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ON THE SOLUTION OF SOME NON-LOCAL PROBLEMS

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Abstract. This paper deals with two types of non-local problems for the Poisson equation in the disc. The first of them deals with the situation when the function value on the circle is given as a combination of unknown function values in the disc. The other type deals with the situation when a combination of the value of the function and its derivative by radius on the circle are given as a combination of unknown function values in the disc. The existence and uniqueness of the classical solution of these problems is proved. The solutions are constructed in an explicit form.

Keywords: non-local problem, Poisson equation, discrete Fourier transform

MSC 2000: 35J25, 35J05

Introduction

This paper investigates non-local boundary problems for the Poisson equation in the disc. The non-local problem for harmonic functions in the two-dimensional case was first investigated by O. Sjöstrand [9]. Unique existence theorems were obtained by using the theory of Fredholm integral equations. Analogous problems were posed by A. Bitsadze and A. Samarski. Unique existence theorems for a harmonic function were obtained in a rectangle [1]. A. Bitsadze [2] also constructed the harmonic function $u(r,\vartheta)$ in the disc $(r \leqslant 1)$ satisfying the condition

$$u(1,\vartheta) = u(h,\vartheta) + f(\vartheta), \quad 0 \leqslant \vartheta \leqslant 2\pi, \quad 0 < h < 1,$$

where r, ϑ are the polar coordinates of the point, $f(\vartheta)$ is a given function, and h is a given constant. The solution is represented by Fourier series. In this paper this problem is generalized and more effective solutions in the integral form by quadratures are constructed. They may be used for a wider class of functions. Non-local boundary problems arise in connection with mathematical modeling of some processes in physics, chemistry, biology, etc. Applications of these problems can be found in the research of baroclinic sea dynamics [4], in the theory of elasticity and shells [3], [6], [7], [8], etc.

1.1. The first problem.

Let D be a disc with radius R, whose center coincides with the origin of coordinates. Consider a finite number of concentric circles, with radii satisfying the condition $R > R_1 > \ldots > R_m > 0$. Let S be the boundary of D, then $\overline{D} = D \cup S$.

Consider the non-local problem for the Poisson equation in the disc:

$$\Delta u = g(r, \vartheta), \quad r\mathrm{e}^{\mathrm{i}\vartheta} \in D, \tag{1}$$

$$u(R, \vartheta) - \sum_{k=1}^{m} a_k u(R_k, \vartheta) = f(\vartheta), \quad 0 \leqslant \vartheta < 2\pi, \tag{2}$$

where $z = r\mathrm{e}^{\mathrm{i}\vartheta}$ are complex points of the disc, $f \in C^2(S)$, $g \in C^1(D)$ are given functions, a_k $(k = 1, \ldots, m)$ are given real numbers, Δ is the Laplace operator, written in polar coordinates

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}.$$

By a classical solution $u(r, \vartheta)$ of the problem (1)–(2) we mean a function $u(r, \vartheta)$ of class $C^2(D) \cap C(\overline{D})$ satisfying all the conditions of the problem (1)–(2).

Theorem 1. *Let $f \in C^2(S)$, $g \in C^1(D)$ and*

$$1 - k_n \neq 0, \quad n = 0, \pm 1, \pm 2, \ldots$$

where

$$k_n = \frac{1}{2\pi}\int_0^{2\pi} k(\vartheta)\mathrm{e}^{-\mathrm{i}n\vartheta}\,\mathrm{d}\vartheta, \quad k(\vartheta) = \sum_{k=1}^{m} a_k \frac{R^2 - R_k^2}{R^2 + R_k^2 - 2RR_k\cos\vartheta}.$$

Then there exists a unique classical solution of problem (1)–(2), *which is represented as follows:*

$$u(r, \vartheta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\vartheta - \theta)}\Big(F(\theta) + \frac{1}{2\pi}\int_0^{2\pi} k^*(\theta - \varphi)F(\varphi)\,\mathrm{d}\varphi\Big)\,\mathrm{d}\theta + u_1(r, \vartheta),$$

where

$$u_1(r,\vartheta)=\frac{1}{4\pi}\int_0^{2\pi}\int_0^R \ln(r^2+\varrho^2-2r\varrho\cos(\vartheta-\theta))g(\varrho,\theta)\varrho\,\mathrm{d}\varrho\,\mathrm{d}\theta, \tag{3}$$

$$k^*(\vartheta)=\sum_{n=-\infty}^{\infty}\frac{k_n}{1-k_n}\,\mathrm{e}^{in\vartheta},$$

$$F(\theta)=f(\theta)-u_1(R,\theta)+\sum_{k=1}^{m}a_k u_1(R_k,\theta). \tag{4}$$

P r o o f. As is known, the general solution of the equation (1) is represented as follows:

$$u=u_0+u_1, \tag{5}$$

where u_0 is a harmonic function and u_1 is a particular solution, which can be taken as (3).

