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Evgenii Pustylnik

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A WAY OF ESTIMATING THE CONVERGENCE RATE OF THE FOURIER METHOD FOR PDE OF HYPERBOLIC TYPE

Evgeniy Pustylnik, Haifa

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Abstract. The Fourier expansion in eigenfunctions of a positive operator is studied with the help of abstract functions of this operator. The rate of convergence is estimated in terms of its eigenvalues, especially for uniform and absolute convergence. Some particular results are obtained for elliptic operators and hyperbolic equations.

Keywords: elliptic operators, eigenfunctions, Fourier series, hyperbolic equation

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1. Introduction

Operator methods for studying series convergence are based on the following simple idea. Let E, F be two Banach spaces and T a linear operator $T\colon E \to F$. If $u = \sum u_i$ (convergence in E) then $Tu = \sum Tu_i$ (convergence in F). Assume, in addition, that this operator T can be split into a product $T = T_1T_2$ where $T_1\colon E \to F$ and T_2 is bounded in E. Then the second operator T_2 can be used to establish some extra properties of the series $\sum Tu_i$ such as the rate of convergence.

As an example of such splitting, we may take a representation $T = \varphi(T)\psi(T)$ when $\varphi(t)\psi(t) = t$ $(t > 0)$. The operators $\varphi(T)$, $\psi(T)$ can be defined via a suitable operator calculus such as the Riesz calculus for self-adjoint operators in a Hilbert space or the calculus from [1] for positive operators in a Banach space. It makes sense to stress that these calculi are defined only for one space E and do not give the answer what functions $\varphi(T)$ act from E to another given space F—this is an additional and not simple problem.

Although the main results below are valid for a large set of functions $\varphi(t)$, just this problem causes restrictions of this set in applications. That is why we consider in Section 3 only positive concave functions $\varphi(t)$ (note that the particular case $\varphi(t) = t^\tau$ and the corresponding splitting $T = T^\tau T^{1-\tau}$ were studied and used in the monograph [2], Section 22).

The Fourier series with respect to the eigenfunctions of an operator T is not a natural object to study if this operator is not symmetric, and should be replaced by the biorthogonal expansion (see, e.g. [2], Section 9.4). We do not consider such series here and confine ourselves only to the class of symmetric positive operators, always starting from some Hilbert space where the convergence of Fourier series is well known.

It will be convenient for us to put $T = A^{-1}$ for some unbounded operator A with a dense domain $D(A)$ in a Hilbert space H, where the operator A^{-1} itself is assumed to be bounded and compact. Thus it must be selfadjoint and its eigenfunctions $e_1, e_2, \dots$ form an orthonormal basis in H. The corresponding eigenvalues $\lambda_i = \|A^{-1}e_i\|$ will be arranged in the descending order and tend to zero. For any $u \in H$ we have representations

$$u = \sum_{i=1}^{\infty}(u, e_i)e_i, \quad A^{-1}u = \sum_{i=1}^{\infty}\lambda_i(u, e_i)e_i, \tag{1}$$

and may define the operator $\varphi(A^{-1})$ for any bounded function $\varphi(t)$ as

$$\varphi(A^{-1})u = \sum_{i=1}^{\infty}\varphi(\lambda_i)(u, e_i)e_i. \tag{2}$$

Functions of the operator A itself can be defined either as the inverse operator $\varphi(A) = [\tilde{\varphi}(A^{-1})]^{-1}$ for $\tilde{\varphi}(t) = 1/\varphi(1/t)$ or via the series

$$\varphi(A)u = \sum_{i=1}^{\infty}\varphi(1/\lambda_i)(u, e_i)e_i, \tag{3}$$

which converges only for u from some subset of H forming the domain $D(\varphi(A))$. This operator calculus is classical and well-studied but only inside the Hilbert space H. The following assertions will be useful for the exit to other spaces.

Let $\varphi(t)\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a concave (and thus necessarily increasing) function.

