Czechoslovak Mathematical Journal

Eva Špániková

Oscillation of differential systems of neutral type

Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 1, 263–271

Persistent URL: http://dml.cz/dmlcz/127975

OSCILLATION OF DIFFERENTIAL SYSTEMS OF NEUTRAL TYPE

Eva Špániková, Žilina

(Received July 11, 2002)

Abstract. We study oscillatory properties of solutions of systems

$$\begin{aligned}[y_1(t) - a(t)y_1(g(t))]' &= p_1(t)y_2(t),\\ y_2'(t) &= -p_2(t)f(y_1(h(t))), \quad t \geqslant t_0.\end{aligned}$$

Keywords: differential system of neutral type, oscillatory solution

MSC 2000: 34K15, 34K40

1. Introduction

In this paper we consider neutral differential systems of the form

$$\begin{aligned}[y_1(t) - a(t)y_1(g(t))]' &= p_1(t)y_2(t),\\ y_2'(t) &= -p_2(t)f(y_1(h(t))), \quad t \geqslant t_0.\end{aligned} \tag{S}$$

The following conditions are assumed to hold throughout the paper:

(a) $a\colon [t_0,\infty) \to (0,\infty)$ is a continuous function;

(b) $g\colon [t_0,\infty) \to \mathbb{R}$ is a continuous and increasing function and $\lim\limits_{t\to\infty} g(t) = \infty$;

(c) $p_i\colon [t_0,\infty) \to [0,\infty)$, $i = 1,2$ are continuous functions not identically equal to zero in every neighbourhood of infinity,

$$\int^{\infty} p_1(t)\,\mathrm{d}t = \infty;$$

This work was supported by the Grant Agency of the Slovak Academy of Sciences under Grant No. 2/3205/23.

(d) $h\colon [t_0,\infty)\to\mathbb{R}$ is a continuous and increasing function and $\lim\limits_{t\to\infty} h(t)=\infty$;

(e) $f\colon \mathbb{R}\to\mathbb{R}$ is a continuous function, $uf(u)>0$ for $u\neq 0$ and $|f(u)|\geqslant K|u|$, where $0<K=\text{const}$.

Let $p_1(t)\equiv 1$ on $[t_0,\infty)$ and $f(u)=u$, $u\in\mathbb{R}$. Then the system (S) is equivalent to the equation

$$\text{(E)}\qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}[y_1(t)-a(t)y_1(g(t))]+p_2(t)y_1(h(t))=0,\quad t\geqslant t_0.$$

In the paper [6] sufficient conditions are given for all bounded solutions and all solutions of the equation (E) to be oscillatory. In this paper we generalize Theorem 1 and Theorem 2 from [6] to the system (S). Our results are new and extend and improve the known criteria for the oscillation of differential systems of neutral type. The oscillatory theory of neutral differential systems has been studied for example in the papers [1]–[10] and in the references given therein.

Let $t_1\geqslant t_0$. Denote

$$\tilde{t}_1=\min\{t_1,g(t_1),h(t_1)\}.$$

A function $y=(y_1,y_2)$ is a solution of the system (S) if there exists a $t_1\geqslant t_0$ such that y is continuous on $[\tilde{t}_1,\infty)$, $y_1(t)-a(t)y_1(g(t))$, $y_2(t)$ are continuously differentiable on $[t_1,\infty)$ and y satisfies (S) on $[t_1,\infty)$.

Denote by W the set of all solutions $y=(y_1,y_2)$ of the system (S) which exist on some ray $[T_y,\infty)\subset[t_0,\infty)$ and satisfy

$$\sup\{|y_1(t)|+|y_2(t)|\colon\ t\geqslant T\}>0\quad\text{for any } T\geqslant T_y.$$

A solution $y\in W$ is nonoscillatory if there exists a $T_y\geqslant t_0$ such that its every component is different from zero for all $t\geqslant T_y$. Otherwise a solution $y\in W$ is said to be oscillatory.

Denote

$$P_1(t)=\int_{t_0}^{t}p_1(x)\,\mathrm{d}x,\quad t\geqslant t_0.$$

For any $y_1(t)$ we define $z_1(t)$ by

$$\text{(1)}\qquad z_1(t)=y_1(t)-a(t)y_1(g(t)).$$

2. Some basic lemmas

Lemma 1 ([4, Lemma 1]). *Let* $y \in W$ *be a solution of the system* (S) *with* $y_1(t) \neq 0$ *on* $[t_1, \infty)$, $t_1 \geqslant t_0$. *Then* y *is nonoscillatory and* $z_1(t)$, $y_2(t)$ *are monotone on some ray* $[T, \infty)$, $T \geqslant t_1$.

