}$. In order to prove the (non-)oscillation of (E_i) for some $i \in \{0, 1, 2, \ldots\}$ we take into acount the generalized Euler equation

$$(r_i(t) y'(t))' + \frac{c}{r_i(t) R_i^2(t)} y(t) = 0, \quad t > t_0. \tag{12}$$

If we now apply the Sturm comparison theorem to equations (E_i) and (12) we observe that equation (E_i) is non-oscillatory if the condition (10) is satisfied and that (E_i) is oscillatory if the condition (11) is satisfied. □

It is clear that using the notation (6) and (7), the equation (E_0) is the equation (1), the equation (E_1) has the form

$$(R(t)r(t)u'(t))' + \Big(R(t)p(t) - \frac{1}{4R(t)r(t)}\Big)u(t) = 0,$$

the equation (E_2) has the form

$$(R(t)r(t)(\ln R(t))u'(t))' + \Big(R(t)p(t)\ln R(t) - \frac{\ln^2 R(t) + 1}{4R(t)r(t)\ln R(t)}\Big)u(t) = 0,$$

and so on.

Now we specify Theorem 1 for $i = 0, 1, 2$.

In the case $i = 0$ Theorem 1 yields the following corollary.

Corollary 1. *Let* (i) *and* (ii) *be satisfied. The equation* (1) *is non-oscillatory if*

$$\limsup_{t\to\infty} R^2(t)r(t)p(t) < \frac{1}{4}$$

and it is oscillatory if

$$\liminf_{t\to\infty} R^2(t)r(t)p(t) > \frac{1}{4}.$$

We see that Corollary 1 gives the same result as Theorem C.

In the case $i = 1$ Theorem 1 has the following form.

Corollary 2. *Let* (i) *and* (ii) *be satisfied. The equation* (1) *is non-oscillatory if*

$$\limsup_{t\to\infty}\Big(R^2(t)r(t)p(t) - \frac{1}{4}\Big)\ln^2 R(t) < \frac{1}{4}$$

and it is oscillatory if

$$\liminf_{t\to\infty}\Big(R^2(t)r(t)p(t) - \frac{1}{4}\Big)\ln^2 R(t) > \frac{1}{4}.$$

Here we can notice that Corollary 2 is an analogue or an extension of Theorem B to equation (1) in the canonical form.

Finally, one can see that for $i = 2$ Theorem 1 has the following form.

Corollary 3. *Let* (i) *and* (ii) *be satisfied. The equation* (1) *is non-oscillatory if*

$$\limsup_{t\to\infty}\Big(R^2(t)r(t)p(t)\ln^2 R(t)-\frac{\ln^2 R(t)+1}{4}\Big)\ln^2(\ln R(t))<\frac{1}{4}$$

and it is oscillatory if

$$\liminf_{t\to\infty}\Big(R^2(t)r(t)p(t)\ln^2 R(t)-\frac{\ln^2 R(t)+1}{4}\Big)\ln^2(\ln R(t))>\frac{1}{4}.$$

It is clear that Corollary 1 can not be used if e.g.

$$\lim_{t\to\infty} R^2(t)r(t)p(t)=\frac{1}{4}$$

and an analogous situation occurs in using of Corollary 2 or Corollary 3. The following example demonstrates the utility of Corollary 2 in the situation when Corollary 1 can not be used.

Example 1. Consider the equation

$$\Big(\frac{t}{(t+1)^2}u'(t)\Big)'+\frac{t-1}{(t+1)^4}u(t)=0,\quad t\geqslant 1. \tag{13}$$

Then $R(t)=t^2/2+2t+\ln t$.

Since $\lim\limits_{t\to\infty} R^2(t)r(t)p(t)=1/4$ Corollary 1 is ineffective but

$$\lim_{t\to\infty}\Big[R^2(t)r(t)p(t)-\frac{1}{4}\Big]\ln^2 R(t)=0$$

and thus by Corollary 2 we know that our equation is non-oscillatory. Note that one solution of (13) is the function $u(t)=t+1$.

Note that if we apply Theorem 1 to equation (2) then for $i=0$ we have Theorem A and for $i=1$ we obtain Theorem B.

