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COMPLETELY CONTINUOUS FUNCTIONS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, after giving the basic results related to the product of functions and the graph of functions in intuitionistic fuzzy topological spaces, we introduce and study the concept of fuzzy completely continuous functions between intuitionistic fuzzy topological spaces.

Keywords: intuitionistic fuzzy set, intuitionistic fuzzy topological space, fuzzy completely continuous function, fuzzy near compactness, fuzzy product space

MSC 2000: 54A40, 54D20

1. Introduction

In [1], [2], Atanassov introduced the fundamental concept of an intuitionistic fuzzy set. Coker in [4], [5] introduced the notion of an intuitionistic fuzzy topological space, fuzzy continuity, fuzzy near compactness and some other related concepts. Completely continuous functions and results related to the product in fuzzy topological spaces were introduced in [7] and [3], respectively. In this paper, some of results related to the product of functions and the graph of functions are obtained in intuitionistic fuzzy topological spaces. Mainly we introduce and study completely continuous functions between intuitionistic fuzzy topological spaces. Some counterexamples are given and also, Theorem 3.14 in [4] is strengthened.

2. Preliminaries

Throughout this section, we shall present the fundamental definitions and results of intuitionistic fuzzy sets as given by Atanassov [2] and Coker [5].

Definition 1 ([2]). Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS, for short) A is an object having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \colon x \in X\}$ where the functions $\mu_A \colon X \to I$ and $\gamma_A \colon X \to I$ denote respectively the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, and $0 \leqslant \mu_A(x) + \gamma_A(x) \leqslant 1$ for each $x \in X$.

Obviously, every fuzzy set A on a nonempty set X is an IFS having the form $A = \{\langle x, \mu_A(x), 1 - \gamma_A(x)\rangle \colon x \in X\}$.

Definition 2 ([2]). Let X be a nonempty set and let the IFS's A and B be in the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \colon x \in X\}$, $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle \colon x \in X\}$ and let $\{A_j \colon j \in J\}$ be an arbitrary family of IFS's in X. Then

(i) $A \leqslant B$ iff $\forall x \in X$ $[\mu_A(x) \leqslant \mu_B(x)$ and $\gamma_A(x) \geqslant \gamma_B(x)]$;
(ii) $\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle \colon x \in X\}$;
(iii) $\bigcap A_j = \{\langle x, \bigwedge \mu_{A_j}(x), \bigvee \gamma_{A_j}(x)\rangle \colon x \in X\}$;
(iv) $\bigcup A_j = \{\langle x, \bigvee \mu_{A_j}(x), \bigwedge \gamma_{A_j}(x)\rangle \colon x \in X\}$;
(v) $\underset{\sim}{1} = \{\langle x, 1, 0\rangle \colon x \in X\}$ and $\underset{\sim}{0} = \{\langle x, 0, 1\rangle \colon x \in X\}$;
(vi) $\overline{\overline{A}} = A$, $\overline{\underset{\sim}{0}} = \underset{\sim}{1}$ and $\overline{\underset{\sim}{1}} = \underset{\sim}{0}$.

Definition 3 ([5]). Let X and Y be two nonempty sets and $f \colon X \to Y$ a function.

(i) If $B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle \colon y \in Y\}$ is an IFS in Y, then the preimage of B under f is denoted and defined by $f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}\gamma_B)(x)\rangle \colon x \in X\}$.
(ii) If $A = \{\langle x, \lambda_A(x), \upsilon_A(x)\rangle \colon x \in X\}$ is an IFS in X, then the image of A under f is denoted and defined by $f(A) = \{\langle y, f(\lambda_A)(y), f_-(\upsilon_A)(y)\rangle \colon y \in Y\}$ where $f_-(\upsilon_A) = 1 - f(1 - \upsilon_A)$.

In (i), (ii), since μ_B, γ_B, λ_A, υ_A are fuzzy sets, we explain that

$$f^{-1}(\mu_B)(x) = \mu_B(f(x)),$$

and

$$f(\lambda_A)(y) = \begin{cases} \sup \lambda_A(x) & \text{if } f^{-1}(y) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4. An intuitionistic fuzzy topology (IFT, for short) on a nonempty set X is a family Ψ of IFS's in X satisfying the following axioms:

(i) $\underset{\sim}{0}, \underset{\sim}{1} \in \Psi$;
(ii) $A_1 \cap A_2 \in \Psi$ for any $A_1, A_2 \in \Psi$;
(iii) $\bigcup A_j \in \Psi$ for any $\{A_j\colon\ j \in J\} \subseteq \Psi$.

