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PERIODIC PROBLEMS AND PROBLEMS WITH DISCONTINUITIES FOR NONLINEAR PARABOLIC EQUATIONS

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Abstract. In this paper we study nonlinear parabolic equations using the method of upper and lower solutions. Using truncation and penalization techniques and results from the theory of operators of monotone type, we prove the existence of a periodic solution between an upper and a lower solution. Then with some monotonicity conditions we prove the existence of extremal solutions in the order interval defined by an upper and a lower solution. Finally we consider problems with discontinuities and we show that their solution set is a compact R_δ-set in $(CT, L^2(Z))$.

Keywords: pseudomonotone operator, L-pseudomonotonicity, operator of type $(S)_+$, operator of type L-$(S)_+$, coercive operator, surjective operator, evolution triple, compact embedding, multifunction, upper solution, lower solution, extremal solution, R_δ-set

MSC 2000: 35K55

1. Introduction

The method of upper and lower solutions turned out to be a powerful tool in the analysis of nonlinear partial differential equations and in the context of semilinear problems it produced monotone iterative schemes which generate the extremal solutions. This is exemplified by the work of Sattinger [35]. Later, in an interesting paper, Deuel-Hess [12] used upper and lower solutions to establish the existence of periodic solutions for a class of nonlinear parabolic problems. The periodic problem in the context of abstract evolution equations, was also addressed recently by Vrabie [37] and Hirano [19], but their hypotheses on the nonlinear perturbation term are strong and exclude the possibility of fitting in their model evolution equation, second order problems with the right hand side term f depending also on the gradient of the solution, as is the case here.

Our work here is closely related to that of Deuel-Hess [12]. However we do not require the upper and lower solutions to be bounded and in return this allows us to impose a more general growth condition on the perturbation term f. Moreover, our approach is different from that of Deuel-Hess. Instead of associating our problem to a parabolic variational inequality with a stationary constraint set (see also Puel [33]), for the analysis of which it is crucial that the upper and lower solutions be bounded (see the proof of the main theorem, p. 101, of Deuel-Hess [12]), here we rely on a fixed point theorem for set-valued maps defined on partially ordered metric spaces, due to Heikkila-Hu [17]. In addition, under some extra hypotheses on the data, which make the problem monotonic (hence guarantee the existence of a unique solution for an auxiliary initial boundary-value problem used in the proof), we show that the problem has extremal periodic solutions in the interval determined by the upper and lower solutions. Finally we also examine problems with discontinuous nonlinearities.

2. Mathematical preliminaries

In our approach we will use evolution triples, some function spaces related to them and evolution equations defined on such triples. So in this section we recall some basic definitions and facts concerning evolution triples. Detailed proofs and additional results can be found in Zeidler [38].

Let H be a Hilbert space and X a dense subspace of H carrying the structure of a separable reflexive Banach space, which embeds into H continuously. Identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$, with all embeddings being continuous and dense. Such a triple of spaces is known in the literature as "evolution triple" or "Gelfand triple". By $|\cdot|$ (resp. $\|\cdot\|, \|\cdot\|_*$), we denote the norm of H (resp. of X, X^*). Also by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*) and by $(\cdot,\cdot)$ the inner product of H. The two are compatible in the sense that $\langle\cdot,\cdot\rangle\,|_{X\times H} = (\cdot,\cdot)$. We will need the following generalization of the notion of a maximal monotone operator (see Zeidler [38], p. 585).

Definition. An operator $A\colon X \to X^*$ is said to be "pseudomonotone", if $x_n \xrightarrow{w} x$ in X as $n \to \infty$ and $\overline{\lim}\,\langle A(x_n), x_n - x\rangle \leqslant 0$, imply that $\langle A(x), x - y\rangle \leqslant \underline{\lim}\,\langle A(x_n), x_n - y\rangle$ for all $y \in X$.

Remark. A monotone hemicontinuous operator or a completely continuous operator $A\colon X \to X^*$, is pseudomonotone. Pseudomonotonicity is preserved under addition and it is easy to see that it implies property (M); i.e., if $x_n \xrightarrow{w} x$ in X, $A(x_n) \xrightarrow{w} u^*$ in X^* as $n \to \infty$ and $\overline{\lim}\,\langle A(x_n), x_n - x\rangle \leqslant 0$, then $A(x) = u^*$.

A related notion, useful in the context of parabolic problems, is the following:

Definition. Let Y be a reflexive Banach space, $L\colon D(L) \subseteq Y \to Y^*$ a linear maximal monotone operator and $V\colon Y \to Y^*$ a bounded nonlinear operator. We say that $V(\cdot)$ is "L-pseudomonotone", if for $\{y_n\}_{n\geqslant 1} \subseteq D(L)$ such that $y_n \xrightarrow{w} y$ in Y, $L(y_n) \xrightarrow{w} L(y)$ in Y^* as $n \to \infty$ and $\overline{\lim}(V(y_n), y_n - y)_{Y^*,Y} \leqslant 0$, we have $V(y_n) \xrightarrow{w} V(y)$ in Y^* and $(V(y_n), y_n)_{Y^*,Y} \to (V(y), y)_{Y^*,Y}$ as $n \to \infty$.

Remark. Recall that a linear operator $L\colon D(L) \subseteq Y \to Y^*$ is maximal monotone if and only if L is densely defined, closed and both L and L^* are monotone (see Zeidler [38], Theorem 321, p. 897).

To see how these two pseudomonotonicity notions are related, we need to introduce a function space, which plays a central role in the analysis of evolution equations. So let $W_{pq}(T) = \{x \in L^p(T,X)\colon \dot{x} \in L^q(T,X^*)\}$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. The time derivative of $x(\cdot)$, is understood in the sense of vector valued distributions. The space $W_{pq}(T)$ embeds continuously in $C(T,H)$ and if X embeds compactly in H, then so does $W_{pq}(T)$ in $L^p(T,H)$. Let $L_1\colon D_1 \subseteq L^p(T,X) \to L^q(T,X^*)$ be defined by $L_1(x) = \dot{x}$ for all $x \in D_1 = \{x \in W_{pq}(T)\colon x(0) = x(b)\}$. By virtue of the continuous embedding of $W_{pq}(T)$ in $C(T,H)$, the pointwise evaluation at $t = 0$ and $t = b$ makes sense. Since the space $C_0^1(T,X)$ is dense in $L^p(T,X)$, we see at once that D_1 is dense in $L^p(T,X)$. Also since $L_1^*\colon D_1^* \subseteq L^p(T,X) \to L^q(T,X^*)$ is defined by $Lv = -\dot{v}$ for all $v \in D_1^* = D_1$, we see that both L_1 and L_1^* are monotone operators (indeed $(\!(L_1x, x)\!) = (\!(L_1^*v, v)\!) = 0$, where $(\!(\cdot,\cdot)\!)$ denotes the duality brackets for the pair $(L^p(T,X), L^q(T,X^*) = L^p(T,X)^*)$) and clearly L_1 is closed. Therefore L_1 is maximal monotone.

The next proposition relates the two pseudomonotonicity notions introduced earlier and it can be found in Papageorgiou [31].

Proposition 1. *If $A\colon T \times X \to X^*$ is an operator such that,*

(i) *for every $x \in X$, $t \to A(t,x)$ is measurable;*

(ii) *for almost all $t \in T$, $x \to A(t,x)$ is demicontinuous and pseudomonotone;*

(iii) $\|A(t,x)\|_* \leqslant a_1(t) + c_1\|x\|^{p-1}$ *a.e. on T, with $a_1 \in L^q(T)$, $c_1 > 0$, $2 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$;*

(iv) $\langle A(t,x), x\rangle \geqslant c\|x\|^p - \eta\|x\|^r - \theta(t)$ *for almost all $t \in T$, with $c, \eta > 0$, $1 \leqslant r < p$ and $\theta \in L^1(T)$; and if $\widehat{A}\colon L^p(T,X) \to L^q(T,X^*)$ is the Nemitsky operator corresponding to A (i.e. $\widehat{A}(x)(\cdot) = A(\cdot, x(\cdot))$),*

then $\widehat{A}$ is demicontinuous and L-pseudomonotone.