4. The non-canonical case

Now we consider the equation (1) under the assumption $\int^\infty \mathrm{d}t/r(t)<\infty$ and thus we put

$$\varrho(t)=\int_t^\infty\frac{\mathrm{d}s}{r(s)},\quad t\geqslant t_0.$$

Theorem 2. *Let* (i) *and* (ii) *be satisfied. Let the equation* (1) *be in the non-canonical form. Then the equation* (1) *is non-oscillatory if*

$$\limsup_{t\to\infty} \varrho^2(t)r(t)p(t) < \frac{1}{4}$$

and it is oscillatory provided

$$\liminf_{t\to\infty} \varrho^2(t)r(t)p(t) > \frac{1}{4}.$$

Proof. Taking into account Lemma 2 and using the Sturm comparison theorem for the equations (1) and (5) we have conclusions of our theorem and the proof is complete. □

It is clear that Theorem 2 is a generalization of Theorem A and also it is an analogue of Theorem C for equation (1) in the non-canonical form.

Now we assign to equation (1) in the non-canonical form an equation of the same form but in the canonical form.

Lemma 4. *Let the equation* (1) *be in the non-canonical form. Then the equation*

$$(\varrho(t)r(t)y'(t))' + \Big(\varrho(t)p(t) - \frac{1}{4\varrho(t)r(t)}\Big)y(t) = 0, \quad t \geqslant t_0 \tag{14}$$

is in the canonical form and a function $u(t)$ is a solution of (1) *if and only if the function $y(t) = \varrho^{-1/2}(t)u(t)$ is a solution of* (14).

Proof. We denote by φ the inverse function to the function ϱ and transform the equation (1) by the change of the independent and of the dependent variable of the form

$$s = \frac{1}{\varrho(t)}, \quad y(s) = su\Big(\varphi\Big(\frac{1}{s}\Big)\Big) \tag{15}$$

for $t \geqslant t_0$ and $s \geqslant s_0$ ($s_0 = \frac{1}{\varrho_0}$, $\varrho_0 = \varrho(t_0)$). Then from equation (1) we obtain the equation

$$s(sy'(s))' - sy'(s) + \frac{1}{s^2}r\Big(\varphi\Big(\frac{1}{s}\Big)\Big)p\Big(\varphi\Big(\frac{1}{s}\Big)\Big)y(s) = 0, \quad s \geqslant s_0. \tag{16}$$

The transformation of the equation (16) by the change of the independent variable in the form

$$x = \ln s, \quad w(x) = y(s) \tag{17}$$

enables us to write it in the form

$$w''(x) - w'(x) + \mathrm{e}^{-2x} r(\varphi(\mathrm{e}^{-x}))p(\varphi(\mathrm{e}^{-x}))w(x) = 0, \quad x \geqslant \ln s_0. \tag{18}$$

Finally, the transformation of the equation (18) by the change of the dependent variable in the form

$$z(x) = \mathrm{e}^{-\frac{x}{2}} w(x), \quad x \geqslant \ln s_0 \tag{19}$$

changes the equation (18) to the form

$$z''(x) + \Big[\mathrm{e}^{-2x} r(\varphi(\mathrm{e}^{-x}))p(\varphi(\mathrm{e}^{-x})) - \frac{1}{4}\Big] z(x) = 0, \quad x \geqslant \ln s_0. \tag{20}$$

Now, the v-transformation of (20) with $v(t) = -\ln \varrho(t)$ yields the equation (14). Since $\int^t \mathrm{d}s/(\varrho(s)r(s)) = -\ln \varrho(t) \to \infty$ as $t \to \infty$ so the equation (14) is in the canonical form. The relation between the solutions of (1) and (14) can be easily obtained from (15), (17), (19) and the v-transformation of (20). The proof is complete. $\square$

Remark 1. Because of the transformations done by (15), (17), (19) preserve the oscillatory character of solutions and the same holds for the v-transformation with $v(t) = -\ln \varrho(t)$, we see that equation (1) is oscillatory if and only if equation (14) is oscillatory.

If we now use the notation

$$r^*(t) = \varrho(t)r(t), \quad p^*(t) = \varrho(t)p(t) - \frac{1}{4\varrho(t)r(t)}$$

and define a function R^* by the rule

$$R^*(t) = \int^t \frac{\mathrm{d}s}{r^*(s)}$$

we can write equation (14) in the form

$$(r^*(t)y'(t))' + p^*(t)y(t) = 0. \tag{21}$$

Since equation (21) is in the canonical form so all what we have derived in the previous part of this paper about equation (1) in the canonical form holds true for equation (21).