In this case the pair (X, Ψ) is called an intuitionistic fuzzy topological space (IFTS, for short) and each IFS in Ψ is known as an intuitionistic fuzzy open set (IFOS, for short) in X.

Definition 5. The complement $\overline{A}$ of IFOS A in IFTS (X, Ψ) is called an intuitionistic fuzzy closed set (IFCS, for short).

Definition 6. Let (X, Ψ) be an IFTS and $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ an IFS in X. Then the fuzzy interior and the fuzzy closure of A are defined by

$\mathrm{cl}(A) = \bigcap\{K\colon\ K$ is an IFCS in X and $A \leqslant K\}$ and
$\mathrm{int}(A) = \bigcup\{G\colon\ G$ is an IFOS in X and $G \leqslant K\}$.

Definition 7. An IFS A of all IFTS X is called
(i) an intuitionistic fuzzy regular open set (IFROS, for short) of X if $\mathrm{int}\,\mathrm{cl}(A) = A$.
(ii) an intuitionistic fuzzy regular closed set (IFRCS, for short) of X if $\mathrm{cl}\,\mathrm{int}(A) = A$.

Obviously, an IFS A in an IFTS X is IFROS iff $\overline{A}$ is IFRCS.

Definition 8. Let (X, Ψ) and (Y, Φ) be two IFTS's and $f\colon\ X \to Y$ a function. Then
(i) f is fuzzy continuous iff the preimage of each IFS in Φ is an IFS in Ψ ([5]);
(ii) f is fuzzy strongly continuous iff for each IFS A in X, $f(\mathrm{cl}(A)) \leqslant f(A)$ ([4]);
(iii) f is fuzzy almost open iff the image of each IFROS in X is an IFOS in Y ([4]).

Definition 9. An IFTS (X, Ψ) is called fuzzy nearly compact iff every fuzzy open cover of X has a finite subcollection such that the interior of closures of IFS's in this subcollection covers X.

3. Basic results

Definition 10. A subfamily β of IFTS (X, Ψ) is called a base for Ψ if each IFS of Ψ is a union of some members of β.

Definition 11. Let X, Y be nonempty sets and $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$, $B = \langle y, \mu_B(y), \gamma_B(y)\rangle$ IFS's of X and Y, respectively. Then $A \times B$ is an IFS of $X \times Y$ defined by

$$(A \times B)(x, y) = \langle (x, y), \min(\mu_A(x), \mu_B(y)), \max(\gamma_A(x), \gamma_B(y))\rangle.$$

Notice that

$$\underset{\sim}{1} - (A \times B)(x, y) = \langle (x, y), \max(\gamma_A(x), \gamma_B(y)), \min(\mu_A(x), \mu_B(y)) \rangle.$$

Lemma 12. *If A is an IFS of X and B is an IFS of Y, then*
(i) $(A \times \underset{\sim}{1}) \cap (\underset{\sim}{1} \times B) = A \times B$;
(ii) $(A \times \underset{\sim}{1}) \cup (\underset{\sim}{1} \times B) = \underset{\sim}{1} - \overline{A} \times \overline{B}$;
(iii) $\underset{\sim}{1} - A \times B = (\overline{A} \times \underset{\sim}{1}) \cup (\underset{\sim}{1} \times \overline{B})$.

Proof. Let $A = \langle x, \mu_A(x), \gamma_A(x) \rangle$, $B = \langle y, \mu_B(y), \gamma_B(y) \rangle$.

(i) Since $A \times \underset{\sim}{1} = \langle x, \min(\mu_A, \underset{\sim}{1}), \max(\gamma_A, \underset{\sim}{0}) \rangle = \langle x, \mu_A(x), \gamma_A(x) \rangle = A$ and similarly $\underset{\sim}{1} \times B = \langle y, \min(\underset{\sim}{1}, \mu_B), \max(\underset{\sim}{0}, \gamma_B) \rangle = B$, we have

$$\begin{aligned}(A \times \underset{\sim}{1}) \cap (\underset{\sim}{1} \times B) &= A(x) \cap B(y) \\ &= \langle (x, y), \mu_A(x) \wedge \mu_B(y), \gamma_A(x) \vee \gamma_B(y) \rangle = A \times B.\end{aligned}$$

(ii) Similarly to (i).