For L-pseudomonotone operators, we have the following basic surjectivity result (see B-A. Ton [36], or Lions [27], p. 319).

Proposition 2. *If Y is a reflexive Banach space, $L\colon D(L) \subseteq Y \to Y^*$ is a linear maximal monotone operator and $G\colon Y \to Y^*$ is a bounded, demicontinuous, L-pseudomonotone, coercive operator (i.e. $\frac{(G(y),y)_{Y^*,Y}}{\|y\|_Y} \to +\infty$ as $\|y\|_Y \to \infty$), then $(L+G)(\cdot)$ is surjective.*

Another monotonicity type notion that we will need, is the following:

Definition. An operator $A\colon X \to X^*$ is said to be of "type $(S)_+$", if $x_n \xrightarrow{w} x$ in X as $n \to \infty$ and $\overline{\lim}\, \langle A(x_n), x_n - x\rangle \leqslant 0$, then $x_n \to x$ in X as $n \to \infty$.

Remark. A uniformly monotone operator is of type $(S)_+$. Also a demicontinuous operator of type $(S)_+$, is pseudomonotone (see Zeidler [38]).

Also in analogy with L-pseudomonotonicity, we introduce the notion of an operator of "type L-$(S)_+$".

Definition. Let Y be a reflexive Banach space, $L\colon D(L) \subseteq Y \to Y^*$ is a linear densely defined maximal monotone operator and $V\colon Y \to Y^*$. We say that $V(\cdot)$ is of "type L-$(S)_+$", if for $\{y_n\}_{n\geqslant 1} \subseteq D(L)$, $y_n \xrightarrow{w} y$ in Y, $L(y_n) \xrightarrow{w} L(y)$ in Y^* and $\overline{\lim}(V(y_n), y_n - y)_{Y^*,Y} \leqslant 0$, we have $y_n \to y$ in Y as $n \to \infty$.

A slight modification of the proof of Proposition 1, gives the following "lifting" property for condition $(S)_+$.

Proposition 3. *If $A\colon T \times X \to X^*$ is an operator such that,*
(i) *for every $x \in X$, $t \to A(t,x)$ is measurable;*
(ii) *for almost all $t \in T$, $x \to A(t,x)$ is demicontinuous and of type $(S)_+$;*
(iii) *$\|A(t,x)\|_* \leqslant a_1(t) + c_1\|x\|^{p-1}$ a.e. on T, with $a_1 \in L^q(T)$, $c_1 > 0$, $2 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$;*
(iv) *$\langle A(t,x), x\rangle \geqslant c\|x\|^p - \eta\|x\|^r - \theta(t)$ for almost all $t \in T$, with $c, \eta > 0$, $1 \leqslant r < p$ and $\theta \in L^1(T)$; and if $\widehat{A}\colon L^p(T,X) \to L^q(T,X^*)$ is the Nemitsky operator corresponding to A,*

then $\widehat{A}$ is demicontinuous and of type L-$(S)_+$.

On the evolution triple (X,H,X^*) we consider the following evolution equation:

$$\begin{aligned} &\dot{x}(t) + A(t,x(t)) = h(t) \quad \text{a.e. on } T \\ &x(0) = x_0. \end{aligned} \tag{1}$$

On $A(t,x)$ we impose the following conditions:
H(A): $A\colon T \times X \to X^*$ is a map such that
(i) for every $x \in X$, $t \to A(t,x)$ is measurable;
(ii) for almost all $t \in T$, $x \to A(t,x)$ is demicontinuous, monotone;

(iii) $\|A(t,x)\|_* \leqslant a_1(t) + c_1\|x\|^{p-1}$ a.e. on T with $a_1 \in L^q(T)$, $c_1 > 0$, $2 \leqslant p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$;

(iv) $\langle A(t,x), x\rangle \geqslant c\|x\|^p - \eta\|x\|^r - \theta(t)$ for almost all $t \in T$, with $c > 0$, $\eta > 0$, $\theta \in L^1(T)$ and $1 \leqslant r < p$.

It is well-known that under these hypotheses, for every $h \in L^q(T,H)$ and every $x_0 \in H$, problem (1) has a unique solution $x \in W_{pq}(T) \subseteq C(T,H)$. Let $\widehat{p}$: $L^q(T,H) \times H \to C(T,H)$ be the map which to each pair $(h,x_0) \in L^q(T,H) \times H$ assigns the unique solution $x = \widehat{p}(h,x_0)$ of (1). The next proposition determines the continuity properties of $\widehat{p}(\cdot,\cdot)$ and can be found in Papageorgiou-Shahzad [32] (see also Avgerinos-Papageorgiou [3]). In what follows by $L^q(T,H)_w$, we denote the Lebesgue-Bochner space $L^q(T,H)$ furnished with the weak topology.

Proposition 4. *If hypotheses $H(A)$ hold and X embeds compactly in H, then $\widehat{p}$: $L^q(T,H)_w \times H \to C(T,H)$ is sequentially continuous.*

3. Existence of solutions

Let $T = [0,b]$ and let $Z \subseteq \mathbb{R}^N$ be a bounded domain in $\mathbb{R}^N$ with C^1-boundary Γ. We consider the following nonlinear parabolic boundary value problem defined on $T \times Z$:

$$(2) \qquad \begin{aligned} &\frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t,z,x,Dx) + f(t,z,x(t,z),Dx(t,z)) = h(t,z) \quad \text{in } T \times Z \\ &x(0,z) = x(b,z) \quad \text{a.e. on } Z, \quad x|_{T\times\Gamma} = 0. \end{aligned}$$

Here as usual $D_k = \frac{\partial}{\partial z_k}$, $k \in \{1,2,\ldots,N\}$, and $D = (D_k)_{k=1}^N$ (the gradient operator). We will need the following hypotheses on the data of (2):

H(a): $a_k\colon T \times Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $k \in \{1,2,\ldots,N\}$, are functions such that

(i) for every $(x,\xi) \in \mathbb{R} \times \mathbb{R}^N$, $(t,z) \to a_k(t,z,x,\xi)$ is measurable;

(ii) for every $(t,z) \in T \times Z$, $(x,\xi) \to a_k(t,z,x,\xi)$ is continuous;

(iii) for every $(x,\xi) \in \mathbb{R} \times \mathbb{R}^N$, $|a_k(t,z,x,\xi)| \leqslant \beta_1(t,z) + c_1(|x|^{p-1} + \|\xi\|^{p-1})$ a.e. on $T \times Z$ with $\beta_1 \in L^q(T \times Z)$, $c_1 > 0$, $2 \leqslant p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$;

(iv) $\sum\limits_{k=1}^{N} \Big(a_k(t,z,x,\xi) - a_k(t,z,x,\xi')\Big)(\xi_k - \xi_k') > 0$ a.e. on $T \times Z$, for every $x \in \mathbb{R}$ and every $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$;

(v) $\sum\limits_{k=1}^{N} a_k(t,z,x,\xi)\xi_k \geqslant c\|\xi\|^p$ a.e. on $T \times Z$ for every $(x,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and with $c > 0$.

Remark. Hypotheses $H(a)$ are the well-known Leray-Lions conditions on the coefficients a_k (see Lions [27]).