Lemma 2 ([7, Lemma 1]). *In addition to the conditions* (a) *and* (b) *suppose that*

$$a(t) \leqslant 1 \quad \text{for } t \geqslant t_0.$$

Let $y_1(t)$ *be a continuous nonoscillatory solution of the functional inequality*

$$y_1(t)[y_1(t) - a(t)y_1(g(t))] < 0$$

defined in a neighbourhood of infinity.

(i) *Suppose that* $g(t) < t$ *for* $t \geqslant t_0$. *Then* $y_1(t)$ *is bounded.*

(ii) *Suppose that* $g(t) > t$ *for* $t \geqslant t_0$. *Then* $y_1(t)$ *is bounded away from zero, that is, there exists a positive constant* C *such that* $|y_1(t)| \geqslant C$ *for all large* t.

Lemma 3 ([7, Lemma 3]). *Assume that* $q\colon [t_0, \infty) \to [0, \infty)$, $\delta\colon [t_0, \infty) \to \mathbb{R}$ *are continuous functions,* $\lim\limits_{t \to \infty} \delta(t) = \infty$ *and*

$$\delta(t) < t \quad \text{for } t \geqslant t_0, \qquad \liminf_{t \to \infty} \int_{\delta(t)}^{t} q(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}.$$

Then the functional inequality

$$x'(t) + q(t)x(\delta(t)) \leqslant 0, \quad t \geqslant t_0$$

cannot have an eventually positive solution and

$$x'(t) + q(t)x(\delta(t)) \geqslant 0, \quad t \geqslant t_0$$

cannot have an eventually negative solution.

3. Oscillation theorems

In this section we shall study the oscillation of solutions of systems (S). In the next theorems $g^{-1}(t)$ and $h^{-1}(t)$ denote the inverse functions of $g(t)$, $h(t)$ and $\alpha\colon [t_0, \infty) \to \mathbb{R}$ is a continuous function.

Theorem 1. *Suppose that $a(t)$ is bounded, $h(t) < t$, $t < \alpha(t)$, $h(\alpha(t)) < g(t)$ for $t \geqslant t_0$ and*

$$\limsup_{t\to\infty}\Big\{KP_1(t)\int_{h^{-1}(t)}^{\infty} p_2(s)\,\mathrm{d}s\Big\} > 1, \tag{2}$$

$$\liminf_{t\to\infty}\int_{g^{-1}(h(\alpha(t)))}^{t} Kp_1(s)\int_{s}^{\alpha(s)}\frac{p_2(v)\,\mathrm{d}v}{a(g^{-1}(h(v)))}\,\mathrm{d}s > \frac{1}{\mathrm{e}}. \tag{3}$$

Then every solution $y \in W$ of (S) *with $y_1(t)$ bounded is oscillatory.*

P r o o f. Let $y = (y_1, y_2) \in W$ be a nonoscillatory solution of (S) with $y_1(t)$ bounded. Without loss of generality we may suppose that $y_1(t)$ is positive and bounded for $t \geqslant t_1$. From the second equation of (S) and by the assumptions (c), (d), (e) we get

$$y_2'(t) \leqslant 0 \quad \text{for sufficiently large } t_2 \geqslant t_1.$$

In view of Lemma 1 we have two cases for sufficiently large $t_3 \geqslant t_2$:

1) $y_2(t) < 0$, $t \geqslant t_3$;

2) $y_2(t) > 0$, $t \geqslant t_3$.

Case 1. Because $y_2(t)$ is negative and nonincreasing we have

$$y_2(t) \leqslant -L, \quad t \geqslant t_3, \quad 0 < L = \text{const}. \tag{4}$$

Since $y_1(t)$ and $a(t)$ are bounded hence also $z_1(t)$ defined by (1) is bounded. Integrating the first equation of (S) from t_3 to t and then using (4) we get

$$z_1(t) - z_1(t_3) \leqslant -L\int_{t_3}^{t} p_1(s)\,\mathrm{d}s, \quad t \geqslant t_3. \tag{5}$$

From (5) and (c) we have $\lim\limits_{t\to\infty} z_1(t) = -\infty$, which contradicts the fact that $z_1(t)$ is bounded. The Case 1 cannot occur.

Case 2. We shall consider two possibilities.