(iii) Obvious by putting A, B instead of $\overline{A}$, $\overline{B}$ in (ii). □

Definition 13. Let (X, Ψ) and (Y, Φ) be IFTS's. The intuitionistic fuzzy product space (IFPTS, for short) of (X, Ψ) and (Y, Φ) is the cartesian product $X \times Y$ of IFS's X and Y together with the IFT ξ of $X \times Y$ which is generated by the family $\{P_1^{-1}(A_i), P_2^{-1}(B_j) \colon A_i \in \Psi, B_j \in \Phi$ and P_1, P_2 are projections of $X \times Y$ onto X and Y, respectively$\}$ (i.e. the family $\{P_1^{-1}(A_i), P_2^{-1}(B_j) \colon A_i \in \Psi, B_j \in \Phi\}$ is a subbase for IFT ξ of $X \times Y$).

Remark 1. In the above definition, since $P_1^{-1}(A_i) = A_i \times \underset{\sim}{1}$ and $P_2^{-1}(B_j) = \underset{\sim}{1} \times B_j$ and $A_i \times \underset{\sim}{1} \cap \underset{\sim}{1} \times B_j = A_i \times B_j$, the family $\beta = \{A_i \times B_j \colon A_i \in \Psi, \ B_j \in \Phi\}$ forms a base for IFPTS ξ of $X \times Y$.

Definition 14. Let $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$. The product $f_1 \times f_2 \colon X_1 \times X_2 \to Y_1 \times Y_2$ is defined by

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)) \quad \forall (x_1, x_2) \in X_1 \times X_2.$$

Definition 15. Let $f \colon X \to Y$ be a function. The graph $g \colon X \to X \times Y$ of f is defined by

$$g(x) = (x, f(x)) \quad \forall x \in X.$$

Lemma 16. *Let $f_i\colon X_i \to Y_i$ $(i = 1,2)$ be functions and A, B IFS's of Y_1, Y_2, respectively, then*

$$(f_1 \times f_2)^{-1} = f_1^{-1}(A) \times f_2^{-1}(B).$$

P r o o f. Let $A = \langle x_1, \mu_A(x_1), \gamma_A(x_1)\rangle$, $B = \langle x_2, \mu_B(x_2), \gamma_B(x_2)\rangle$. For each $(x_1, x_2) \in X_1 \times X_2$, we have

$$\begin{aligned}(f_1 \times f_2)^{-1}(A,B)(x_1,x_2) &= (A \times B)(f_1 \times f_2)(x_1,x_2)\\ &= (A \times B)(f_1(x_1), f_2(x_2))\\ &= \langle (f_1(x_1), f_2(x_2)), \min(\mu_A(f_1(x_1)), \mu_B(f_2(x_2))),\\ &\qquad \max(\gamma_A(f_1(x_1)), \gamma_B(f_2(x_2)))\rangle\\ &= \langle (x_1,x_2), \min(f_1^{-1}(\mu_A)(x_1), f_2^{-1}(\mu_B)(x_2)),\\ &\qquad \max(f_1^{-1}(\gamma_A)(x_1), f_2^{-1}(\gamma_B)(x_2))\rangle\\ &= (f_1^{-1}(A) \times f_2^{-1}(B))(x_1,x_2).\end{aligned}$$

□

Lemma 17. *Let $g\colon X \to X \times Y$ be the graph of a function $f\colon X \to Y$. If A is an IFS of X and B is an IFS of Y, then*

$$g^{-1}(A \times B)(x) = (A \cap f^{-1}(B))(x).$$

P r o o f. Let $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$, $B = \langle x, \mu_B(x), \gamma_B(x)\rangle$. For each $x \in X$, we have

$$\begin{aligned}g^{-1}(A \times B)(x) &= (A \times B)g(x) = (A \times B)(x, f(x))\\ &= \langle (x, f(x)), \min(\mu_A(x), \mu_B(f(x)), \max(\gamma_A(x), \gamma_B(f(x))\rangle\\ &= \langle (x, f(x)), \min(\mu_A(x), f^{-1}(\mu_B)(x)), \max(\gamma_A(x), f^{-1}(\gamma_B)(x))\rangle\\ &= (A \cap f^{-1}(B))(x)\end{aligned}$$

□

Lemma 18. *Let A, B, C and D be IFS's in X. Then*

$$A \leqslant B,\ C \leqslant D \Rightarrow A \times C \leqslant B \times D.$$

P r o o f. Let $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$, $B = \langle x, \mu_B(x), \gamma_B(x)\rangle$, $C = \langle x, \mu_C(x), \gamma_C(x)\rangle$ and $D = \langle x, \mu_D(x), \gamma_D(x)\rangle$ be IFS's. Since $A \leqslant B \Rightarrow \mu_A \leqslant \mu_B$, $\gamma_A \geqslant \gamma_B$ and also $C \leqslant D \Rightarrow \mu_C \leqslant \mu_D$, $\gamma_C \geqslant \gamma_D$, we have $\min(\mu_A, \mu_C) \leqslant \min(\mu_B, \mu_D)$ and $\max(\gamma_A, \gamma_C) \geqslant \max(\gamma_B, \gamma_D)$. Hence the result. □