Because of hypotheses $H(a)$, we can define the semilinear Dirichlet form a: $L^p(T, W^{1,p}(Z)) \times L^p(T, W^{1,p}(Z)) \to \mathbb{R}$, by

$$a(x, y) = \int_0^b \int_Z \sum_{k=1}^N a_k(t, z, x, Dx) D_k y(t, z) \, \mathrm{d}z \, \mathrm{d}t.$$

In what follows by $(\!(\cdot, \cdot)\!)$ we denote the duality brackets for the pairs

$$(L^p(T, W^{1,p}(Z)), L^q(T, W^{1,q}(Z)^*)) \qquad \text{and} \qquad (L^p(T, W_0^{1,p}(Z)), L^q(T, W^{-1,q}(Z)));$$

i.e. $(\!(x, v)\!) = \int_0^b \langle x(t), v(t) \rangle \, \mathrm{d}t$. Recall that if Y is a reflexive Banach space (or more generally if Y^* has the Radon-Nikodym property) and $1 \leqslant p < \infty$, then $L^p(T, Y)^* = L^q(T, Y^*)$, with $\frac{1}{p} + \frac{1}{q} = 1$ (see Diestel-Uhl [13], Theorem 1, p. 98).

In what follows the following two particular instances of $W_{pq}(T)$ introduced earlier, will be very useful in our considerations:

$$\widehat{W}_{pq}(T) = \left\{ f \in L^p(T, W^{1,p}(Z)) \colon \frac{\partial f}{\partial t} \in L^q(T, W^{1,p}(Z)^*) \right\}$$

and

$$W_{pq}(T) = \left\{ f \in L^p(T, W_0^{1,p}(Z)) \colon \frac{\partial f}{\partial t} \in L^q(T, W^{-1,q}(Z)) \right\}.$$

In these definitions, the derivative $\frac{\partial f}{\partial t}$ is defined in the sense of vector-valued distributions. Both spaces become separable reflexive Banach spaces, when we furnish them with the norm $\|f\|_{pq} = \|f\|_p + \left\| \frac{\partial f}{\partial t} \right\|_q$. Moreover, they embed continuously in $C(T, L^2(Z))$ and compactly in $L^p(T \times Z)$.

Now we introduce the notions of upper and lower solutions, which will be our main analytical tools in what follows:

Definition. A function $\varphi \in \widehat{W}_{pq}(T)$ is said to be an "upper solution" of (2), if

$$\left(\!\!\left(\frac{\partial \varphi}{\partial t}, y\right)\!\!\right) + a(\varphi, y) + \int_0^b \int_Z f(t, z, \varphi, D\varphi) y(t, z) \, \mathrm{d}z \, \mathrm{d}t \geqslant \int_0^b \int_Z h(t, z) y(t, z) \, \mathrm{d}z \, \mathrm{d}t$$

for all $y \in L^p(T, W_0^{1,p}(Z)) \cap L^p(T \times Z)_+$, and $\varphi(0, z) \geqslant \varphi(b, z)$ a.e. on Z, $\varphi|_{T \times \Gamma} = 0$. Similarly a function $\psi \in \widehat{W}_{pq}(T)$, is a "lower solution" of (1), if the inequalities in the above definition are reversed.

Remark. The hypotheses on f (see $H(f)$ below) justify the integrations $\int_0^b \int_Z f(t, z, \varphi, D\varphi) y(t, z) \, \mathrm{d}z \, \mathrm{d}t$ and $\int_0^b \int_Z f(t, z, \psi, D\psi) y(t, z) \, \mathrm{d}z \, \mathrm{d}t$ and so φ and ψ are well-defined.

$\mathbf{H_0}$: There exist an upper solution $\varphi \in \widehat{W}_{pq}(T)$ and a lower solution $\psi \in \widehat{W}_{pq}(T)$ for the problem (2) and $\psi(t,z) \leqslant \varphi(t,z)$ a.e. on $T \times Z$.

Remark. In contrast to Deuel-Hess [12], we do not require that $\varphi, \psi \in L^\infty(T\times Z)$.

Definition. A function $x \in W_{pq}(T)$ is a "solution" of (2), if

$$\left(\left(\frac{\partial x}{\partial t}, y\right)\right) + a(x,y) + \int_0^b \int_Z f(t,z,x,Dx)y(t,z)\,\mathrm{d}z\,\mathrm{d}t = \int_0^b \int_Z h(t,z)y(t,z)\,\mathrm{d}z\,\mathrm{d}t$$

for all $y \in L^p(T, W_0^{1,p}(Z))$.

The hypotheses on $f(t,z,x,\xi)$ are the following:

$\mathbf{H(f)}$: $f\colon T \times Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, is a function such that

(i) for every $(x,\xi) \in \mathbb{R} \times \mathbb{R}^N$, $(t,z) \to f(t,z,x,\xi)$ is measurable;

(ii) for every $(t,z) \in T \times Z$, $(x,\xi) \to f(t,z,x,\xi)$ is continuous;

(iii) for almost all $(t,z) \in T \times Z$, for every $x \in [\psi(t,z), \varphi(t,z)]$ and every $\xi \in \mathbb{R}^N$, $|f(t,z,x,\xi)| \leqslant \beta_2(t,z) + c_2(|x|^{p-1} + \|\xi\|^{p-1})$ with $\beta_2 \in L^q(T \times Z)$, $c_2 > 0$.

The approach that we employ here uses truncation and penalization techniques. So we introduce the truncation operator $\tau(x)(\cdot,\cdot)$, defined by

$$\tau(x)(t,z) = \begin{cases} \varphi(t,z) & \text{if } \varphi(t,z) \leqslant x(t,z) \\ x(t,z) & \text{if } \psi(t,z) \leqslant x(t,z) \leqslant \varphi(t,z) \\ \psi(t,z) & \text{if } x(t,z) \leqslant \psi(t,z). \end{cases}$$

The following lemma can be found in Cardinali-Fiacca-Papageorgiou [6].

Lemma 5. $\tau\colon L^p(T, W^{1,p}(Z)) \to L^p(T, W^{1,p}(Z))$ *is continuous.*

The penalty function $u\colon T \times Z \times \mathbb{R} \to \mathbb{R}$ *is defined by*

$$u(t,z,x) = \begin{cases} (x - \varphi(t,z))^{p-1} & \text{if } \varphi(t,z) \leqslant x \\ 0 & \text{if } \psi(t,z) \leqslant x \leqslant \varphi(t,z) \\ -(\psi(t,z) - x)^{p-1} & \text{if } x \leqslant \psi(t,z). \end{cases}$$

A straightforward, elementary calculation reveals that the following is true for $u(t,z,x)$ *(see also Deuel-Hess [12]).*

Lemma 6. $u\colon T \times Z \times \mathbb{R} \to \mathbb{R}$ *is a function such that,*

(a) *for every* $x \in \mathbb{R}$, $(t,z) \to u(t,z,x)$ *is measurable;*

(b) *for every* $(t,z) \in T \times Z$, $x \to u(t,z,x)$ *is continuous;*

(c) $|u(t,z,x)| \leqslant \beta_3(t,z) + c_3|x|^{p-1}$ *for almost all* $(t,z) \in T \times Z$ *and all* $x \in \mathbb{R}$, *with* $\beta_3 \in L^q(T \times Z)$, $c_3 > 0$; *and*

(d) $\int_0^b\int_Z u(t,z,x(t,z))x(t,z)\,\mathrm{d}z\,\mathrm{d}t \geqslant c_4\|x\|^p_{L^p(T\times Z)} - c_5\|x\|^{p-1}_{L^p(T\times Z)}$ *for some* c_4 *and* $c_5 > 0$.

A final auxiliary result that we will need in the proof of the existence theorem of this section is the next proposition. Let $A\colon\ T\times W_0^{1,p}(Z)\to W^{-1,q}(Z)$ be defined by

$$\langle A(t,x),y\rangle=\sum_{k=1}^N\int_Z a_k(t,z,\tau(x),Dx)D_ky(t,z)\,\mathrm{d}z\,\mathrm{d}t$$

for all $y\in W_0^{1,p}(Z)$.