Proposition 1.1 (see [3]). *If $A^{-1}\colon L_2 \to L_p$, $p > 2$, then $\varphi(A^{-1})\colon L_2 \to L_\Phi$, where L_Φ is the Orlicz space generated by the function $\Phi(u)$ inverse to the function $t^{1/2}\varphi(t^{1/p-1/2})$.*

Proposition 1.2 (see [1]). *Let $\varphi(t)$ have a representation*

$$\varphi\left(\frac{1}{t}\right) = \int_0^\infty \frac{\mathrm{d}\sigma(s)}{t+s}, \quad \sigma(s)\uparrow, \quad \int_0^\infty \frac{\mathrm{d}\sigma(s)}{1+s} < \infty. \tag{4}$$

Then

$$\varphi(A^{-1}) = \int_0^\infty (A+sI)^{-1}\,\mathrm{d}\sigma(s). \tag{5}$$

Proposition 1.3. *Let $\varphi(0) = \varphi'(\infty) = 0$. Then*

$$\varphi\left(\frac{1}{t}\right) \sim \int_0^\infty \frac{\mathrm{d}\varphi'(1/s)}{t+s}. \tag{6}$$

Proof. An integration by parts gives immediately that

$$\int_0^\infty \frac{\mathrm{d}\varphi'(1/s)}{t+s} = \int_0^\infty \frac{\varphi'(1/s)\,\mathrm{d}s}{(t+s)^2}.$$

Since the derivative of a concave function is decreasing, we obtain further that for each $t > 0$

$$\int_0^\infty \frac{\varphi'(1/s)\,\mathrm{d}s}{(t+s)^2} \leqslant \int_0^\infty \frac{\varphi'(1/(t+s))\,\mathrm{d}s}{(t+s)^2} = \int_0^{1/t} \varphi'(\tau)\,\mathrm{d}\tau = \varphi\left(\frac{1}{t}\right).$$

On the other hand $\varphi'(t) \geqslant 0$, thus

$$\int_0^\infty \frac{\varphi'(1/s)\,\mathrm{d}s}{(t+s)^2} \geqslant \int_t^\infty \frac{\varphi'(1/s)\,\mathrm{d}s}{4s^2} = \frac{1}{4}\varphi\left(\frac{1}{t}\right),$$

and the required equivalence is proved. □

The last two propositions show that, even for selfadjoint operators in a Hilbert space, we may use all properties of the calculus from [1], where just the equality (5) is taken as a definition of operator functions. This equality is especially useful for a second order elliptic operator A because its resolvent $(A+sI)^{-1}$ has well-known properties (see, e.g. [4]).

In this section we give the main results which enable us to estimate the rate of convergence of Fourier series and, consequently, of the Fourier method. Here we do not need the above-mentioned restrictions on the set of functions. For simplicity we denote below any unessential constant by the same letter M.

Theorem 2.1. *Let the eigenfunctions $e_i(x)$ of a positive selfadjoint operator A form a basis in a Hilbert space H and let the positive functions $\alpha(t)$, $\beta(t)$ be such that the function $\kappa(t) = (\alpha(t)\beta(1/t))^{-1}$ is monotone increasing. Suppose that the operator $\alpha(A^{-1})$ acts from H to some other Banach space E and that a function $u(x) \in D(\beta(A))$. Then $u \in E$ and can be expanded as a Fourier series $u = \sum(u, e_i)e_i$ which converges in E. The rate of convergence can be characterized by the estimate*

$$\left\| u - \sum_{i=1}^{n}(u, e_i)e_i \right\|_E = o(\kappa(\lambda_n)). \tag{7}$$

Proof. First we remark that each $e_i(x)$ is an eigenfunction of the operator $\alpha(A^{-1})$ and thus $e_i(x) \in E$. Hence the partial sums of the Fourier series with respect to $\{e_i\}$ for any $u \in H$ always belong to E. And if $u \in D(\beta(A))$, it also belongs to E, since it can be represented as

$$u = \beta^{-1}(A)v = \alpha(A^{-1})\kappa(A^{-1})v$$

for some $v \in H$, and $\kappa(A^{-1})$ is a bounded operator in H.