Lemma 19. *Let A and B be* IFCS*'s in IFTS's X and Y, respectively. Then $A \times B$ is an* IFCS *in the* IFPTS *of $X \times Y$.*

Proof. Let $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$, $B = \langle y, \mu_B(y), \gamma_B(y)\rangle$. From Lemma 12 we have $(\underset{\sim}{1} - A \times B)(x, y) = (\overline{A} \times \underset{\sim}{1}) \cup (\underset{\sim}{1} \times \overline{B})(x, y)$. Since $\overline{A} \times \underset{\sim}{1}$ and $\underset{\sim}{1} \times \overline{B}$ are IFOS's in X and Y respectively, hence $\overline{A} \times \underset{\sim}{1} \cup \underset{\sim}{1} \times \overline{B}$ is IFOS of $X \times Y$. Hence $\underset{\sim}{1} - A \times B$ is an IFOS of $X \times Y$ and consequently $A \times B$ is an IFCS of $X \times Y$. $\square$

Remark 2. In ordinary topology, it is well known that the closure of the product is the product of the closures, while this property is not true in fuzzy setting (see [3]). The next example shows that

(i) this property is not true in intuitionistic fuzzy setting, either.

(ii) If A, B are IFS's of X and C, D are IFS's of Y such that $A \times \underset{\sim}{1} \cup \underset{\sim}{1} \times C \geqslant B \times D$, then it need not be true that $A \geqslant B$ or $C \geqslant D$. However, $A \geqslant B$, $C \geqslant D$ implies that $A \times \underset{\sim}{1} \cup \underset{\sim}{1} \times C \geqslant B \times D$.

Example 20. Let $X = Y = I = [0, 1]$ and consider the IFS's $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$, $B = \langle x, \mu_B(x), \gamma_B(x)\rangle$, $C = \langle x, \mu_C(x), \gamma_C(x)\rangle$ and $D = \langle x, \mu_D(x), \gamma_D(x)\rangle$ as follows:

$$\mu_A(x) = \begin{cases} -\frac{4}{3}x + 1 & \text{if } 0 \leqslant x \leqslant \frac{3}{4}, \\ 0 & \text{if } \frac{3}{4} \leqslant x \leqslant 1, \end{cases}$$

$$\gamma_A(x) = \begin{cases} \frac{x}{6} & \text{if } 0 \leqslant x \leqslant \frac{3}{4}, \\ \frac{1}{8} & \text{if } \frac{3}{4} \leqslant x \leqslant 1, \end{cases}$$

$$\mu_B(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \frac{3}{4}, \\ 4x - 3 & \text{if } \frac{3}{4} \leqslant x \leqslant 1, \end{cases}$$

$$\gamma_B(x) = \frac{3}{4} \quad \text{if } 0 \leqslant x \leqslant 1,$$

$$\mu_C(x) = \begin{cases} \frac{1}{6} & \text{if } x = \frac{2}{3}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\gamma_C(x) = \begin{cases} \frac{1}{6} & \text{if } x = \frac{2}{3}, \\ \frac{5}{6} & \text{otherwise}, \end{cases}$$

$$\mu_D(x) = \begin{cases} \frac{2}{5} & \text{if } x = \frac{4}{5}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\gamma_D(x) = \begin{cases} \frac{1}{5} & \text{if } x = \frac{4}{5}, \\ \frac{4}{5} & \text{otherwise}. \end{cases}$$

Then $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, \overline{A}\}$ and $\Phi = \{\underset{\sim}{0}, \underset{\sim}{1}, \overline{B}\}$ are IFTS's on X and Y, respectively. Now, we notice that

(i) $A \not\geqslant C$ and $B \not\geqslant D$;

(ii) $\mathrm{cl}(C) = \langle x, 1, 0\rangle = \underset{\sim}{1}$ in X,
$\mathrm{cl}(D) = \langle x, 1, 0\rangle = \underset{\sim}{1}$ in Y and
$\mathrm{cl}(C) \times \mathrm{cl}(D) = \langle (x, y), 1, 0\rangle = \underset{\sim}{1}$;