Proposition 7. *If hypotheses $H(a)$ hold and $A\colon\ T\times W_0^{1,p}(Z)\to W^{-1,q}(Z)$ is defined as above, then for every $x\in W_0^{1,p}(Z)$ $t\to A(t,x)$ is measurable and for every $t\in T$ $x\to A(t,x)$ is demicontinuous and of type $(S)_+$.*

Proof. In what follows, for notational simplicity, let $X=W_0^{1,p}(Z)$ and $X^*=W^{-1,q}(Z)$. By Fubini's theorem, for every $y\in X$, $t\to\langle A(t,x),y\rangle$ is measurable. So $t\to A(t,x)$ is weakly measurable and since X^* is separable, from the Pettis measurability theorem (see Diestel-Uhl [13], Theorem 2, p. 42), we infer that $t\to A(t,x)$ is measurable.

Next fix $t\in T$ and let $x_n\to x$ in X as $n\to\infty$. Then by passing to a subsequence if necessary, we may assume that $\tau(x_n)(t,z)\to\tau(x)(t,z)$ and $Dx_n(z)\to Dx(z)$ a.e. on Z as $n\to\infty$. By virtue of hypothesis $H(a)$ (ii), $a_k(t,z,\tau(x_n)(t,z),Dx_n(z))\to a_k(t,z,\tau(x)(t,z),Dx(z))$ a.e. on Z as $n\to\infty$ for all $k\in\{1,2,\dots,N\}$. So applying the dominated convergence theorem (see hypothesis $H(a)$ (v)), it follows that for all $y\in X$

$$\begin{aligned}\langle A(t,x_n),y\rangle&=\int_Z\sum_{k=1}^N a_k(t,z,\tau(x_n),Dx_n)D_ky(z)\,\mathrm{d}z\\ \to\int_Z\sum_{k=1}^N a_k(t,z,\tau(x),Dx)D_ky(z)\,\mathrm{d}z&=\langle A(t,x),y\rangle\quad\text{as}\quad n\to\infty.\end{aligned}$$

Since $y\in X$ was arbitrary, we infer that $A(t,x_n)\xrightarrow{w}A(t,x)$ in X^* as $n\to\infty$ and this proves the demicontinuity of $A(t,\cdot)$.

Finally we will show that $A(t,\cdot)$ is of type $(S)_+$. To this end let $x_n\xrightarrow{w}x$ in X and assume that $\overline{\lim}\,\langle A(t,x_n)-A(t,x),x_n-x\rangle\leqslant 0$. Since X embeds compactly in $L^p(Z)$, by passing to a subsequence if necessary, we may assume that $x_n\to x$ in $L^p(Z)$ and $x_n(z)\to x(z),\tau(x_n)(t,z)\to\tau(x)(t,z)$ a.e. on Z. Then we have

$$\begin{aligned}&\overline{\lim}\int_Z\sum_{k=1}^N\left(a_k(t,z,\tau(x_n),Dx_n)-a_k(t,z,\tau(x_n),Dx)\right)D_k(x_n-x)(z)\,\mathrm{d}z\\&+\underline{\lim}\int_Z\sum_{k=1}^N\left(a_k(t,z,\tau(x_n),Dx)-a_k(t,z,\tau(x),Dx)\right)D_k(x_n-x)(z)\,\mathrm{d}z\leqslant 0.\end{aligned}$$

By virtue of the continuity of $a_k(t,z,\cdot,\cdot)$, we have

$$\underline{\lim} \int_Z \sum_{k=1}^{N} \left(a_k(t,z,\tau(x_n),Dx) - a_k(t,z,\tau(x),Dx)\right) D_k(x_n - x)(z)\,\mathrm{d}z = 0$$
$$\Rightarrow \overline{\lim} \int_Z \sum_{k=1}^{N} \left(a_k(t,z,\tau(x_n),Dx_n) - a_k(t,z,\tau(x_n),Dx)\right) D_k(x_n - x)(z)\,\mathrm{d}z \leqslant 0.$$

By hypothesis $H(a)$ (iv), we have

$$\int_Z \sum_{k=1}^{N} \left(a_k(t,z,\tau(x_n),Dx_n) - a_k(t,z,\tau(x_n),Dx)\right) D_k(x_n - x)(z)\,\mathrm{d}z \to 0$$

and by passing to an appropriate subsequence if necessary, we may also assume that

$$\sum_{k=1}^{N} \left(a_k(t,z,\tau(x_n),Dx_n) - a_k(t,z,\tau(x_n),Dx)\right) D_k(x_n - x)(z) \to 0$$

and

$$\sum_{k=1}^{N} \left(a_k(t,z,\tau(x_n),Dx_n) - a_k(t,z,\tau(x_n),Dx)\right) D_k(x_n - x)(z) \leqslant h_1(z)$$

for all $z \in Z \setminus N_1$, $\lambda(N_1) = 0$ (λ being the Lebesque measure on Z) and with $h \in L^1(Z)$. Using hypothesis $H(a)$ (v), we see that for every $z \in Z \setminus N_1$ and every $n \geqslant 1$, we have

$$\begin{aligned} (3) \qquad h_1(z) &\geqslant \sum_{k=1}^{N} \left(a_k(t,z,\tau(x_n),Dx_n) - a_k(t,z,\tau(x_n),Dx)\right) D_k(x_n - x)(z) \\ &\geqslant c_1 \left(\|Dx_n(z)\|^p + \|Dx(z)\|^p\right) - 2\beta_1(t,z) \\ &\quad - \sum_{k=1}^{N} |D_k x(z)| \left(\beta_2(t,z) + c_2(|\tau(x_n)(t,z)|^{p-1} + |D_k x_n(z)|^{p-1})\right) \\ &\quad - \sum_{k=1}^{N} |D_k x_n(z)| \left(\beta_2(t,z) + c_2(|\tau(x_n)(t,z)|^{p-1} + |D_k x(z)|^{p-1})\right). \end{aligned}$$

Recall that $x_n(z) \to x(z)$ and $\tau(x_n)(t,z) \to \tau(x)(t,z)$ as $n \to \infty$ and moreover $|\tau(x_n)(t,z)| \leqslant \max[|\varphi(t,z)|, |\psi(t,z)|]$ for all $t \in T$ and all $z \in Z \setminus N_3$, $\lambda(N_3) = 0$. From (3) above it follows that for every $z \in Z \setminus N$, $N = \bigcup_{k=1}^{3} N_k$, the sequence $\{\|Dx_n(z)\|\}_{n \geqslant 1}$ is bounded. So for every $z \in Z \setminus N$, we can find a subsequence

$\{x_m(z)\}_{m\geqslant 1}$ of $\{x_n(z)\}_{n\geqslant 1}$, such that $\tau(x_m)(z) \to \tau(x)(z)$ and $D_k x_m(z) \to y_k(z)$ as $m \to \infty$. Hence in the limit as $m \to \infty$, we have for all $z \in Z \setminus N$ and for $y(z) = (y_k(z))_{k=1}^N$

$$\sum_{k=1}^{N} \left(a_k(t,z,\tau(x)(t,z),y(z)) - a_k(t,z,\tau(x)(t,z),Dx)\right)(y_k - D_k x)(z) = 0$$
$$\Rightarrow y_k(z) = D_k x(z) \quad \text{for all } k = 1,2,\ldots,N \quad \text{(see hypothesis } H(a) \text{ (iv))}.$$