(A) Let $z_1(t) > 0$ for $t \geqslant t_4$, where $t_4 \geqslant t_3$ is sufficiently large. Because $z_1(t)$ is nondecreasing we get

$$z_1(t) \geqslant M, \quad t \geqslant t_4, \quad 0 < M = \text{const}. \tag{6}$$

From (1) we have $z_1(t) < y_1(t)$ and using (e) we get

$$p_2(t)z_1(h(t)) \leqslant \frac{p_2(t)f(y_1(h(t)))}{K}, \quad t \geqslant t_5, \tag{7}$$

where $t_5 \geqslant t_4$ is sufficiently large.

Integrating the second equation of (S) from t to $t^\star$, using (7) and then letting $t^\star \to \infty$ we obtain

$$y_2(t) \geqslant K \int_t^\infty p_2(s) z_1(h(s))\,\mathrm{d}s, \quad t \geqslant t_5. \tag{8}$$

With regard to (2) we get

$$\frac{1}{K} < \limsup_{t\to\infty} \left\{ P_1(t) \int_{h^{-1}(t)}^\infty p_2(s)\,\mathrm{d}s \right\} \leqslant \limsup_{t\to\infty} \int_t^\infty P_1(s) p_2(s)\,\mathrm{d}s. \tag{9}$$

We claim that the condition (2) implies

$$\int_T^\infty P_1(s) p_2(s)\,\mathrm{d}s = \infty, \quad T \geqslant t_0. \tag{10}$$

Otherwise if

$$\int_T^\infty P_1(s) p_2(s)\,\mathrm{d}s < \infty,$$

we can choose $T_1 \geqslant T$ so large that

$$\int_{T_1}^\infty P_1(s) p_2(s)\,\mathrm{d}s < \frac{1}{K},$$

which is a contradiction with (9).

Integrating $\int_T^t P_1(s) y_2'(s)\,\mathrm{d}s$ by parts we have

$$\int_T^t P_1(s) y_2'(s)\,\mathrm{d}s = P_1(t) y_2(t) - P_1(T) y_2(T) - z_1(t) + z_1(T). \tag{11}$$

Using (6), (7) and the second equation of (S), by virtue of (11) we get

$$\int_T^t P_1(s) y_2'(s)\,\mathrm{d}s \leqslant -MK \int_T^t P_1(s) p_2(s)\,\mathrm{d}s, \quad t \geqslant T \geqslant t_5$$

and

$$MK \int_T^t P_1(s) p_2(s)\,\mathrm{d}s \leqslant -P_1(t) y_2(t) + P_1(T) y_2(T) + z_1(t) - z_1(T), \qquad t \geqslant T \geqslant t_5. \tag{12}$$

Combining (10) with (12) we get $\lim\limits_{t\to\infty} (z_1(t) - P_1(t) y_2(t)) = \infty$ and

$$z_1(t) \geqslant P_1(t) y_2(t), \quad t \geqslant t_6, \quad \text{where } t_6 \geqslant t_5 \text{ is sufficiently large.}$$

The last inequality together with (8) and the monotonicity of $z_1(t)$ implies

$$
\begin{aligned}
z_1(t) &\geqslant KP_1(t)\int_t^\infty p_2(s)z_1(h(s))\,\mathrm{d}s \geqslant KP_1(t)\int_{h^{-1}(t)}^\infty p_2(s)z_1(h(s))\,\mathrm{d}s\\
&\geqslant KP_1(t)z_1(t)\int_{h^{-1}(t)}^\infty p_2(s)\,\mathrm{d}s, \quad t\geqslant t_6
\end{aligned}
$$

and

$$
1\geqslant KP_1(t)\int_{h^{-1}(t)}^\infty p_2(s)\,\mathrm{d}s, \quad t\geqslant t_6,
$$

which contradicts (2). This case cannot occur.

(B) Let $z_1(t)<0$ for $t\geqslant t_4$. Denote $\beta(t)=g^{-1}(h(t))$.

From (1) we have $z_1(\beta(t))>-a(\beta(t))y_1(h(t))$, $t\geqslant t_5\geqslant t_4$, where t_5 is sufficiently large and

$$
\frac{-Kp_2(t)z_1(\beta(t))}{a(\beta(t))}\leqslant Kp_2(t)y_1(h(t)), \quad t\geqslant t_5.
$$

In view of (e) and the second equation of (S) the last inequality implies

$$
y_2'(t)-\frac{Kp_2(t)z_1(\beta(t))}{a(\beta(t))}\leqslant 0, \quad t\geqslant t_5. \tag{13}
$$

Integrating (13) from t to $\alpha(t)$ and then using $y_2(\alpha(t))>0$, we have

$$
y_2(t)+\int_t^{\alpha(t)}\frac{Kp_2(s)z_1(\beta(s))\,\mathrm{d}s}{a(\beta(s))}\geqslant 0, \quad t\geqslant t_5. \tag{14}
$$