(iii) $(C \times D)(x, y) = \langle (x, y), \min(\mu_C, \mu_D), \max(\gamma_C, \gamma_D)\rangle = \langle (x, y), \mu_{C\times D}, \gamma_{C\times D}\rangle$ where $\mu_{C\times D} = \frac{1}{6}$ if $(x, y) = (\frac{2}{3}, \frac{4}{5})$; $\gamma_{C\times D} = \frac{1}{5}$ if $(x, y) = (\frac{2}{3}, \frac{4}{5})$ and $(A \times \underset{\sim}{1} \vee \underset{\sim}{1} \times B)(\frac{2}{3}, \frac{4}{5}) = (\underset{\sim}{1} - \overline{A} \times \overline{B})(\frac{2}{3}, \frac{4}{5}) = \langle (\frac{2}{3}, \frac{4}{5}), \max(\mu_A, \mu_B), \min(\gamma_A, \gamma_B)\rangle = \langle (\frac{2}{3}, \frac{4}{5}), \mu_{A\times B}, \gamma_{A\times B}\rangle$ where $\mu_{A\times B} = \frac{1}{5}$ if $(x, y) = (\frac{2}{3}, \frac{4}{5})$; $\gamma_{A\times B} = \frac{1}{8}$ if $(x, y) = (\frac{2}{3}, \frac{4}{5})$. Then we have $(A \times \underset{\sim}{1} \cup \underset{\sim}{1} \times B) \geqslant C \times D$;

(iv) $A \times \underset{\sim}{1} \cup \underset{\sim}{1} \times B$ is IFCS of $X \times Y$, then $\underset{\sim}{1} \neq A \times \underset{\sim}{1} \cup \underset{\sim}{1} \times B \geqslant \mathrm{cl}(C \times D)$, and since $\mathrm{cl}(C) \times \mathrm{cl}(D) = \underset{\sim}{1}$, we observe that $\mathrm{cl}(C \times D) \neq \mathrm{cl}(C) \times \mathrm{cl}(D)$.

Theorem 21. *If A and B are IFS's of IFTS's X and Y, respectively, then*

(i) $\mathrm{cl}(A) \times \mathrm{cl}(B) \geqslant \mathrm{cl}(A \times B)$;

(ii) $\mathrm{int}(A) \times \mathrm{int}(B) \leqslant \mathrm{int}(A \times B)$.

P r o o f. (i) Since $A \leqslant \mathrm{cl}(A)$ and $B \leqslant \mathrm{cl}(B)$, hence $A \times B \leqslant \mathrm{cl}(A) \times \mathrm{cl}(B)$ implies $\mathrm{cl}(A \times B) \leqslant \mathrm{cl}(\mathrm{cl}(A) \times \mathrm{cl}(B))$ and from Lemma 19 we have $\mathrm{cl}(A \times B) \leqslant \mathrm{cl}(A) \times \mathrm{cl}(B)$.

(ii) follows from (i) and the fact that $\underset{\sim}{1} - \mathrm{cl}(A) = \mathrm{int}(\underset{\sim}{1} - A)$. $\square$

Definition 22. Let (X, Ψ), (Y, Φ) be IFTS's and $A \in \Psi$, $B \in \Phi$. We say that (X, Ψ) is product related to (Y, Φ) if for any IFS's C of X and D of Y, whenever $\overline{A} \not\geqslant C$ and $\overline{B} \not\geqslant D) \Rightarrow (\overline{A} \times \underset{\sim}{1} \cup \underset{\sim}{1} \times \overline{B} \geqslant C \times D)$, there exist $A_1 \in \Psi$, $B_1 \in \Phi$ such that $\overline{A_1} \geqslant C$ or $\overline{B_1} \geqslant D$ and $\overline{A_1} \times \underset{\sim}{1} \cup \underset{\sim}{1} \times \overline{B_1} = \overline{A} \times \underset{\sim}{1} \cup \underset{\sim}{1} \times \overline{B}$.

Lemma 23. *For IFS's A_i's and B_j's of IFTS's X and Y, respectively, we have*

(i) $\bigwedge\{A_i, B_j\} = \min(\bigwedge A_i, \bigwedge B_j)$; $\quad \bigvee\{A_i, B_j\} = \max(\bigvee A_i, \bigvee B_j)$.

(ii) $\bigwedge\{A_i, \underset{\sim}{1}\} = (\bigwedge A_i) \times \underset{\sim}{1}$; $\quad \bigvee\{A_i, \underset{\sim}{1}\} = (\bigvee A_i) \times \underset{\sim}{1}$.

(iii) $\bigwedge\{\underset{\sim}{1} \times B_j\} = \underset{\sim}{1} \times (\bigwedge B_j)$; $\quad \bigvee\{\underset{\sim}{1} \times B_j\} = \underset{\sim}{1} \times (\bigvee B_j)$.