So we may consider

$$\left\| u - \sum_{i=1}^{n}(u, e_i)e_i \right\|_E = \left\| \beta^{-1}(A)v - \sum_{i=1}^{n}(\beta^{-1}(A)v, e_i)e_i \right\|_E .$$

Here

$$\begin{aligned}(\beta^{-1}(A)v, e_i)e_i = (\kappa(A^{-1})v, \alpha(A^{-1})e_i)e_i &= (\kappa(A^{-1})v, e_i)\alpha(\lambda_i)e_i \\ &= (\kappa(A^{-1})v, e_i)\alpha(A^{-1})e_i,\end{aligned}$$

hence

$$\begin{aligned}\left\| u - \sum_{i=1}^{n}(u, e_i)e_i \right\|_E &= \left\| \alpha(A^{-1})\left[\kappa(A^{-1})v - \sum_{i=1}^{n}(\kappa(A^{-1})v, e_i)e_i\right] \right\|_E \\ &\leqslant M\left\| \kappa(A^{-1})v - \sum_{i=1}^{n}(\kappa(A^{-1})v, e_i)e_i \right\|_H ,\end{aligned} \tag{8}$$

and it remains only to estimate the rate of convergence in H of the Fourier series for the function $\kappa(A^{-1})v$. We have

$$
\begin{aligned}
(9)\qquad \Bigl\|\kappa(A^{-1})v - \sum_{i=1}^{n}(\kappa(A^{-1})v, e_i)e_i\Bigr\|_H &= \Bigl\|\sum_{i=n+1}^{\infty}(\kappa(A^{-1})v, e_i)e_i\Bigr\|_H \\
= \Bigl\|\sum_{i=n+1}^{\infty}(v, \kappa(A^{-1})e_i)e_i\Bigr\|_H &= \Bigl\|\sum_{i=n+1}^{\infty}\kappa(\lambda_i)(v, e_i)e_i\Bigr\|_H \\
\leqslant \kappa(\lambda_{n+1})\Bigl\|\sum_{i=n+1}^{\infty}(v, e_i)e_i\Bigr\|_H &= o(\kappa(\lambda_n))
\end{aligned}
$$

due to the convergence of the Fourier series for each function $v \in H$. □

Varying the space E gives a possibility to obtain different kinds of convergence: mean convergence (with some exponent p), uniform convergence, or convergence together with the differentiated series. This convergence inherits also the properties of the basis $\{e_i\}$ in the space H such as permutability etc. If, for example, we consider the case $E = C$, the convergence will be not only uniform but also absolute (in usual pointwise sense).

Consider now an abstract hyperbolic equation

$$
(10)\qquad \frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2} + Au(t) = f(t), \quad u(0) = u_0, \quad u_t'(0) = v_0
$$

where A is a positive selfadjoint operator in H and t varies on $[0, T]$. The simplest example is $A = -\Delta$ on some bounded domain $\Omega \subset \mathbb{R}^N$ with zero boundary conditions, which gives us the classical oscillation equation. In any case, the solution of the equation (10) can be written in an operator form

$$
(11)\quad u(t) = \cos(A^{1/2}t)u_0 + A^{-1/2}\sin(A^{1/2}t)v_0 + A^{-1/2}\int_0^t \sin(A^{1/2}t - A^{1/2}s)f(s)\,\mathrm{d}s.
$$

If e_i, $i = 1, 2, \ldots$ are as in Theorem 2.1, the Fourier method for this equation consists in constructing successive approximations

$$
u_n(x) = P_n u = \sum_{i=1}^{n}(u, e_i)e_i
$$

which must converge to the exact solution.