By means of the equality (5), the problem (1)–(2) reduces to the following problem:

$$\Delta u_0=0,\quad r\mathrm{e}^{\mathrm{i}\vartheta}\in D, \tag{6}$$

$$u_0(R,\vartheta)-\sum_{k=1}^{m}a_k u_0(R_k,\vartheta)=F(\vartheta),\quad 0\leqslant\vartheta<2\pi, \tag{7}$$

where $F(\vartheta)$ is given by the formula (4). Since u_0 is a function harmonic in D and continuous in $\overline{D}$, it is possible to use the Poisson formula

$$u_0(r,\vartheta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)u_0(R,\theta)\,\mathrm{d}\theta}{R^2+r^2-2Rr\cos(\vartheta-\theta)},\quad 0\leqslant\vartheta<2\pi. \tag{8}$$

Using the formula (8) for the condition (7) one can obtain

$$u_0(R,\vartheta)-\frac{1}{2\pi}\sum_{k=1}^{m}a_k\int_0^{2\pi}\frac{(R^2-R_k^2)u_0(R,\theta)\,\mathrm{d}\theta}{R^2+R_k^2-2RR_k\cos(\vartheta-\theta)}=F(\vartheta),\quad 0\leqslant\vartheta<2\pi. \tag{9}$$

Let us introduce the following notations

$$v(\vartheta)=u_0(R,\vartheta),\quad k(\vartheta)=\sum_{k=1}^{m}a_k\frac{R^2-R_k^2}{R^2+R_k^2-2RR_k\cos\vartheta}. \tag{10}$$

By virtue of (10) the equation (9) can be written in the following way:

$$v(\vartheta)-\frac{1}{2\pi}\int_0^{2\pi}k(\vartheta-\theta)v(\theta)\,\mathrm{d}\theta=F(\vartheta),\quad 0\leqslant\vartheta<2\pi. \tag{11}$$

The equation (11) represents a convolution type equation, whose kernel has the same period as the function $F(\vartheta)$ in the right-hand side of the equation (11) and the unknown function $v(\vartheta)$. Therefore in this case the solution of the equation (11) can be sought in quadratures by applying the discrete Fourier transform. Let us introduce the following notations

$$v_n = \frac{1}{2\pi}\int_0^{2\pi} v(\vartheta)\mathrm{e}^{-in\vartheta}\,\mathrm{d}\vartheta, \quad k_n = \frac{1}{2\pi}\int_0^{2\pi} k(\vartheta)\mathrm{e}^{-in\vartheta}\,\mathrm{d}\vartheta, \quad F_n = \frac{1}{2\pi}\int_0^{2\pi} F(\vartheta)\mathrm{e}^{-in\vartheta}\,\mathrm{d}\vartheta. \tag{12}$$

Multiplying the equation (11) by $\frac{1}{2\pi}\mathrm{e}^{-in\vartheta}$, integrating from 0 to 2π and changing the integration order (as the sub-integral functions are continuous), one can obtain

$$\frac{1}{2\pi}\int_0^{2\pi} v(\vartheta)\mathrm{e}^{-in\vartheta}\,\mathrm{d}\vartheta - \frac{1}{2\pi}\int_0^{2\pi} v(\vartheta)\,\mathrm{d}\vartheta\frac{1}{2\pi}\int_0^{2\pi} k(\varphi-\vartheta)\mathrm{e}^{-in\varphi}\,\mathrm{d}\varphi = \frac{1}{2\pi}\int_0^{2\pi} F(\vartheta)\mathrm{e}^{-in\vartheta}\,\mathrm{d}\vartheta. \tag{13}$$

Let us denote

$$\varphi - \vartheta = \gamma.$$

Taking into account this notation and (12), the equation (13) can be rewritten as

$$v_n - \frac{1}{2\pi}\int_0^{2\pi} v(\vartheta)\,\mathrm{d}\vartheta\frac{1}{2\pi}\int_{-\vartheta}^{2\pi-\vartheta} k(\gamma)\mathrm{e}^{-in(\gamma+\vartheta)}\,\mathrm{d}\gamma = F_n. \tag{14}$$