Thus for $t \geqslant t_0$ we put

$$r_0^*(t) = r^*(t), \quad p_0^*(t) = p^*(t) \tag{22}$$

and for $i = 0, 1, 2, \dots$

$$R_i^*(t) = \int^t \frac{\mathrm{d}s}{r_i^*(s)}, \qquad r_{i+1}^*(t) = R_i^*(t) r_i^*(t), \tag{23}$$
$$p_{i+1}^*(t) = R_i^*(t) p_i^*(t) - \frac{1}{4R_i^*(t) r_i^*(t)}.$$

Then

$$R_0^*(t) = \int^t \frac{\mathrm{d}s}{r^*(s)} = \int^t \frac{\mathrm{d}s}{\varrho(s) r(s)} = -\ln \varrho(t) = \ln \frac{1}{\varrho(t)},$$

and

$$R_1^*(t) = \int^t \frac{\mathrm{d}s}{r_1^*(s)} = \int^t \frac{\mathrm{d}s}{r_0^*(s)(-\ln \varrho(s))} = \int^t \frac{\mathrm{d}s}{r(s)\varrho(s)(-\ln \varrho(s))}$$
$$= \ln(-\ln \varrho(t)) = \ln\Bigl(\ln \frac{1}{\varrho(t)}\Bigr).$$

Now we can write without proof the following analogue of Lema 3.

Lemma 5. *Let $i \in \{0, 1, 2, \dots\}$. Let* (i) *and* (ii) *be satisfied. If the equation*

$$(r_i^*(t) u'(t))' + p_i^*(t) u(t) = 0, \quad t > t_0 \tag{E$_i^*$}$$

is in the canonical form then also the equation

$$(r_{i+1}^*(t) u'(t))' + p_{i+1}^*(t) u(t) = 0, \quad t > t_0 \tag{E$_{i+1}^*$}$$

is such and moreover, a function $y(t)$ is a solution of ($\mathrm{E}_{\mathrm{i}}^*$) *if and only if the function $w(t) = (R_i^*(t))^{-1/2} y(t)$ is a solution of* ($\mathrm{E}_{\mathrm{i+1}}^*$).

Also the following analogue of Theorem 1 will be introduced without proof.

Theorem 3. *Let* (i) *and* (ii) *be satisfied. Let equation* (1) *be in the non-canonical form. Then equation* (1) *is non-oscillatory if for some $i \in \{0, 1, 2, \dots\}$*

$$\limsup_{t \to \infty} (R_i^*(t))^2 r_i^*(t) p_i^*(t) < \frac{1}{4},$$

and the equation (1) *is oscillatory if for some $i \in \{0, 1, 2, \dots\}$*

$$\liminf_{t \to \infty} (R_i^*(t))^2 r_i^*(t) p_i^*(t) > \frac{1}{4}.$$

We see that for $i = 0$ Theorem 3 gives the following assertion.

Corollary 4. *Let* (i) *and* (ii) *be satisfied. Let equation* (1) *be in the non-canonical form. Then equation* (1) *is non-oscillatory if*

$$\limsup_{t\to\infty}\Big(\varrho^2(t)r(t)p(t)-\frac{1}{4}\Big)\ln^2\varrho(t)<\frac{1}{4}$$

and it is oscillatory if

$$\liminf_{t\to\infty}\Big(\varrho^2(t)r(t)p(t)-\frac{1}{4}\Big)\ln^2\varrho(t)>\frac{1}{4}.$$

In a similar manner, for $i=1$ we obtain

Corollary 5. *Let* (i) *and* (ii) *be satisfied. Let equation* (1) *be in the non-canonical form. Then equation* (1) *is non-oscillatory if*

$$\limsup_{t\to\infty}\Big(\varrho^2(t)r(t)p(t)\ln^2\varrho(t)-\frac{\ln^2\varrho(t)+1}{4}\Big)\ln^2(-\ln\varrho(t))<\frac{1}{4}$$

and it is oscillatory if

$$\liminf_{t\to\infty}\Big(\varrho^2(t)r(t)p(t)\ln^2\varrho(t)-\frac{\ln^2\varrho(t)+1}{4}\Big)\ln^2(-\ln\varrho(t))>\frac{1}{4}.$$

It is easy to see that problems with one quarter are now the same as in the canonical case. Here we present an example of an equation for which Theorem 2 can not be used and the problem of oscillation is answered by Corollary 4.

Example 2. Consider the equation

$$\Big(\frac{\mathrm{e}^t}{t}u'(t)\Big)'+\mathrm{e}^t\Big(\frac{2}{t^3}-\frac{1}{2t^2}+\frac{1}{4t}\Big)u(t)=0,\quad t\geqslant 1. \tag{24}$$

Then $\varrho(t)=(t+1)\mathrm{e}^{-t}$.

Since $\lim\limits_{t\to\infty}\varrho^2(t)r(t)p(t)=1/4$ Theorem 2 can not be used but

$$\lim_{t\to\infty}\Big[\varrho^2(t)r(t)p(t)-\frac{1}{4}\Big]\ln^2\varrho(t)=\frac{5}{4}$$

and thus by Corollary 4 we know that our equation is oscillatory. Note that one solution of (24) is the function $u(t)=t\mathrm{e}^{-t/2}\sin(\ln t)$.