Theorem 2.2. *Let $\alpha(s)$, $\beta(s)$, $\kappa(s)$ be as in Theorem* 2.1 *and let $\gamma(s) = \beta(s)/\sqrt{s}$, $g(t) = \gamma(A)f(t)$. Suppose that*

$$
u_0 \in D(\beta(A)), \quad v_0 \in D(\gamma(A)), \quad f(t) \in D(\gamma(A))
$$

for all $t \in [0,T]$ and that the function $g(t)$ is bounded in H. Then the successive approximations of the Fourier method $u_n(t) \to u(t)$ in E uniformly on $[0,T]$ and the rate of convergence is characterized by the estimate

$$\sup_{0\leqslant t\leqslant T} \|u(t)-u_n(t)\|_E = O(\kappa(\lambda_n)). \tag{12}$$

P r o o f. Set $Q_n = I - P_n$. In fact, we have to estimate $\|Q_n u\|_E$. The operators Q_n commute with all functions of the operator A, so the formula (11) implies that

$$\begin{aligned} \|Q_n u\|_E \leqslant \|\cos(A^{1/2}t)Q_n u_0\|_E &+ \|A^{-1/2}\sin(A^{1/2}t)Q_n v_0\|_E \\ &+ \int_0^t \|\sin(A^{1/2}t - A^{1/2}s)\beta^{-1}(A)Q_n g(s)\|_E \,\mathrm{d}s. \end{aligned} \tag{13}$$

The elements $u_1 = \beta(A)u_0$ and $v_1 = A^{-1/2}\beta(A)v_0$ belong to H and the operators $\cos(A^{1/2}t)$ and $\sin(A^{1/2}t)$ have the norms in H not exceeding 1 for all t, hence, analogously to (8) and (9), we can easily show that

$$\|\cos(A^{1/2}t)Q_n u_0\|_E = \|\beta^{-1}(A)\cos(A^{1/2}t)Q_n u_1\|_E = o(\kappa(\lambda_n))$$

and

$$\|A^{-1/2}\sin(A^{1/2}t)Q_n v_0\|_E = \|\beta^{-1}(A)\sin(A^{1/2}t)Q_n v_1\|_E = o(\kappa(\lambda_n))$$

uniformly on $[0,T]$. For any $t\in[0,T]$ we also have

$$\|\beta^{-1}(A)Q_n g(t)\|_E \leqslant \|\alpha(A^{-1})\|_{H\to E}\|\kappa(A^{-1})Q_n g(t)\|_H \leqslant M\|g(t)\|_H \kappa(\lambda_{n+1}),$$

and the integral in (13) can be estimated independently of t:

$$\int_0^t \|\sin(A^{1/2}t - A^{1/2}s)\beta^{-1}(A)Q_n g(s)\|_E \,\mathrm{d}s \leqslant MT \sup_{0\leqslant t\leqslant T}\|g(t)\|_H \kappa(\lambda_{n+1}).$$

It remains only to substitute these estimates into (13). □

Remark. Both theorems can be also adapted to the case when the function $(\alpha(t)\beta(1/t))^{-1}$ is not monotone increasing. It is sufficient to take for $\kappa(t)$ any increasing majorant of this function—for example,

$$\kappa(t) = \sup_{s\leqslant t}(\alpha(s)\beta(1/s))^{-1}.$$

There are certain problems in which a Fourier series undergoes some additional operations (differentiation, special integral transforms etc.) and the rate of its convergence is considered only thereafter. One way to investigate such problems is to add these operations to the definition of the norm in the space E, but such spaces could get too complicated for searching for a function $\alpha(t)$ satisfying the condition $\alpha(A^{-1})\colon H \to E$. Another way is to assume the subordination of these operations to some functions of the operator A^{-1}.