As $k(\gamma)\mathrm{e}^{-in(\gamma+\vartheta)}$ is a periodic function with period 2π, one gets

$$\int_{-\vartheta}^{2\pi-\vartheta} k(\gamma)\mathrm{e}^{-in(\gamma+\vartheta)}\,\mathrm{d}\gamma = \int_0^{2\pi} k(\gamma)\mathrm{e}^{-in(\gamma+\vartheta)}\,\mathrm{d}\gamma.$$

Consequently, from (14) one can obtain

$$v_n - \frac{1}{2\pi}\int_0^{2\pi} v(\vartheta)\mathrm{e}^{-in\vartheta}\,\mathrm{d}\vartheta\frac{1}{2\pi}\int_0^{2\pi} k(\gamma)\mathrm{e}^{-in\gamma}\,\mathrm{d}\gamma = F_n.$$

Hence, one gets

$$v_n(1-k_n) = F_n, \quad n = 0, \pm 1, \pm 2, \dots. \tag{15}$$

The last equation is solvable for any F_n only when

$$1 - k_n \neq 0, \quad n = 0, \pm 1, \pm 2, \dots. \tag{16}$$

In this case the equation (15) has the unique solution:

$$v_n = \frac{F_n}{1-k_n}, \quad n = 0, \pm 1, \pm 2, \dots . \tag{17}$$

Let us rewrite (17) in the following way

$$v_n = F_n + k_n^* F_n, \quad n = 0, \pm 1, \pm 2, \dots \tag{18}$$

where

$$k_n^* = \frac{k_n}{1-k_n} = k_n\Big(\frac{1}{1-k_n} - 1\Big) + k_n.$$

Since k_n is the discrete Fourier transform of the periodic function $k(\vartheta)$, $k_n\big((1-k_n)^{-1}-1\big)$ will be the discrete Fourier transform of a periodic function.

Taking into account the notations (10) and (12) one can obtain

$$k_n = \sum_{j=1}^{m} \frac{a_j}{2\pi} \int_0^{2\pi} \frac{(R^2 - R_j^2)\,\mathrm{e}^{-in\vartheta}\,\mathrm{d}\vartheta}{R^2 + R_j^2 - 2RR_j\cos\vartheta}.$$

Introducing the notation

$$t = R\mathrm{e}^{\mathrm{i}\vartheta},$$

one gets

$$\cos\vartheta = \frac{\mathrm{e}^{\mathrm{i}\vartheta} + \mathrm{e}^{-\mathrm{i}\vartheta}}{2} = \frac{R^2 + t^2}{2Rt}.$$

According to the residue theory one obtains

$$\begin{aligned} k_n &= -\sum_{j=1}^{m} (R^2 - R_j^2) \frac{a_j R^n}{2\pi\mathrm{i} R_j} \int_s \frac{\mathrm{d}t}{(t-R_j)(t-R^2/R_j)t^n} \\ &= \sum_{j=1}^{m} a_j \begin{cases} \Big(\dfrac{R_j}{R}\Big)^n, & n \geqslant 0, \\ \Big(\dfrac{R}{R_j}\Big)^n, & n \leqslant -1, \end{cases} \end{aligned}$$

thus

$$k_n = \sum_{j=1}^{m} a_j \Big(\frac{R_j}{R}\Big)^{|n|}, \quad n = 0, \pm 1, \pm 2, \dots . \tag{19}$$

It is obvious that k_n^* vanishes at the infinity just as fast as k_n does, therefore $k^*(\vartheta)$ is an analytic function. The solution of the equation (11) can be obtained by multiplying (18) by $\mathrm{e}^{in\vartheta}$ and summing over the interval $(-\infty, \infty)$:

$$v(\vartheta) = F(\vartheta) + \frac{1}{2\pi} \int_0^{2\pi} k^*(\vartheta - \theta) F(\theta)\,\mathrm{d}\theta, \tag{20}$$

where on the basis of (19) one gets:

$$k^*(\vartheta) = \sum_{n=-\infty}^{\infty} \frac{k_n}{1-k_n} \mathrm{e}^{in\vartheta} = \sum_{n=-\infty}^{\infty} \frac{\sum\limits_{j=1}^{m} a_j \left(\frac{R_j}{R}\right)^{|n|} \mathrm{e}^{in\vartheta}}{1 - \sum\limits_{j=1}^{m} a_j \left(\frac{R_j}{R}\right)^{|n|}}. \tag{21}$$