So we deduce that $Dx_n(z) \to Dx(z)$ for all $z \in Z \setminus N$ as $n \to \infty$. Moreover, from (3) we have that

$$\begin{aligned}(4)\quad \|Dx_n(z)\|^p \leqslant h_1(z) + c_1\|Dx(z)\|^p + 2\beta_1(t,z) \\ + \sum_{k=1}^{N} |D_k x(z)| \left(\beta_2(t,z) + c_2(|\tau(x_n)(t,z)|^{p-1} + |D_k x_n(z)|^{p-1})\right) \\ + \sum_{k=1}^{N} |D_k x_n(z)| \left(\beta_2(t,z) + c_2(|\tau(x_n)(t,z)|^{p-1} + |D_k x(z)|^{p-1})\right)\end{aligned}$$

for all $z \in Z \setminus N$. Note that for $C \subseteq Z$ measurable, we have

$$\begin{aligned}(5)\quad &\sum_{k=1}^{N} \int_C |D_k x(z)| \left(\beta_2(t,z) + c_2(|\tau(x_n)(t,z)|^{p-1} + |D_k x_n(z)|^{p-1})\right) \mathrm{d}z \\ &\leqslant \sum_{k=1}^{N} \|\chi_C D_k x\|_p \left(\|\beta_2(t,\cdot)\|_q^q + \int_Z \left(c_2(|\tau(x_n)(t,z)|^{p-1} + |D_k x_n(z)|^{p-1})\right)^q \mathrm{d}z\right)^{1/q} \\ &\leqslant c_4 \left(\sum_{k=1}^{N} \|\chi_C D_k x_n\|_p^p\right)^{1/p} \left(\sum_{k=1}^{N} \|\beta_2(t,\cdot)\|_q^q + c_2\|\tau(x_n)(t,\cdot)\|_p^p + c_2\|Dx_n(\cdot)\|_p^p\right)^{1/q} \\ &\quad \text{for some } c_4 > 0 \\ &\leqslant c_5 \left(\sum_{k=1}^{N} \int_C |D_k x_n(z)|^p \,\mathrm{d}z\right)^{1/p} \quad \text{for some } c_5 > 0.\end{aligned}$$

Also we have

$$\begin{aligned}(6)\quad &\sum_{k=1}^{N} \int_C |D_k x_n(z)| \left(\beta_2(t,z) + c_2(|\tau(x)(t,z)|^{p-1} + |D_k x(z)|^{p-1})\right) \mathrm{d}z \\ &\leqslant \sum_{k=1}^{N} \left\|\chi_C \left(\beta_2(t,\cdot) + c_2(|\tau(x)(t,\cdot)|^{p-1} + |D_k x(\cdot)|^{p-1})\right)\right\|_q \|D_k x_n\|_p \\ &\leqslant c_6 \sum_{k=1}^{N} \int_C \left(\beta_2(t,z) + c_2(|\tau(x)(t,z)|^{p-1} + |D_k x(z)|^{p-1})\right) \mathrm{d}z \\ &\quad \text{for some } c_6 > 0.\end{aligned}$$

From (4), (5) and (6) it follows that $\{\|Dx_n(\cdot)\|^p\}$ is uniformly integrable. So from the extended dominated convergence theorem (see for example Ash [2], Theorem 7.5.2, p. 295), we infer that $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$ as $n \to \infty$. Therefore $x_n \to x$ in X as $n \to \infty$ and this proves that $A(t, \cdot)$ is of type $(S)_+$. □

Now we are ready for the existence theorem of this section.

Theorem 8. *If hypotheses $H(a)$, H_0 hold and $h \in L^q(T \times Z)$, then problem* (2) *admits a solution $x \in W_{pq}(T)$ such that $\psi(t,z) \leqslant x(t,z) \leqslant \varphi(t,z)$ a.e. on $T \times Z$.*

Proof. Let $x_0 \in [\psi(0,\cdot), \varphi(0,\cdot)] = \{y \in L^2(Z)\colon\ \psi(0,z) \leqslant y(z) \leqslant \varphi(0,z)$ a.e. on $Z\}$ and consider the following initial-boundary value problem

$$
(7)\qquad \begin{aligned}
&\frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t,z,\tau(x),Dx) + f(t,z,\tau(x)(t,z),D\tau(x)(t,z)) + \lambda u(t,z,x(t,z))\\
&\qquad = h(t,z) \text{ in } T \times Z\\
&x(0,z) = x_0(z) \text{ a.e. on } Z, x|_{T\times\Gamma} = 0.
\end{aligned}
$$

Here $\lambda > 0$ and is going to be fixed in the process of the proof. In what follows we consider the evolution triple $X = W_0^{1,p}(Z)$, $H = L^2(Z)$ and $X^* = W^{-1,q}(Z)$. Note that in this case the embeddings are compact. Let $L\colon\ D(L) \subseteq L^p(T,X) \to L^q(T,X^*)$ be defined by $L(x) = \dot{x}$ for all $x \in D(L) = \{x \in W_{pq}(T)\colon\ x(0) = 0\}$. Using the integration by parts formula for functions in $W_{pq}(T)$ (see Zeidler [38], Proposition 23.23, pp. 422–423), we obtain that $L^*\colon\ D^* \subseteq L^p(T,X) \to L^q(T,X^*)$ is defined by $L^*(v) = -\dot{v}$ for all $v \in D^* = \{v \in W_{pq}(T)\colon\ v(b) = 0\}$. So L is densely defined (since $C_0^\infty(T,X)$ is dense in $L^p(T,X)$), closed and both L and L^* are monotone. Therefore L is maximal monotone operator.

We will show that problem (7) has a solution. First assume that $x_0 \in X \cap [\psi(0,\cdot), \varphi(0,\cdot)]$. Define $A\colon\ T \times X \to X^*$ by

$$
\langle A(t,x), y\rangle = \int_Z \sum_{k=1}^{N} a_k(t,z,\tau(x),Dx) D_k y(z)\,\mathrm{d}z.
$$

Let $A_1\colon\ T \times X \to X^*$ be defined by $A_1(t,x) = A(t, x + x_0)$. Using Proposition 7, we have that $t \to A_1(t,x)$ is measurable, while $x \to A_1(t,x)$ is demicontinuous and of type $(S)_+$, thus demicontinuous and pseudomonotone. Let $\widehat{A}_1\colon\ L^p(T,X) \to L^q(T,X^*)$ be the Nemitsky operator corresponding to $A_1(\cdot,\cdot)$; i.e. $\widehat{A}_1(x)(\cdot) = A_1(\cdot, x(\cdot))$. By virtue of Proposition 1, $\widehat{A}_1(\cdot)$ is L-pseudomonotone. Also let $\widehat{F}\colon\ L^p(T,X) \to L^q(T \times Z)$ be defined by

$$
\widehat{F}(x)(t,z) = f(t,z,\tau(x)(t,z),D\tau(x)(t,z)) + \lambda u(t,z,x(t,z))
$$

By Lemmas 5 and 6, $\widehat{F}(\cdot)$ is continuous. Hence so is $\widehat{F}_1(x) = \widehat{F}(x + x_0)$. Now if $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ and $\overline{\lim}(\!(\widehat{A}_1(x_n) + \widehat{F}_1(x_n), x_n - x)\!) \leqslant 0$, we have

$$\begin{aligned} &\overline{\lim}(\!(\widehat{A}_1(x_n), x_n - x)\!) + \underline{\lim}(\!(\widehat{F}_1(x_n), x_n - x)\!) \leqslant 0 \\ \Rightarrow\ &\overline{\lim}(\!(\widehat{A}_1(x_n), x_n - x)\!) + \underline{\lim}(\widehat{F}_1(x_n), x_n - x)_{pq} \leqslant 0 \end{aligned}$$

with $(\cdot,\cdot)_{pq}$ being the duality brackets for the pair $(L^p(T \times Z), L^q(T \times Z))$. Since $W_{pq}(T)$ embeds compactly in $L^p(T \times Z)$, it follows that $x_n \to x$ in $L^p(T \times Z)$ and so $(\widehat{F}_1(x_n), x_n - x)_{pq} \to 0$. Therefore $\overline{\lim}(\!(\widehat{A}_1(x_n), x_n - x)\!) \leqslant 0$ and since by Proposition 3, $\widehat{A}_1(\cdot)$ is demicontinuous and of type L-$(S)_+$, it follows that $x_n \to x$ in $L^p(T, X)$ and $\widehat{A}_1(x_n) \xrightarrow{w} \widehat{A}_1(x)$. Then exploiting the continuity of $\widehat{F}_1(\cdot)$, we have $\widehat{A}_1(x_n) + \widehat{F}_1(x_n) \xrightarrow{w} \widehat{A}_1(x) + \widehat{F}_1(x)$ in $L^q(T, X^*)$ and $(\!(\widehat{A}_1(x_n) + \widehat{F}_1(x_n), x_n)\!) \to (\!(\widehat{A}_1(x) + \widehat{F}_1(x), x)\!)$ as $n \to \infty$, which proves the L-pseudomonotonicity of the bounded demicontinuous operator $\widehat{G}_1(x) = \widehat{A}_1(x) + \widehat{F}_1(x)$.