Theorem 2.4. *Let a linear operator B be defined on the domain $D(\tilde{\alpha}(A))$, where $\tilde{\alpha}(t) = 1/\alpha(1/t)$, and satisfy the inequality*

$$\|B\alpha(A^{-1})v\|_E \leqslant M\|v\|_H \quad \text{for all } v \in H. \tag{14}$$

Let $u \in D(\beta(A))$ and let $\kappa(t)$ be the same as in Theorems 2.1 and 2.2. Then one may apply to the first series from (1), term by term, the operator B and the resulting series

$$Bu = \sum_{i=1}^{\infty}(u, e_i)Be_i \tag{15}$$

converges in E with the rate which can be estimated as

$$\left\|Bu - \sum_{i=1}^{n}(u, e_i)Be_i\right\|_E = o(\kappa(\lambda_n)).$$

P r o o f. The proof is analogous to that of Theorem 2.1. Note, as before, that the series (15) admits any permutations of its terms and so, if $E = C$, we again obtain a uniform and absolute convergence. In order to obtain the inequality (14) we can use the so-called inverse moment inequalities, for example, a result from [1]. It states that the inequality (14) holds if the operator B is closed and

$$\|Bv\|_E \leqslant \|v\|_H\delta(\|Av\|_H/\|v\|_H)$$

for all $v \in D(A)$ and some function $\delta(t)$ such that

$$\int_0^1 \alpha(t)\delta\left(\frac{1}{t}\right)\frac{\mathrm{d}t}{t} < \infty.$$

$\square$

3. Special part

In this section we give some applications of the main results. We consider the usual second order hyperbolic equations in $L_2(\Omega)$ for some bounded set $\Omega \subset \mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. This means that we take in (9) as A a positive selfadjoint second order elliptic operator with some admissible boundary conditions on $\partial\Omega$. We will be interested in absolute and uniform convergence on Ω of the Fourier method which can be obtained if we take $E = C(\Omega)$. For the application of Theorem 2.2 to our case, we must solve the following problems:

1) Under what conditions on the function $\alpha(t)$ does the corresponding operator $\alpha(A^{-1})$ act continuously from $L_2(\Omega)$ to $C(\Omega)$?
2) Under what conditions on the function $\beta(t)$ does a given function $u = u(x)$, $x \in \Omega$, belong to the domain $D(\beta(A))$?

In the discussion below we consider functions $\alpha(t)$, $\beta(t)$, having the form $t^m\varphi(t)$ for some $m \geqslant 0$ and some positive concave function $\varphi(t)$. Note that in the solution of the first problem we may take the space L_∞ instead of the space C, because all eigenfunctions $e_i(x)$ of elliptic operators are continuous on Ω.

Theorem 3.1. *Let A be a second order positive elliptic operator with sufficiently smooth coefficients, boundary and boundary conditions (cf.* [5]*), and let $\alpha(t) = t^{N/4}\psi(t)$ with some positive concave function $\psi(t)$. Then the condition*

$$\int_0^1 (\psi(t)^{N/(N-1)} \frac{\mathrm{d}t}{t} < \infty \tag{16}$$

ensures that $\alpha(A^{-1})\colon\ L_2 \to C$.

Proof. Set $m = N/4 - 1/2$, $\varphi(t) = t^{1/2}\psi(t)$, then $\alpha(A^{-1}) = A^{-m}\varphi(A^{-1})$. As shown in [2], Section 16.5, the operator $A^{-m}\colon\ L_2 \to L_N$. At the same time for any $\varepsilon > 0$, the operator $A^{-N/4}A^{-\varepsilon}\colon\ L_2 \to C$. So we need not consider functions $\psi(t)$ which are increasing faster then t^ε and, without loss of generality, we may assume that the function $\varphi(t)$ is also concave. Moreover, due to (6), we may assume that the operator $\varphi(A^{-1})$ has a representation (5) and can use Lemma 1 from [5] stating that $\varphi(A^{-1})\colon\ \Lambda \to L_\infty$, where Λ is the so-called Lorentz space with the fundamental function $\varphi(t^{2/N})$ (a full description of such spaces and their properties can be found e.g. in [6]). If we show that $L_N \subset \Lambda$, the proof of the theorem will be finished. The conditions for such embeddings are given in [7]; in our case, it is sufficient that $\varphi(t^{2/N})/t \in L_{N/(N-1)}[0,1]$. But