Taking into account (10) and substituting $u_0(R,\theta)$ into the Poisson formula (8) for the function $v(\theta)$ defined from the formula (20), one obtains unknown harmonic function $u_0(r,\vartheta)$, $0 < r < R$, expressed as

$$\begin{aligned} &u_0(r,\vartheta) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\vartheta - \theta)} \left(F(\theta) + \frac{1}{2\pi} \int_0^{2\pi} k^*(\theta - \varphi) F(\varphi) \,\mathrm{d}\varphi \right) \mathrm{d}\theta, \end{aligned} \tag{22}$$

where $F(\theta)$ is the function defined by the formula (4). Thus, on the basis of (22), (3) and (5) the solution of the problem (1)–(2) can be represented as follows:

$$\begin{aligned} u(r,\vartheta) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\vartheta - \theta)} \left(F(\theta) + \frac{1}{2\pi} \int_0^{2\pi} k^*(\theta - \varphi) F(\varphi) \,\mathrm{d}\varphi \right) \mathrm{d}\theta \\ &\quad + \frac{1}{4\pi} \int_0^{2\pi} \int_0^{R} \ln(r^2 + \varrho^2 - 2r\varrho\cos(\vartheta - \theta)) g(\varrho,\theta) \varrho \,\mathrm{d}\varrho \,\mathrm{d}\theta. \end{aligned}$$

□

Remark. Since $k^*(\vartheta)$ is an analytic function, the function $v(\vartheta)$ obtained by the formula (20) will belong to the same class as $F(\vartheta)$ and the formula (20) is true not only for continuous $F(\vartheta)$, but also for an integrable function $F(\vartheta)$.

1.2. The second problem.

Consider the non-local problem for the Poisson equation in the disc:

$$\Delta u = g(r,\vartheta), \quad r\mathrm{e}^{\mathrm{i}\vartheta} \in D, \tag{23}$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=R} + \alpha u(R,\vartheta) = \sum_{k=1}^{m} \beta_k u(R_k,\vartheta) + f(\vartheta), \quad 0 \leqslant \vartheta < 2\pi, \tag{24}$$

where Δ, D, S, R and R_k $(k = 1,\dots,m)$ are defined as in the first problem, $f \in C^2(S)$, $g \in C^1(D)$ are given functions, α, β_k $(k = 1,\dots,m)$ are given real numbers.

By a classical solution $u(r,\vartheta)$ of the problem (23)–(24) we mean a function $u(r,\vartheta)$ of class $C^2(D) \cap C^1(\overline{D})$ satisfying all the conditions of the problem (23)–(24).

Theorem 2. *Let* $f \in C^2(S)$, $g \in C^1(D)$ *and* $\alpha \neq \sum_{k=1}^{m} \beta_k$,

$$1 - k_n \neq 0, \quad n = 0, \pm 1, \pm 2, \ldots$$

where

$$k_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\vartheta) \mathrm{e}^{-in\vartheta} \, \mathrm{d}\vartheta,$$

$$k(\vartheta) = \alpha R \ln(2R^2(1 - \cos\vartheta)) - \sum_{k=1}^{m} \beta_k R \ln(R^2 + R_k^2 - 2RR_k \cos\vartheta).$$

Then there exists a unique classical solution of the problem (23)–(24), *which is represented as follows:*

$$u(r, \vartheta) = -\frac{R}{2\pi} \int_0^{2\pi} \ln(R^2 + r^2 - 2Rr\cos(\vartheta - \theta)) w(\theta) \, \mathrm{d}\theta + c + u_1(r, \vartheta),$$

where

$$u_1(r, \vartheta) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^R \ln(r^2 + \varrho^2 - 2r\varrho\cos(\vartheta - \theta)) g(\varrho, \theta) \varrho \, \mathrm{d}\varrho \, \mathrm{d}\theta, \tag{25}$$

$$w(\vartheta) = F(\vartheta) + \frac{1}{2\pi} \int_{-\pi}^{\pi} k^*(\vartheta - \theta) F(\theta) \, \mathrm{d}\theta + \Big(\sum_{k=1}^{m} \beta_k - \alpha \Big) c \Big(1 + \frac{k_0}{1 - k_0} \Big),$$