Multiplying (14) by $p_1(t)$ and then using the monotonicity of $z_1(t)$ and the first equation of (S), we get

$$
z_1'(t)+\left(Kp_1(t)\int_t^{\alpha(t)}\frac{p_2(s)\,\mathrm{d}s}{a(\beta(s))}\right)z_1(\beta(\alpha(t)))\geqslant 0, \quad t\geqslant t_5.
$$

By condition (3) and Lemma 3 the last inequality cannot have an eventually negative solution and this contradicts the hypothesis that $z_1(t)<0$. The proof is complete.

□

Theorem 2. *Suppose that* $a(t) \leqslant 1$, $g(t) < t$, $h(t) < t$, $t < \alpha(t)$, $h(\alpha(t)) < g(t)$ *for* $t \geqslant t_0$ *and the conditions* (2), (3) *are satisfied. Then all solutions of* (S) *are oscillatory.*

P r o o f. Let $y = (y_1, y_2) \in W$ be a nonoscillatory solution of (S). Without loss of generality we may suppose that $y_1(t)$ is positive for $t \geqslant t_1$. As in the proof of Theorem 1 we get two cases — Case 1 and Case 2.

Case 1. Analogously to Case 1 of the proof of Theorem 1 we can show that $\lim\limits_{t\to\infty} z_1(t) = -\infty$. By Lemma 2 $y_1(t)$ is bounded and thereby $z_1(t)$ is bounded, which is a contradiction. Case 1 cannot occur.

Case 2. We can treat this case in the same way as in the proof of Theorem 1. The proof is complete. □

Example 1. We consider the system

$$
\begin{aligned}
&\left[y_1(t) - \frac{1}{2}y_1\Big(\frac{t}{2}\Big)\right]' = t\,y_2(t),\\
(15)\qquad &y_2'(t) = -\frac{c}{t^3}y_1\Big(\frac{t}{6}\Big), \quad t \geqslant 1,
\end{aligned}
$$

where c is a positive constant. In this example $f(t) = t$ and $K = 1$. We choose $\alpha(t) = 2t$ and calculate the conditions (2) and (3) as follows:

$$
\begin{aligned}
\limsup_{t\to\infty}\left\{\frac{(t^2-1)}{2}\int_{6t}^{\infty}\frac{c}{s^3}\,\mathrm{d}s\right\} &= \frac{c}{144},\\
\liminf_{t\to\infty}\int_{\frac{2}{3}t}^{t} s\int_{s}^{2s}\frac{2c\,\mathrm{d}v}{v^3}\,\mathrm{d}s &= \frac{3c}{4}\ln\frac{3}{2}.
\end{aligned}
$$

For $c > 144$ all conditions of Theorem 2 are satisfied and so all solutions of (15) are oscillatory.

Theorem 3. *Suppose that* $a(t) \leqslant 1$, $t < g(t)$, $g(t) < h(t)$ *for* $t \geqslant t_0$ *and*

$$
\limsup_{t\to\infty}\int_{h^{-1}(g(t))}^{t}\frac{K(P_1(t)-P_1(s))p_2(s)\,\mathrm{d}s}{a(g^{-1}(h(s)))} > 1, \tag{16}
$$

$$
\int_T^{\infty} p_1(s)\int_s^{\infty} p_2(v)\,\mathrm{d}v\,\mathrm{d}s = \infty, \quad T \geqslant t_0, \tag{17}
$$

$$
\limsup_{t\to\infty}\left\{KP_1(t)\int_t^{\infty} p_2(s)\,\mathrm{d}s\right\} > 1. \tag{2'}
$$

Then all solutions of (S) *are oscillatory.*

P r o o f. Let $y=(y_1,y_2)\in W$ be a nonoscillatory solution of (S). Without loss of generality we may suppose that $y_1(t)$ is positive for $t\geqslant t_1$. As in the proof of Theorem 1 we get two cases — Case 1 and Case 2.

Case 1. From (1) we have

$$z_1(t) > -a(t)y_1(g(t)) \quad \text{for } t\geqslant t_3$$

and

$$f(y_1(h(t)))\geqslant Ky_1(h(t)) > -\frac{Kz_1(g^{-1}(h(t)))}{a(g^{-1}(h(t)))}, \quad t\geqslant t_4 \tag{18}$$

where $t_4\geqslant t_3$ is sufficiently large.