P r o o f. Obvious. $\square$

Theorem 24. *Let (X, Ψ) and (Y, Φ) be IFTS's such that X is product related to Y. Then for IFS's A of X and B of Y, we have*

(i) $\mathrm{cl}(A \times B) = \mathrm{cl}(A) \times \mathrm{cl}(B)$;

(ii) $\mathrm{int}(A \times B) = \mathrm{int}(A) \times \mathrm{int}(B)$.

P r o o f. (i) Since $\mathrm{cl}(A\times B)\leqslant \mathrm{cl}(A)\times \mathrm{cl}(B)$ (see Theorem 21) it is sufficient to show that $\mathrm{cl}(A\times B)\geqslant \mathrm{cl}(A)\times \mathrm{cl}(B)$. Let $A_i\in\Psi$ and $B_j\in\Phi$. Then

$$
\begin{aligned}
&\mathrm{cl}(A\times B)\\
&\quad=\Big\langle (x,y),\bigwedge\overline{\{A_i\times B_j\}}\colon\ \overline{\{A_i\times B_j\}}\geqslant A\times B,\\
&\qquad\bigvee\{A_i\times B_j\}\colon\ \{A_i\times B_j\}\leqslant A\times B\Big\rangle\\
&\quad=\Big\langle (x,y),\bigwedge(\overline{A_i}\times \underset{\sim}{1}\cup \underset{\sim}{1}\times\overline{B_j})\colon\ \overline{A_i}\times \underset{\sim}{1}\cup \underset{\sim}{1}\times\overline{B_j}\geqslant A\times B,\\
&\qquad\bigvee(A_i\times \underset{\sim}{1}\cap \underset{\sim}{1}\times B_j)\colon\ A_i\times \underset{\sim}{1}\cap \underset{\sim}{1}\times B_j\leqslant A\times B\Big\rangle\\
&\quad=\Big\langle (x,y),\bigwedge(\overline{A_i}\times \underset{\sim}{1}\cup \underset{\sim}{1}\times\overline{B_j})\colon\ \overline{A_i}\geqslant A\ \text{ or }\ \overline{B_j}\geqslant B,\\
&\qquad\bigvee(A_i\times \underset{\sim}{1}\cap \underset{\sim}{1}\times B_j)\colon\ A_i\leqslant A\ \text{ and }\ B_j\leqslant B\Big\rangle\\
&\quad=\Big\langle (x,y),\min\Big(\bigwedge\{\overline{A_i}\times \underset{\sim}{1}\cup \underset{\sim}{1}\times\overline{B_j}\colon\ \overline{A_i}\geqslant A\},\bigwedge\{\overline{A_i}\times \underset{\sim}{1}\vee \underset{\sim}{1}\times\overline{B_j}\colon\ \overline{B_j}\geqslant B\}\Big),\\
&\qquad\max\Big(\bigvee\{A_i\times \underset{\sim}{1}\cap \underset{\sim}{1}\times B_j\colon\ A_i\leqslant A\},\bigvee\{A_i\times \underset{\sim}{1}\cap \underset{\sim}{1}\times B_j\colon\ B_j\leqslant B\}\Big)\Big\rangle.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\Big\langle (x,y),\bigwedge\{\overline{A_i}\times \underset{\sim}{1}\cup \underset{\sim}{1}\times\overline{B_j}\colon\ \overline{A_i}\geqslant A\},\bigwedge\{\overline{A_i}\times \underset{\sim}{1}\cup \underset{\sim}{1}\times\overline{B_j}\colon\ \overline{B_j}\geqslant B\}\Big\rangle\\
&\qquad\geqslant\Big\langle (x,y),\bigwedge\{\overline{A_i}\times \underset{\sim}{1}\colon\ \overline{A_i}\geqslant A\},\bigwedge\{\underset{\sim}{1}\times\overline{B_j}\colon\ \overline{B_j}\geqslant B\}\Big\rangle\\
&\qquad=\Big\langle (x,y),\bigwedge\{\overline{A_i}\colon\ \overline{A_i}\geqslant A\}\times \underset{\sim}{1},\underset{\sim}{1}\times\bigwedge\{\overline{B_j}\colon\ \overline{B_j}\geqslant B\}\rangle\\
&\qquad=\langle (x,y),\mathrm{cl}(A)\times \underset{\sim}{1},\underset{\sim}{1}\times\mathrm{cl}(B)\Big\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\langle (x,y),\bigvee\{A_i\times \underset{\sim}{1}\cap \underset{\sim}{1}\times B_j\colon\ A_i\leqslant A,\bigvee\{A_i\times \underset{\sim}{1}\cap \underset{\sim}{1}\times B_j\colon\ B_j\leqslant B\}\Big\rangle\\
&\qquad\leqslant\Big\langle (x,y),\bigvee\{A_i\times \underset{\sim}{1}\colon\ A_i\leqslant A\},\bigvee\{\underset{\sim}{1}\times B_j\colon\ B_j\leqslant B\}\Big\rangle\\
&\qquad=\Big\langle (x,y),\bigvee\{A_i\colon\ A_i\leqslant A\}\times \underset{\sim}{1},\underset{\sim}{1}\times\bigvee\{B_j\colon\ B_j\leqslant B\}\Big\rangle\\
&\qquad=\langle (x,y),\mathrm{int}(A)\times \underset{\sim}{1},\underset{\sim}{1}\times\mathrm{int}(B)\rangle,
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathrm{cl}(A\times B)&\geqslant\langle (x,y),\min(\mathrm{cl}(A)\times \underset{\sim}{1},\underset{\sim}{1}\times\mathrm{cl}(B)),\max(\mathrm{int}(A)\times \underset{\sim}{1},\underset{\sim}{1}\times\mathrm{int}(B))\rangle\\
&=\langle (x,y),\min(\mathrm{cl}(A),\mathrm{cl}(B)),\max(\mathrm{int}(A),\mathrm{int}(B))\rangle\\
&=\ \mathrm{cl}(A)\times\mathrm{cl}(B).
\end{aligned}
$$