Next we will show that $\widehat{G}_1(\cdot)$ is coercive; i.e. $\frac{(\!(\widehat{G}_1(x), x)\!)}{\|x\|_{L^p(T,X)}} \to +\infty$ as $\|x\|_{L^p(T,X)} \to \infty$. To this end let $\widehat{G}(x) = \widehat{A}(x) + \widehat{F}(x)$, where $F(x)(t,z) = f(t,z,\tau(x)(t,z), D\tau(x)(t,z))$. Because of hypothesis $H(a)$ (v), we have

$$\int_0^b\!\!\int_Z \sum_{k=1}^N a_k(t,z,\tau(x),Dx) D_k x \,\mathrm{d}z\,\mathrm{d}t \geqslant c \int_0^b\!\!\int_Z \|Dx(t,z)\|^p \,\mathrm{d}z\,\mathrm{d}t = c\|x\|^p_{L^p(T,X)} \tag{8}$$

(recall that $\left(\sum\limits_{k=1}^N \|D_k x\|_p^p\right)^{1/p}$ is an equivalent norm on $W_0^{1,p}(Z)$). Also from Lemma 6, we have that

$$\lambda \int_0^b\!\!\int_Z u(t,z,x(t,z))x(t,z)\,\mathrm{d}z\,\mathrm{d}t \geqslant \lambda c_4 \|x\|^p_{L^p(T\times Z)} - \lambda c_5 \|x\|^{p-1}_{L^p(T\times Z)}. \tag{9}$$

In addition, because of hypothesis $H(f)$ (iii), we have

$$\begin{aligned} &\left| \int_0^b\!\!\int_Z f(t,z,\tau(x)(t,z), D\tau(x)(t,z))x(t,z)\,\mathrm{d}z\,\mathrm{d}t \right| \\ \leqslant\ &\int_0^b\!\!\int_Z \left(\beta_2(t,z) + c_2\left(|\tau(x)(t,z)|^{p-1} + \|D\tau(x)(t,z)\|^{p-1}\right)\right)|x(t,z)|\,\mathrm{d}z\,\mathrm{d}t. \end{aligned}$$

From Gilbarg-Trudinger [16] (p. 145), we know that

$$D\tau(x)(t,z) = \begin{cases} D\varphi(t,z) & \text{if } \varphi(t,z) \leqslant x(t,z) \\ Dx(t,z) & \text{if } \psi(t,z) \leqslant x(t,z) \leqslant \varphi(t,z) \\ D\psi(t,z) & \text{if } x(t,z) \leqslant \psi(t,z). \end{cases}$$

So via Hölder's inequality, we obtain

$$
\begin{aligned}
(10)\qquad & \left|\int_0^b\int_Z f(t,z,\tau(x)(t,z),D\tau(x)(t,z))x(t,z)\,\mathrm{d}z\,\mathrm{d}t\right| \\
&\leqslant \left(\|\beta_2\|_{L^q(T\times Z)}+c_6+\|Dx\|^{p-1}_{L^p(T\times Z,\mathbb{R}^N)}\right)\|x\|_{L^p(T\times Z)} \quad \text{(for some } c_6>0\text{)} \\
&\leqslant \left(c_7+\|x\|^{p-1}_{L^p(T,X)}\right)\|x\|_{L^p(T\times Z)} \quad \text{(for some } c_7>0\text{)} \\
&\leqslant c_8(\varepsilon)+c_9(\varepsilon)\|x\|^p_{L^p(T,X)}+c_{10}(\varepsilon)\|x\|_{L^p(T\times Z)} \\
&\quad \text{(by Young's inequality with } \varepsilon>0 \text{ and } c_8(\varepsilon),c_9(\varepsilon),c_{10}(\varepsilon)>0\text{)} \\
&\quad \Rightarrow \int_0^b\int_Z f(t,z,\tau(x)(t,z),D\tau(x)(t,z))\,\mathrm{d}z\,\mathrm{d}t \\
&\geqslant -c_8(\varepsilon)-c_9(\varepsilon)\|x\|^p_{L^p(T,X)}-c_{10}(\varepsilon)\|x\|^p_{L^p(T\times Z)}.
\end{aligned}
$$

From (8), (9) and (10), it follows that

$$
\begin{aligned}
(11)\qquad & ((G(x),x)) \geqslant (c-c_9(\varepsilon))\|x\|^p_{L^p(T,X)}+(\lambda c_4-c_{10}(\varepsilon))\|x\|^p_{L^p(T,X)} \\
&\quad -\lambda c_5\|x\|^p_{L^p(T,X)}-c_8(\varepsilon) \\
&\Rightarrow ((G_1(x),x)) \geqslant (c-\widehat{c}_9(\varepsilon))\|x \\
&\quad +x_0\|^p_{L^p(T,X)}+(\lambda c_4-c_{10}(\varepsilon))\|x+x_0\|^p_{L^p(T,X)}-\lambda\widehat{c}_5\|x \\
&\quad + x_0\|^p_{L^p(T,X)}-\widehat{c}_8(\varepsilon)
\end{aligned}
$$

for some $\widehat{c}_9(\varepsilon),\widehat{c}_5,\widehat{c}_8(\varepsilon)>0$. Let $\varepsilon>0$ be such that $c-\widehat{c}_9(\varepsilon)>0$. Then for this choice of $\varepsilon>0$, we choose $\lambda>0$ large enough so that $\lambda c_4-c_{10}(\varepsilon)>0$. Therefore (11) implies that $G_1(\cdot)$ is coercive.

Apply Proposition 2 to obtain $x\in D(L)$ such that $L(x)+G_1(x)=h$. Evidently $x+x_0=y$ solves (7) when $x_0\in X\cap[\psi(0,\cdot),\varphi(0,\cdot)]$. For the general case let $x_0\in[\psi(0,\cdot),\varphi(0,\cdot)]$ and let Proposition 2 $x_0^n\in X\cap[\psi(0,\cdot),\varphi(0,\cdot)]$ be such that $x_0^n\to x_0$ in H as $n\to\infty$. To see that such a sequence exists, let $\{y_0^n\}_{n\geqslant 1}\subseteq X$ be such that $y_0^n\to x_0$ in H as $n\to\infty$. Set $x_0^n=(y_0^n\vee\psi(0))\wedge\varphi(0)=(y_0^n\wedge\varphi(0))\vee\psi(0)$. From Gilbarg-Trudinger [16] (p. 145), we have that $x_0^n\in X$ for every $n\geqslant 1$. Moreover, from the continuity of the lattice operations in H, it follows that $x_0^n\to x_0$ in H as $n\to\infty$. Let $x_n\in W_{pq}(T)$ $n\geqslant 1$ be a solution of (7) with initial condition x_0^n. We have

$$
\begin{aligned}
(12)\qquad & \dot{x}_n+\widehat{A}(x_n)+\widehat{F}(x_n)=h,\ x_n(0)=x_0^n \quad n\geqslant 1 \\
&\Rightarrow ((\dot{x}_n,x_n))+((\widehat{A}(x_n),x_n))+((\widehat{F}(x_n),x_n))=((h,x_n)).
\end{aligned}
$$