The following example can be interesting as well.

Example 3. Consider the equation

$$(25)\qquad (\mathrm{e}^t u'(t))' + \mathrm{e}^t\Big(\frac{1}{4} + \frac{1}{4t^2} + \frac{b^2}{t^2}\Big)u(t) = 0, \quad t > 0, \quad b \in \mathbb{R}.$$

Since $\int^\infty \mathrm{d}t/\mathrm{e}^t < \infty$ equation (25) is in the non-canonical form and $\varrho(t) = \mathrm{e}^{-t}$.

One can observe that

$$\lim_{t\to\infty} \varrho^2(t)r(t)p(t) = \lim_{t\to\infty} e^{-2t}\mathrm{e}^t\mathrm{e}^t\Big(\frac{1}{4} + \frac{1}{4t^2} + \frac{b^2}{t^2}\Big) = \frac{1}{4}$$

for any value of the parameter b. It means that Theorem 2 can not be used.

Other calculation shows that

$$\lim_{t\to\infty}\Big(\varrho^2(t)r(t)p(t) - \frac{1}{4}\Big)\ln^2\varrho(t) = \lim_{t\to\infty}\Big(\mathrm{e}^{-2t}\mathrm{e}^t\mathrm{e}^t\Big[\frac{1}{4} + \frac{1}{4t^2} + \frac{b^2}{t^2}\Big] - \frac{1}{4}\Big)t^2 = b^2 + \frac{1}{4}.$$

Now we see that if $b \neq 0$ so according to Corollary 4 we know that equation (25) is oscillatory. On the other hand, Corollary 4 can not be used in the case $b = 0$.

Finally we have

$$\begin{aligned}
&\lim_{t\to\infty}\Big(\varrho^2(t)r(t)p(t)\ln^2\varrho(t) - \frac{\ln^2\varrho(t) + 1}{4}\Big)\ln^2(-\ln\varrho(t))\\
&\quad = \lim_{t\to\infty}\Big(\mathrm{e}^{-2t}\mathrm{e}^t\mathrm{e}^t\Big(\frac{1}{4} + \frac{1}{4t^2} + \frac{b^2}{t^2}\Big)t^2 - \frac{t^2+1}{4}\Big)\ln^2 t\\
&\quad = \lim_{t\to\infty} b^2\ln^2 t = \begin{cases} 0 & \text{if } b = 0,\\ \infty & \text{if } b \neq 0\end{cases}
\end{aligned}$$

and with regard to Corollary 5 we know that equation (25) is non-oscillatory for $b = 0$ and it is oscillatory for any $b \neq 0$. Note that one solution of (25) is the function

$$u(t) = \sqrt{\frac{t}{\mathrm{e}^t}}\sin\ln(at^b), \quad t > 0$$

for arbitrary $a > 0$ and $b \in \mathbb{R}$. And indeed, this solution is non-oscillatory if $b = 0$, and oscillatory otherwise.

We conclude the paper by the following example which relies on a relation of Theorem 2 to another result known for equation (1).

Example 4. We consider the non-canonical differential equation

$$(26)\qquad ((t+2)^2u'(t))' + (t+2)^2u(t) = 0, \quad t \geqslant 0.$$

It is easy to see that $I_r = \int_0^\infty \mathrm{d}t/r(t) = 1/2 < \infty$, $I_{pr} = \int_0^\infty p(t) \int_0^t \mathrm{d}s/r(s)\,\mathrm{d}t = \infty$, $I_{rp} = \int_0^\infty 1/r(t) \int_0^t p(s)\mathrm{d}s\mathrm{d}t = \infty$ in the case of equation (26) and thus by [2, Theorem 1, (i_4)] we know that every solution $u(t)$ of (26) is either oscillatory or such that $u(t)u'(t) < 0$ for sufficiently large t and $\lim\limits_{t\to\infty} u(t) = 0$. On the other hand, since $\lim\limits_{t\to\infty} \varrho^2(t)r(t)p(t) = \infty$ so by our Theorem 2 it is clear that equation (26) is oscillatory, which is a stronger assertion than the previous one. Note that one solution of equation (26) is the function $u(t) = (t+2)^{-1}\sin(t+2)$.

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Author's address: Department of Mathematics, P. J. Šafárik University, Jesenná 5, 041 54 Košice, Slovak Republic, e-mail: `ohriska@kosice.upjs.sk`.