(ii) follows from (i). □

4. Completely continuous functions

Throughout this section (X, Ψ), (Y, Φ) will denote IFTS's and $f\colon X \to Y$ will denote a function.

Definition 25. $f\colon (X, \Psi) \to (Y, \Phi)$ is called a fuzzy completely continuous function if $f^{-1}(B)$ is an IFROS in X for each $B \in \Phi$.

Remark 3. For $f\colon X \to Y$ the following implications hold: fuzzy strongly continuous $\Rightarrow$ fuzzy completely continuous $\Rightarrow$ fuzzy continuous.

The following examples show that none of the above implications is reversible.

Example 26. A fuzzy continuous functions need not be fuzzy completely continuous.

Let $X = \{a, b\}$ and $A = \langle x, (\frac{a}{0.4}, \frac{b}{0.5}), (\frac{a}{0.4}, \frac{b}{0.4})\rangle$. Then the family $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, A\}$ of IFS's in X is an IFT on X and the identity function $f\colon (X, \Psi) \to (X, \Psi)$ is fuzzy continuous, but not fuzzy completely continuous. (Indeed, $f^{-1}(A) = A$ and $\operatorname{int}\operatorname{cl}(A) = \underset{\sim}{1} \neq A$.)

Remark 4. For an IFTS (X, Ψ) it is obvious that the identity function $f\colon (X, \Psi) \to (X, \Psi)$ is fuzzy strongly continuous iff Ψ is the intuitionistic fuzzy discrete topology on X (i.e., every IFS of X is IFOS).

Example 27. Let $X = \{a, b, c\}$ and $A = \langle x, (\frac{a}{0.3}, \frac{b}{0.2}, \frac{c}{0.1}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.6})\rangle$. Then the family $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, A\}$ of IFS's in X is an IFT on X, but it is not an intuitionistic fuzzy discrete topology. Also notice that if $A \in \Phi$ and $\operatorname{cl}(A) = \overline{A}$, $\operatorname{int}\operatorname{cl}(A) = \operatorname{int}(\overline{A}) = A$, then A is an IFROS in X. Hence the identity function $f\colon (X, \Psi) \to (X, \Psi)$ is fuzzy completely continuous but not fuzzy strongly continuous.

Theorem 28. *A function $f\colon X \to Y$ is fuzzy completely continuous iff $f^{-1}(B)$ is an IFRCS in X for each IFCS B in Y.*

P r o o f. Since $f^{-1}(\overline{B}) = \overline{(f^{-1}(B))}$ for any IFS B of Y, and a fuzzy set B is IFROS iff $\overline{B}$ is IFRCS, the proof is obvious. □

Theorem 29. *For any two fuzzy completely continuous functions $f_1, f_2\colon (X, \Psi) \to (Y, \Phi)$, the function $(f_1, f_2)\colon (X, \Psi) \to (Y \times Y, \Phi \times \Phi)$ is also a fuzzy completely continuous function, where $(f_1, f_2)(x) = (f_1(x), f_2(x))$ $\forall x \in X$.*