$$k^*(\vartheta) = \sum_{n=-\infty}^{\infty} \frac{k_n}{1 - k_n} \mathrm{e}^{in\vartheta},$$

$$F(\vartheta) = -\frac{\partial u_1}{\partial r}\Big|_{r=R} - \alpha u_1(R, \vartheta) + \sum_{k=1}^{m} \beta_k u_1(R_k, \vartheta) + f(\vartheta),$$

$$c = \frac{-\Big(\int_{-\pi}^{\pi} \big(F(\vartheta) + \frac{1}{2\pi} \int_{-\pi}^{\pi} k^*(\vartheta - \theta) F(\theta) \, \mathrm{d}\theta \big) \, \mathrm{d}\vartheta \Big)}{\Big(\sum_{k=1}^{m} \beta_k - \alpha \Big) 2\pi \big(1 + k_0 (1 - k_0)^{-1} \big)}.$$

P r o o f. To solve this problem one cannot use the Poisson formula, since for determining the value $\partial u_1 / \partial r \big|_{r=R}$ the boundary value of the kernel obtained as a result of differentiation of the integral kernel has a second order singularity at $\theta = \vartheta$. Therefore for investigating this problem it is more convenient to use Dini's formula [5], which gives the solution of Neumann's problem to the Laplace equation.

As in the case of the solution of the first problem, the general solution of the equation (23) is represented as follows:

$$u = u_0 + u_1, \tag{26}$$

where u_1 is a particular solution, which can be taken as (3), and u_0 is a harmonic function satisfying the following problem:

$$\Delta u_0 = 0, \quad r e^{i\vartheta} \in D, \tag{27}$$

$$\frac{\partial u_0}{\partial r}\Big|_{r=R} + \alpha u_0(R,\vartheta) - \sum_{k=1}^{m} \beta_k u_0(R_k,\vartheta) = F(\vartheta), \quad 0 \leqslant \vartheta < 2\pi, \tag{28}$$

where $F(\vartheta)$ is given by the formula (25). If $f \in C^2(S)$, $g \in C^1(D)$, then $F(\vartheta) \in C^2(S)$. By virtue of (28) the solvability condition of the Neumann problem will be

$$\int_0^{2\pi} \Big(\alpha u_0(R,\vartheta) - \sum_{k=1}^{m} \beta_k u_0(R_k,\vartheta)\Big)\,d\vartheta = \int_0^{2\pi} F(\vartheta)\,d\vartheta. \tag{29}$$

Provided (29), $u_0(r,\vartheta)$ is represented by Dini's formula [5]:

$$u_0(r,\vartheta) = -\frac{R}{2\pi}\int_0^{2\pi} \ln(R^2 + r^2 - 2Rr\cos(\vartheta-\theta))w(\theta)\,d\theta + c, \tag{30}$$

where

$$w(\vartheta) = \frac{\partial u_0}{\partial r}\Big|_{r=R}, \quad c = \text{const}.$$

Thus the function u_0 defined by the formula (30) represents the solution of the problem (27), (28).

Substituting into the condition (28) the value of u_0 defined by the formula (30) one obtains an integral equation with respect to $w(\vartheta)$:

$$w(\vartheta) - \frac{1}{2\pi}\int_0^{2\pi} k(\vartheta-\theta)w(\theta)\,d\theta = F(\vartheta) + \widetilde{c}, \quad 0 \leqslant \vartheta < 2\pi, \tag{31}$$

where

$$\widetilde{c} = c\Big(\sum_{k=1}^{m} \beta_k - \alpha\Big), \tag{32}$$

$$k(\gamma) = \alpha R \ln(2R^2(1-\cos\gamma)) - \sum_{k=1}^{m} \beta_k R \ln(R^2 + R_k^2 - 2RR_k\cos\gamma),$$

where $k(\gamma)$ is a periodic function with period 2π, which is continuous except at $\gamma = 0$, where it has the logarithmic singularity. Since $w(\theta)$ is a periodic function, the equation (31) can be expressed as follows:

$$w(\vartheta) - \frac{1}{2\pi}\int_{-\pi}^{\pi} k(\vartheta-\theta)w(\theta)\,d\theta = F(\vartheta) + \widetilde{c}, \quad -\pi \leqslant \vartheta < \pi. \tag{33}$$