From the integration by parts formula for functions in $W_{pq}(T)$, we have

$$
(13)\qquad ((\dot{x}_n,x_n))=\frac{1}{2}|x_n(b)|^2-\frac{1}{2}|x_0^n|^2\geqslant -c_{11}
$$

for some $c_{11} > 0$. Also from the previous estimations which established the coercivity of $\widehat{G}(\cdot)$, we have

$$(14) \qquad (\!(\widehat{A}(x_n) + \widehat{F}(x_n), x_n)\!) \geqslant c_{12}\|x_n\|^p_{L^p(T,X)} - \lambda c_{13}\|x_n\|^{p-1}_{L^p(T,X)} - c_{14}.$$

Using (13) and (14) in (12), we obtain

$$(15) \qquad c_{12}\|x_n\|^p_{L^p(T,X)} \leqslant \|h\|_{L^q(T,H)}\|x_n\|_{L^p(T,X)} + \lambda c_{13}\|x_n\|^{p-1}_{L^p(T,X)} + c_{15}$$

with $c_{15} = c_{11} + c_{14} > 0$. From (15) it follows at once that $\{x_n\}_{n\geqslant 1}$ is bounded in $L^p(T,X)$ and then by virtue of (12), hypotheses $H(a)$ (iii), $H(f)$ (iii) and Lemma 6, we obtain that $\{\dot{x}_n\}_{n\geqslant 1}$ is bounded in $L^q(T,X^*)$. So $\{x_n\}_{n\geqslant 1}$ is bounded in $W_{pq}(T)$ and by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ as $n \to \infty$. Since $W_{pq}(T)$ embeds continuously in $C(T,H)$, we also have $x_n \xrightarrow{w}$ in $C(T,H)$, thus $x_n(0) \xrightarrow{w} x(0)$ in H as $n \to \infty$. Therefore $x(0) = x_0$. Also we have

$$(16) \qquad \overline{\lim}(\!(\widehat{A}(x_n) + \widehat{F}(x_n), x_n - x)\!) \leqslant \overline{\lim}(\!(\dot{x}_n, x - x_n)\!).$$

Once again, employing the integration by parts formula in $W_{pq}(T)$, we obtain

$$(17) \qquad (\!(\dot{x}_n, x - x_n)\!) = -\frac{1}{2}\,|x(b) - x_n(b)|^2 + \frac{1}{2}\,|x(0) - x_0^n|^2 + (\!(\dot{x}, x - x_n)\!)$$
$$\Rightarrow \overline{\lim}(\!(\dot{x}_n, x - x_n)\!) \leqslant 0.$$

Using (17) in (16) and since $(\!(\widehat{F}(x_n), x_n - x)\!) = (\widehat{F}(x_n), x_n - x)_{pq} \to 0$ as $n \to \infty$, we infer that

$$\overline{\lim}(\!(\widehat{A}(x_n), x_n - x)\!) \leqslant 0.$$

But recall that $\widehat{A}(\cdot)$ is demicontinuous and of type L-$(S)_+$. Hence $x_n \to x$ in $L^p(T,X)$ and $\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^q(T,X^*)$. Because $\widehat{F}(\cdot)$ is continuous, we obtain $\widehat{F}(x_n) \to F(x)$ in $L^q(T \times Z)$. Thus in the limit as $n \to \infty$, we have $\dot{x} + \widehat{A}(x) + \widehat{F}(x) = h$, $x(0) = x_0$, which proves that $x \in W_{pq}(T)$ is a solution of (7) for the initial condition $x_0 \in [\psi(0,\cdot), \varphi(0,\cdot)]$. Therefore the solution set $S(x_0) \subseteq W_{pq}(T)$ of (7) is nonempty for every $x_0 \in [\psi(0,\cdot), \varphi(0,\cdot)]$.

Next let $K = [\psi, \varphi] = \{x \in C(T,H)\colon \psi(t,z) \leqslant x(t,z) \leqslant \varphi(t,z)$ a.e. on Z for every $t \in T\}$. We claim that for every $x_0 \in [\psi(0,\cdot), \varphi(0,\cdot)]$, $S(x_0) \subseteq K$. Indeed let $x \in S(x_0)$. Since ψ is a lower solution of (2), with $(\psi - x)_+ \in W_{pq}(T) \cap L^p(T \times Z)_+$

as our test function, we have

$$
\begin{aligned}
(18)\qquad & -\int_0^b \left\langle \frac{\partial\psi}{\partial t}, (\psi - x)_+ \right\rangle \mathrm{d}t - a(\psi, (\psi - x)_+) \\
& - \int_0^b\int_Z f(t, z, \psi, D\psi)(\psi - x)_+(t, z)\,\mathrm{d}z\,\mathrm{d}t \\
\geqslant & - \int_0^b\int_Z h(t, z)(\psi - x)_+(t, z)\,\mathrm{d}z\,\mathrm{d}t, \\
& \psi(0, z) \leqslant \psi(b, z) \quad \text{a.e. on } Z,\ \psi|_{T\times\Gamma} \leqslant 0.
\end{aligned}
$$

Since $x \in S(x_0)$, we have

$$
\begin{aligned}
(19)\qquad & \int_0^b \left\langle \frac{\partial x}{\partial t}, (\psi - x)_+ \right\rangle \mathrm{d}t + a(x, (\psi - x)_+) \\
& + \int_0^b\int_Z f(t, z, \tau(x), D\tau(x))(\psi - x)_+(t, z)\,\mathrm{d}z\,\mathrm{d}t \\
& + \lambda \int_0^b\int_Z u(t, z, x)(\psi - x)_+(t, z)\,\mathrm{d}z\,\mathrm{d}t \\
= & \int_0^b\int_Z h(t, z)(\psi - x)_+(t, z)\,\mathrm{d}z\,\mathrm{d}t, \\
& x(0, z) = x(b, z) \quad \text{a.e. on } Z,\ x|_{T\times\Gamma} = 0.
\end{aligned}
$$

Adding (18) and (19) above, we obtain

$$
\begin{aligned}
(20)\qquad & \int_0^b \left\langle \frac{\partial(x - \psi)}{\partial t}, (\psi - x)_+ \right\rangle \mathrm{d}t + a(x, (\psi - x)_+) - a(\psi, (\psi - x)_+) \\
& + \int_0^b\int_Z (f(t, z, \tau(x), D\tau(x)) - f(t, z, \psi, D\psi))(\psi - x)_+(t, z)\,\mathrm{d}z\,\mathrm{d}t \\
& + \lambda \int_0^b\int_Z u(t, z, x)(\psi - x)_+(t, z)\,\mathrm{d}z\,\mathrm{d}t \geqslant 0.
\end{aligned}
$$

By virtue of the integration by parts formula for functions in $W_{pq}(T)$ and since $x_0 \in [\psi(0,\cdot), \varphi(0,\cdot)]$, hence $(\psi(0,\cdot) - x(0,\cdot))_+ = 0$, we have

$$
(21)\qquad \int_0^b \left\langle \frac{\partial(x - \psi)}{\partial t}, (\psi - x)_+ \right\rangle \mathrm{d}t = -\frac{1}{2}\|\psi(b,\cdot) - x(b,\cdot)\|_2^2 \leqslant 0.
$$

Also because of hypothesis $H(a)$ (ii), we have

$$
\begin{aligned}
(22)\qquad & a(x, (\psi - x)_+) - a(\psi, (\psi - x)_+) \\
& = \int_0^b\int_Z \sum_{k=1}^N (a_k(t, z, \tau(x), Dx) - a_k(t, z, \psi, D\psi))D_k(\psi - x)_+(t, z)\,\mathrm{d}z\,\mathrm{d}t \leqslant 0.
\end{aligned}
$$

Finally from the definition of the truncation map $\tau(\cdot)$, we have

$$\int_0^b \int_Z (f(t,z,\tau(x),D\tau(x)) - f(t,z,\psi,D\psi))(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t = 0. \tag{23}$$

Using (21), (22) and (23) in (20) above, we obtain

$$\begin{aligned}
&\lambda \int_0^b \int_Z u(t,z,x(t,z))(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \geqslant 0 \\
&\Rightarrow 0 \leqslant \lambda \iint_{\{\psi \geqslant x\}} -(\psi - x)^{p-1}(t,z)(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \\
&\Rightarrow \|(\psi - x)_+\|^p_{L^p(T\times Z)} = 0,
\end{aligned}$$

hence $\psi(t,z) \leqslant x(t,z)$ for all $t \in T$ and almost all $z \in Z$.