$$\int_0^1 (\varphi(t^{2/N})/t)^{N/(N-1)}\,\mathrm{d}t = \int_0^1 \bigl(t^{1/N}\varphi(t^{2/N})\bigr)^{N/(N-1)} \frac{\mathrm{d}t}{t} = \frac{N}{2}\int_0^1 (\psi(t))^{N/(N-1)} \frac{\mathrm{d}t}{t},$$

and (16) is proved. □

It is possible to compare the results obtained with previous results from [2], allowing to take $\psi(t) = t^\varepsilon$ with arbitrary $\varepsilon > 0$. The condition (16) allows us to take

$$\psi(t) = \left(\frac{1}{\ln(e/t)}\right)^{1-1/N+\varepsilon}, \quad \varepsilon > 0,$$

which increases essentially slower than any power.

The conditions of Theorem 3.1 can be generalized to studying the convergence of a Fourier series after its differentiation term by term. Let $\mathbf{D}^l$ be a differential operator of order $|l|$. It is known (see e.g. [2], Section 16), that $\mathbf{D}^l$ is subordinate in L_2 to the power $A^{|l|/2}$ of any positive second order elliptic operator if the inverse operator A^{-1} is bounded in L_2. Thus we may study the convergence of the series

$$\mathbf{D}^l u = \sum_{i=1}^{\infty} (u, e_i)\mathbf{D}^l e_i,$$

applying the result of Theorem 2.4. The condition on the function $\alpha(t)$ will be

$$\alpha(t) = t^{N/4+|l|/2}\psi(t)$$

for any $\psi(t)$ satisfying (16).

The second problem, that is, the description of the domains $D(\beta(A))$, is less studied and more difficult than to find $\alpha(t)$. Even for the fractional power t^τ, the domain $D(A^\tau)$ can be easily described only for $\tau = 1/2$, when it coincides with the Sobolev space W_2^1; otherwise we only have the embedding $D(A^\tau) \subset W_2^{2\tau}$ which is useless for our purpose. So the following results on functions $\beta(t)$ will be useful also when these functions have a power form. Note that for equivalent functions $\beta_1(t) \sim \beta_2(t)$, the corresponding domains $D(\beta_1(A))$ and $D(\beta_2(A))$ are the same, thus we may assume that all our positive concave functions $\varphi(t)$ have the form (4) with $\sigma(s) = \varphi'(1/s)$.

We give here two methods of checking that a given function $u(x) \in D(\beta(A))$ in $L_2(\Omega)$. The first of them is suitable for any positive operator A in an arbitrary Hilbert space H and uses a result from [1] dealing with the Peetre K-functional. For a Banach couple (E, F) and every $u \in E + F$, this functional is defined as

$$K(t, u, E, F) = \inf_{u=u_0+u_1} (\|u_0\|_E + t\|u_1\|_F).$$

In our case we take $E = H$ and $F = D(A)$ equipping the last space with the norm $\|u\|_{D(A)} = \|Au\|_H$.

Let $\varphi(t)$ be as above. As shown in [1], one has an embedding $D(\varphi(A)) \supset H_\varphi$, where H_φ is a subspace of H with the norm

$$\|u\|_{H_\varphi} = \int_0^\infty \frac{1}{t} K(t, u, H, D(A)) \, \mathrm{d}\varphi'\left(\frac{1}{t}\right).$$

Hence for a function $\beta(t) = t^m \varphi(t)$ with some integer m, the inclusion $u \in D(\beta(A))$ is ensured by the inclusion $A^m u \in H_\varphi$ which can be written as

$$\int_0^1 \frac{1}{t} K(t, A^m u, H, D(A)) \, \mathrm{d}\varphi'\left(\frac{1}{t}\right) < \infty, \quad u \in D(A^m). \tag{17}$$

Indeed, the corresponding integral from 1 to ∞ is necessarily finite, since $K(t, v, H, D(A))$ is constant for $t > 1/\|A^{-1}\|$ and any $v \in H$.