Applying the discrete Fourier transform to the equation (33), one gets

$$w_n(1-k_n) = F_n + \widetilde{c}_n, \quad n = 0, \pm 1, \pm 2, \dots \tag{34}$$

here also

$$\begin{aligned} w_n &= \frac{1}{2\pi}\int_0^{2\pi} w(\vartheta)\mathrm{e}^{-in\vartheta}\,\mathrm{d}\vartheta, \\ k_n &= \frac{1}{2\pi}\int_0^{2\pi} k(\vartheta)\mathrm{e}^{-in\vartheta}\,\mathrm{d}\vartheta, \\ F_n &= \frac{1}{2\pi}\int_0^{2\pi} F(\vartheta)\mathrm{e}^{-in\vartheta}\,\mathrm{d}\vartheta, \\ \widetilde{c}_n &= \begin{cases} \widetilde{c}, & n = 0, \\ 0, & n \neq 0. \end{cases} \end{aligned}$$

The last equation is solvable for any $F_n + \widetilde{c}_n$ only when

$$1 - k_n \neq 0, \quad n = 0, \pm 1, \pm 2, \dots.$$

In this case the equation (34) has a unique solution which is represented as follows

$$w_n = \frac{F_n + \widetilde{c}_n}{1-k_n}, \quad n = 0, \pm 1, \pm 2, \dots. \tag{35}$$

As in the previous problem, (35) can be expressed as

$$w_n = F_n + \widetilde{c}_n + k_n^*(F_n + \widetilde{c}_n), \quad n = 0, \pm 1, \pm 2, \dots \tag{36}$$

where

$$k_n^* = \frac{k_n}{1-k_n}.$$

Hence, similarly as above, the solution of the problem (33) can be written as follows:

$$w(\vartheta) = F(\vartheta) + \frac{1}{2\pi}\int_{-\pi}^{\pi} k^*(\vartheta-\theta)F(\theta)\,\mathrm{d}\theta + \Big(\sum_{k=1}^{m}\beta_k - \alpha\Big)c\Big(1 + \frac{k_0}{1-k_0}\Big) \tag{37}$$

where

$$k^*(\vartheta) = \sum_{n=-\infty}^{\infty} \frac{k_n}{1-k_n}\,\mathrm{e}^{in\vartheta}.$$

Proceeding from (32) one obtains

$$k_n = \frac{\alpha R}{2\pi}\int_{-\pi}^{\pi} \ln(2R^2(1-\cos\theta))\mathrm{e}^{-in\theta}\,\mathrm{d}\theta - \sum_{k=1}^{m}\frac{\beta_k R}{2\pi}\int_{-\pi}^{\pi}\ln(R^2+R_k^2-2RR_k\cos\theta)\mathrm{e}^{-in\theta}\,\mathrm{d}\theta.$$

Let us calculate the first integral as shown below:

$$\int_{-\pi}^{\pi}\ln(2R^2(1-\cos\theta))\mathrm{e}^{-in\theta}\,\mathrm{d}\theta = \ln R^2\int_{-\pi}^{\pi}\mathrm{e}^{-in\theta}\,\mathrm{d}\theta + \int_{-\pi}^{\pi}2\ln\left|2\sin\frac{\theta}{2}\right|\mathrm{e}^{-in\theta}\,\mathrm{d}\theta = \begin{cases} -2\pi/n, & n=1,2,\dots,\\ 2\pi\ln R^2, & n=0,\\ 2\pi/n, & n=-1,-2,\dots. \end{cases}$$

Let us calculate the second integral when $n=0$:

$$\int_{-\pi}^{\pi}\ln(R^2+R_k^2-2RR_k\cos\theta)\mathrm{e}^{-in\theta}\,\mathrm{d}\theta = 2\pi\ln R^2.$$

Introduce the following notation:

$$t = \mathrm{e}^{\mathrm{i}\theta}R,$$

then

$$\cos\theta = \frac{t^2+R^2}{2Rt},\quad \sin\theta = \frac{t^2-R^2}{2\mathrm{i}Rt}.$$

Let us calculate the second integral when $n\neq 0$:

$$\int_{-\pi}^{\pi}\ln(R^2+R_k^2-2RR_k\cos\theta)\mathrm{e}^{-in\theta}\,\mathrm{d}\theta = \frac{2RR_k}{in}\int_{-\pi}^{\pi}\frac{\sin\theta\mathrm{e}^{-in\theta}\,\mathrm{d}\theta}{R^2+R_k^2-2RR_k\cos\theta} = \frac{R^n}{\mathrm{i}n}\int_S\frac{(t^2-R^2)\,\mathrm{d}t}{t^{n+1}(t-R_k)(t-R^2/R_k)}.$$