P r o o f. Let $A \times B$ be any IFOS in $Y \times Y$. Then

$$
\begin{aligned}
&(f_1, f_2)^{-1}(A \times B)(x) \\
&\qquad = (A \times B)(f_1(x), f_2(x)) \\
&\qquad = \langle x, \min(\mu_A(f_1(x)), \mu_B(f_2(x)), \max(\gamma_A(f_1(x)), \gamma_B(f_2(x))\rangle \\
&\qquad = \langle x, \min(f_1^{-1}(\mu_A)(x), f_2^{-1}(\mu_B)(x)), \max(f_1^{-1}(\gamma_A)(x), f_2^{-1}(\gamma_B)(x))\rangle \\
&\qquad = (f_1^{-1}(A) \cap f_2^{-1}(B))(x)
\end{aligned}
$$

By using the fuzzy complete continuity of f_1 and f_2, we find that $f_1^{-1}(A)$ and $f_2^{-1}(B)$ are IFROS in X and $f_1^{-1}(A) \cap f_2^{-1}(B)$ is also IFROS (see Theorem 2.8 (b) in [6]). Hence (f_1, f_2) is fuzzy completely continuous. □

Theorem 30. *If f is fuzzy completely continuous and g is fuzzy continuous, then $g \circ f$ is fuzzy completely continuous.*

P r o o f. Obvious. □

Corollary 31. *The composite of two fuzzy completely continuous functions is fuzzy completely continuous.*

Corollary 32. *Let $P_i\colon X_1 \times X_2 \to Y_i$ $(i = 1, 2)$ be the projection of $X_1 \times X_2$ into Y_i where X_1, X_2, Y_i are IFTS's. If $f\colon B \to B_1 \times B_2$ is a fuzzy completely continuous function, then $P_i \circ f$ is also a fuzzy completely continuous function.*

P r o o f. Since f is fuzzy completely continuous and P_i is fuzzy continuous, hence $P_i \circ f$ is fuzzy completely continuous by the above theorem. □

Theorem 33. *Let $f\colon (X, \Psi) \to (Y, \Phi)$ be a function and $g\colon X \to X \times Y$ the graph of the function f. Then f is fuzzy completely continuous if g is so.*

P r o o f. Let $B \in \Phi$, then $f^{-1}(B) = f^{-1}(\underset{\sim}{1} \times B) = \underset{\sim}{1} \cap f^{-1}(B) = g^{-1}(\underset{\sim}{1} \times B)$. Since B is IFOS in Y, $\underset{\sim}{1} \times B$ is IFOS in $X \times Y$. Also, the fact that g is fuzzy completely continuous implies that $g^{-1}(\underset{\sim}{1} \times B)$ is IFROS in X. Hence $f^{-1}(B)$ is IFROS in X and so f is fuzzy completely continuous. □

Theorem 34. *Let X, Y and Z be IFTS's. If $f\colon X \to Y$ is a fuzzy almost open and fuzzy completely continuous surjection function and $g\colon Y \to Z$ is a function such that $g \circ f$ is fuzzy completely continuous, hence g is fuzzy continuous.*

P r o o f. Let $B = \langle z, \mu_B, \gamma_B\rangle$ be any IFOS of Z. Since $g \circ f$ is fuzzy completely continuous, hence $(g \circ f)^{-1}(B)$ is IFROS of X. Since f is fuzzy almost open, hence $f((g \circ f)^{-1}(B)) = f(f^{-1}(g^{-1}(B))) = f^{-1}(B)$ is IFOS of Y. Hence f is fuzzy continuous. □

Remark 5. Since every fuzzy strongly continuous function is fuzzy completely continuous, the following theorem improves Theorem 3.14 in [4].

Theorem 35. *The image of a fuzzy nearly compact IFTS under a fuzzy completely continuous surjection function is fuzzy compact.*

Proof. Let $f\colon X \to Y$, and let $B = \{B_j\colon j \in J\}$ be a fuzzy open cover of Y, where $B_j = \langle y, \mu_{B_j}, \gamma_{B_j}\rangle$, $j \in J$. Then from fuzzy completely continuity of f it follows that $A = \{f^{-1}(B_j)\colon j \in J\}$ is a fuzzy regular open cover of X. Since X is fuzzy nearly compact, there exists a finite subfamily $\{B_j\colon j = 1,\dots,n\}$ such that $\bigvee\limits_{i=1}^{n} ff^{-1}(B_j) = \underset{\sim}{1}$. From the surjectivity of f we have

$$f\Bigl(\bigvee_{i=1}^{n} f^{-1}(B_j)\Bigr) = \bigvee_{i=1}^{n} ff^{-1}(B_j) = \bigvee_{i=1}^{n} B_j = f(\underset{\sim}{1}) = \underset{\sim}{1}.$$

Hence Y is fuzzy compact. □

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