The condition (17) is rather abstract and gets usable only if the K-functional can be expressed more explicitly. Such a situation occurs for a positive elliptic operator of order $2k$ in L_2, since its domain $D(A)$ coincides there with W_2^{2k}. The K-functional for the couple (L_2, W_2^{2k}) was found by J. Peetre [8]:

$$K(t, v, L_2, W_2^{2k}) \cong \omega_2^{(2k)}(t^{1/2k}, v) + t\|v\|_{L_2},$$

where $\omega_2^{(n)}(t, v)$ means the n-th modulus of continuity in L_2 of the function $v(x)$:

$$\omega_2^{(n)}(t, v) = \sup_{|h| \leqslant t} \left\| v(x + nh) - \binom{n}{1} v(x + (n-1)h) + \ldots \right\|_{L_2}.$$

The second summand $t\|v\|_{L_2}$ may be ignored, since its contribution to the integral in (17) is always finite. Summarizing, we arrive at the following assertion.

Theorem 3.2. *Let A be a selfadjoint positive elliptic operator of order $2k$ acting in L_2. Let $\beta(t) = t^m \varphi(t)$ with some positive concave function $\varphi(t)$. Then for any $u \in D(A^m)$, the inequality*

$$\int_0^1 \frac{1}{t} \omega_2^{(2k)}(t^{1/2k}, A^m u) \, \mathrm{d}\varphi'\left(\frac{1}{t}\right) < \infty \tag{18}$$

ensures the inclusion $u \in D(\beta(A))$.

Another way to describe $D(\beta(A))$ can be suggested especially for the second order elliptic operators in L_2. In this case we may use the integral representation of operators $\varphi(A^{-1})$ from [5]:

$$\varphi(A^{-1}) u(x) = \int_\Omega G_\varphi(x, y) u(y) \, \mathrm{d}y,$$

with kernels satisfying for $N \geqslant 3$ an inequality

$$0 \leqslant G_\varphi(x,y) \leqslant M|x-y|^{2-N}\varphi'(|x-y|^2). \tag{19}$$

Theorem 3.3. *Let A be a second order positive elliptic operator in $L_2(\Omega)$ as in Theorem 3.1, where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$. Assume the existence of a minimal integer $m \geqslant 1$ such that the function $v = A^m u$ is integrable, but does not belong to L_2 on Ω. If a concave function $\varphi(t)$ satisfies the inequality*

$$\int_\Omega \left[\int_\Omega \frac{|v(y)|\varphi'(|x-y|^2)}{|x-y|^{N-2}}\,\mathrm{d}y\right]^2 \mathrm{d}x < \infty, \tag{20}$$

then $u \in D(\beta(A))$ for $\beta(t) = t^m\varphi(1/t)$.

P r o o f. The condition (20) entails that $\varphi(A^{-1})A^m u \in L_2$ and consequently $u \in D(A^m\varphi(A^{-1})) = D(\beta(A))$. □

The conditions of the last theorem seem to be easier for checking, because we need not deal with the hardly computable modulus of continuity. The operator A^m, used in both theorems, does not make any additional difficulties, being a mere iteration of the implicitly given operator A. The case $N = 2$ is always known as peculiar. It was also considered in [5], where it was proved that the estimate (19) remains valid if the function $\varphi(t)$ is not "very close" to t. Otherwise, $G_\varphi(x,y) \leqslant M\eta(|x-y|)$ where

$$\eta(t) = \int_{t^2}^{a} \varphi'(s)\frac{\mathrm{d}s}{s}$$

with arbitrary $a \geqslant (2\,\mathrm{diam}\,\Omega)^2$. The reader can easily adapt Theorem 3.3 to this case.

The same technique can be used for checking the last condition of Theorem 2.2, the boundedness of the function $g(t)$ in L_2. To compute $A^m f(t)$ for a given A and $f(t)$ is not a difficult task when m is an integer, and then we need once again to examine the inequality (20) with $A^m f(t)$ as v. The only additional problem which we have here is to show that the integral in (20) can be estimated independently of $t \in [0,T]$.

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Author's address: Department of Mathematics, Technion–Israel Institute of Technology, 32000 Haifa, Israel, e-mail `evg@techunix.technion.ac.il`.