According to the residue theorem one gets

$$\int_{-\pi}^{\pi}\ln(R^2+R_k^2-2RR_k\cos\theta)\mathrm{e}^{-in\theta}\,\mathrm{d}\theta = \frac{R^n}{n}\begin{cases} -\dfrac{\left(\dfrac{R^2}{R_k}\right)^2-R^2}{\left(\dfrac{R^2}{R_k}\right)^{n+1}\left(\dfrac{R^2}{R_k}-R_k\right)}2\pi, & n\geqslant 1,\\ \dfrac{R_k^2-R^2}{R_k^n\left(R_k-\dfrac{R^2}{R_k}\right)}2\pi, & n\leqslant -1. \end{cases}$$

We have found that the integral vanishes at the infinity. Therefore k_n^* is represented as a sum of two summands. The first component vanishes at the infinity as $1/n$, and the second one vanishes at the infinity to a higher order. Therefore the kernel $k^*(\vartheta)$ is continuous everywhere except at the point $\vartheta = 0$, where it has a logarithmic singularity. As is known, the convolution of an integrable function with a continuous function is continuous. Proceeding from that, since $F(\vartheta)$ is continuous, the function defined by the formula (37) is continuous, which means that $w(\vartheta)$ is continuous as well.

It is possible to choose the constant c so that $\int_{-\pi}^{\pi} w(\theta)\,\mathrm{d}\theta = 0$. On the basis of (37) we obtain:

$$c = \frac{-\Bigl(\int_{-\pi}^{\pi}\bigl(F(\vartheta) + \frac{1}{2\pi}\int_{-\pi}^{\pi} k^*(\vartheta-\theta)F(\theta)\,\mathrm{d}\theta\Bigr)\,\mathrm{d}\vartheta\bigr)}{\Bigl(\sum_{k=1}^{m}\beta_k - \alpha\Bigr)2\pi\bigl(1 + k_0(1-k_0)^{-1}\bigr)}. \tag{38}$$

Since

$$1 - k_n \neq 0, \quad n = 0, \pm 1, \pm 2, \ldots$$

we have $\bigl(1 + k_0(1-k_0)^{-1}\bigr) \neq 0$.

Thus, one obtains the solution of the problem (23), (24):

$$u(r,\vartheta) = -\frac{R}{2\pi}\int_0^{2\pi} \ln(R^2 + r^2 - 2Rr\cos(\vartheta-\theta))w(\theta)\,\mathrm{d}\theta + c + u_1(r,\vartheta).$$

□

References

[1] *A. Bitsadze and A. Samarski*: On some simplest generalizations of linear elliptic problems. Dokl. Akad. Nauk SSSR *185* (1969), 739–740. (In Russian.)

[2] *A. Bitsadze*: Boundary Value Problems for Elliptic Equations of Second Order. Nauka, Moscow, 1986. (In Russian.)

[3] *D. Gordeziani*: On solvability of some boundary value problems for one variant of equations of thin shells. Dokl. Akad. Nauk SSSR *215* (1974), 1289–1292. (In Russian.)

[4] *D. Gordeziani and T. Z. Djioev*: The generalization of Bitsadze-Samarski problem with reference to the problems of baroclinic sea dynamics. Outlines on Physics and Chemistry of Waters of the Black Sea. IO ANSSSR, Moscow, 1978. (In Russian.)

[5] *U. Dini*: Sulla integrazione della equazione $\Delta^2 u = 0$. Brioschi Ann. *VI* (1873), 305–345. (In Italian.)

[6] *E. Obolashvili*: Solution of nonlocal problems in plane elasticity theory. Current Problems of Mathematical Physics, Vol. 2. Transactions of All-Union Symposium. Gos. Univ., Tbilisi, 1987, pp. 295–302, 394. (In Russian.)

[7] *E. Obolashvili*: Nonlocal problems for some partial differential equations. Complex Variables Theory Appl. *19* (1992), 71–79.

[8] *E. Obolashvili*: P.D.E. in Clifford Analysis. Longman, Addison Wesley, 1998.

[9] *O. Sjöstrand*: Sur une equation aux derivées partielles due type composite. Ark. Mat. Astron. Fys. *25A* (1937), no. 21, 1–11; *26A* (1938), no. 1, 1–10.

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