In this case $y_2(t)<0$ and $z_1(t)<0$ for $t\geqslant t_5$, where $t_5\geqslant t_4$ is sufficiently large. Then the integral identity

$$z_1(t)=z_1(\xi)+(P_1(t)-P_1(\xi))y_2(\xi)+\int_\xi^t (P_1(t)-P_1(s))y_2'(s)\,\mathrm{d}s$$

yields

$$z_1(t) < \int_\xi^t (P_1(t)-P_1(s))y_2'(s)\,\mathrm{d}s, \quad t>\xi\geqslant t_5.$$

Combining the last inequality with the second equation of (S) and (18) we get

$$\begin{aligned} z_1(t) &< \int_\xi^t (P_1(t)-P_1(s))(-p_2(s)f(y_1(h(s))))\,\mathrm{d}s \\ &< \int_\xi^t \frac{K(P_1(t)-P_1(s))p_2(s)z_1(g^{-1}(h(s)))\,\mathrm{d}s}{a(g^{-1}(h(s)))}, \quad t>\xi\geqslant t_5. \end{aligned}$$

Putting $\xi=h^{-1}(g(t))$ and using the monotonicity of $z_1(t)$, from the last inequality we get

$$z_1(t) < z_1(t)\int_{h^{-1}(g(t))}^t \frac{K(P_1(t)-P_1(s))p_2(s)\,\mathrm{d}s}{a(g^{-1}(h(s)))}$$

and

$$1 > \int_{h^{-1}(g(t))}^t \frac{K(P_1(t)-P_1(s))p_2(s)\,\mathrm{d}s}{a(g^{-1}(h(s)))},$$

which contradicts the condition (16).

Case 2. As in the proof of Theorem 1 we shall consider two posibilities A) and B).

A) We can treat the proof in the same way as in Theorem 1 using the condition (2′) instead of the condition (2).

B) In this case $z_1(t)$ is negative and bounded for $t \geqslant t_4$. Then by Lemma 2 it follows that

$$y_1(t) \geqslant C, \quad 0 < C = \text{const. for } t \geqslant t_4. \tag{19}$$

Integrating the second equation of (S) from s to $s^\star$, using (e), (19) and then letting $s^\star \to \infty$, we obtain

$$y_2(s) > KC \int_s^\infty p_2(v)\,\mathrm{d}v \quad \text{for sufficiently large } s. \tag{20}$$

Multiplying (20) by $p_1(s)$ and integrating from T to $T^\star$ and then letting $T^\star \to \infty$ we get

$$-z_1(T) > KC \int_T^\infty p_1(s) \int_s^\infty p_2(v)\,\mathrm{d}v\,\mathrm{d}s \quad \text{for sufficiently large } T$$

and with regard to condition (17) we have a contradiction. The proof is complete. □

References

[1] *I. Györi and G. Ladas*: Oscillation of systems of neutral differential equations. Diff. and Integral Equat. *1* (1988), 281–286.

[2] *Y. Kitamura and T. Kusano*: On the oscillation of a class of nonlinear differential systems with deviating argument. J. Math. Anal. Appl. *66* (1978), 20–36.

[3] *P. Marušiak*: Oscillation criteria for nonlinear differential systems with general deviating arguments of mixed type. Hiroshima Math. J. *20* (1990), 197–208.

[4] *P. Marušiak*: Oscillatory properties of functional differential systems of neutral type. Czechoslovak Math. J. *43(118)* (1993), 649–662.

[5] *B. Mihalíková*: Some properties of neutral differential systems equations. Bolletino U.M.I. *8 5-B* (2002), 279–287.

[6] *H. Mohamad and R. Olach*: Oscillation of second order linear neutral differential equations. In: Proceedings of the International Scientific Conference of Mathematics. University of Žilina, Žilina, 1998, pp. 195–201.

[7] *R. Olach*: Oscillation of differential equation of neutral type. Hiroshima Math. J. *25* (1995), 1–10.

[8] *R. Olach and H. Šamajová*: Oscillation of nonlinear differential systems with retarded arguments. 1st International Conference APLIMAT 2002. Bratislava, 2002, pp. 309–312.

[9] *E. Špániková*: Oscillatory properties of solutions of three-dimensional differential systems of neutral type. Czechoslovak Math. J. *50(125)* (2000), 879–887.

[10] *E. Špániková*: Oscillatory properties of solutions of neutral differential systems. Fasc. Math. *31* (2001), 91–103.

Author's address: Department of Appl. Mathematics, University of Žilina, J. M. Hurbana 15, 010 26 Žilina, Slovak Republic, e-mail: `eva.spanikova@fstroj.utc.sk`.