In a similar fashion, we prove that $x(t,z) \leqslant \varphi(t,z)$ a.e. on Z, $t \in T$. Hence we have proved that for every $x_0 \in [\psi(0,\cdot),\varphi(0,\cdot)]$, $S(x_0) \subseteq K = [\psi,\varphi]$.

Now let $R\colon [\psi(0,\cdot),\varphi(0,\cdot)] \to 2^{[\psi(0,\cdot),\varphi(0,\cdot)]} \setminus \{\emptyset\}$ be the multifunction defined by

$$R(y) = (e_b \circ S)(y) = e_b(S(y)) = S(y)(b) = \{x(b)\colon\ x \in S(y)\}$$

(here $e_b\colon\ C(T,H) \to H$ is the evaluation at $t = b$ map). First note that if $y = \psi(0,\cdot)$, then for every $x \in S(y)$, we have $\psi(0,\cdot) \leqslant x(b,\cdot) \in R(y)$. Next let $v_1 \in R(y_1), y_1 \leqslant v_1$ and $y_1 \leqslant y_2$ (the order being the usual pointwise partial order on H). We claim that there is $v_2 \in R(y_2)$ such that $v_1 \leqslant v_2$. Indeed let $x_1 \in S(y_1)$ such that $x_1(b) = v_1$, $x_1(0) = y_1$ and let τ_1 be the truncation map and u_1 the penalty function corresponding to the pair (x_1,φ). Note that since $y_1 \leqslant v_1$, $x_1(\cdot,\cdot)$ is a lower solution for problem (2). Consider the following initial-boundary value problem:

$$\begin{aligned}
&\frac{\partial x}{\partial t} - \sum_{k=1}^N D_k a_k(t,z,\tau_1(x),Dx) + f(t,z,\tau_1(x),D\tau_1(x)) + \lambda u_1(t,z,x(t,z)) \\
&\qquad = h(t,z) \ \text{ in } T\times Z \\
&x(0,z) = y_2(z) \ \text{ a.e. on } Z, x|_{T\times\Gamma} = 0.
\end{aligned} \tag{24}$$

Exactly as we did for problem (2), we can show that problem (24) above has a nonempty solution set $S_1(y_2) \subseteq W_{pq}(T)$ and that $S_1(y_2) \subseteq K_1 = [x_1,\varphi]$. In particular, if $x_2 \in S_1(y_2)$, then $x_2 \in S(y_2)$ and $x_1 \leqslant x_2$.

Next we claim that $S(x_0)$ is compact in $C(T,H)$ for every $x_0 \in [\psi(0,\cdot),\varphi(0,\cdot)]$. To this end $\{x_n\}_{n\geqslant 1} \subseteq S(x_0)$. From earlier consideration, we know that $\{x_n\}_{n\geqslant 1}$ is bounded in $W_{pq}(T)$. Thus by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ as $n \to \infty$. From an earlier part of the proof, we know that

$x \in S(x_0)$ and $\overline{\lim}(\!(\widehat{A}(x_n), x_n - x)\!) \leqslant 0$. Since $\widehat{A}(\cdot)$ is of type L-$(S)_+$, we have that $x_n \to x$ in $L^p(T, X)$ as $n \to \infty$. From the integration by parts formula for functions in $W_{pq}(T)$, we obtain

$$\begin{aligned} \frac{1}{2}|x_n(t) - x(t)|^2 &\leqslant \int_0^t \langle \dot{x}_n(s) - \dot{x}(s), x_n(s) - x(s) \rangle \, \mathrm{d}s \\ &\leqslant \|\dot{x}_n - \dot{x}\|_{L^p(T,X^*)} \|x_n - x\|_{L^p(T,X)} \\ &\leqslant M\|x_n - x\|_{L^p(T,X)} \end{aligned}$$

for some $M > 0$ since $\{\dot{x}_n\}_{n \geqslant 1} \subseteq L^q(T, X^*)$ is bounded. Therefore

$$\begin{aligned} &\|x_n - x\|_{C(T,H)} \to 0 \quad \text{as } n \to \infty \\ \Rightarrow\ & S(x_0) \ \text{ is compact in } C(T, H) \ \text{ as claimed.} \end{aligned}$$

Then $R = e_b \circ S$ is a multifunction with nonemty compact values in $[\psi(0, \cdot), \varphi(0, \cdot)]$. Since the positive cone in $L^2(Z)$ is regular, we can apply Proposition 2.2 of Heikkila-Hu [17] and produce $y \in [\psi(0, \cdot), \varphi(0, \cdot)]$ such that $y = R(y)$. If $x \in S(y) \subseteq W_{pq}(T)$ is such that $x(b) = x(0) = y$, then $x \in W_{pq}(T)$ is the desired solution of (2). $\square$

4. Extremal solutions

In this section we consider a version of (2) in which the coefficient functions a_k are independent of x and the term f is independent of the gradient Dx. Moreover, the continuity condition on $f(t, z, \cdot)$ is replaced by a weak monotonicity condition. For such a problem we prove the existence of extremal solutions in the order interval $[\psi, \varphi]$; i.e. we show that there exist solutions $x_*, x^* \in [\psi, \varphi]$ such that for every solution $x \in [\psi, \varphi]$, we have $x_*(t, z) \leqslant x(t, z) \leqslant x^*(t, z)$ for all $t \in T$ and almost all $z \in Z$.

So let $T, Z \subseteq \mathbb{R}^N$ be as in section 3. On $T \times Z$ we consider the following nonlinear periodic parabolic problem:

$$(25) \qquad \begin{aligned} &\frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t, z, Dx) = f(t, z, x(t, z)) \ \text{ in } T \times Z \\ &x(0, z) = x(b, z) \ \text{ a.e. on } Z, x|_{T \times \Gamma} = 0. \end{aligned}$$

Our hypotheses concerning the data of problem (25), are the following:
$\mathbf{H(a)_1}$: $a_k \colon T \times Z \times \mathbb{R}^N \to \mathbb{R}$, $k = 1, 2, \ldots, N$, are functions such that
(i) for every $\xi \in \mathbb{R}^N$, $(t, z) \to a_k(t, z, \xi)$ is measurable;
(ii) for every $(t, z) \in T \times Z$, $\xi \to a_k(t, z, \xi)$ is continuous;

(iii) for every $\xi \in \mathbb{R}^N$, $|a_k(t,z,\xi)| \leqslant \beta_1(t,z) + c_1\|\xi\|^{p-1}$ a.e on $T \times Z$, with $\beta_1 \in L^q(T \times Z)$, $c_1 > 0$, $2 \leqslant p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$;

(iv) $\sum\limits_{k=1}^{N} \Big(a_k(t,z,\xi) - a_k(t,z,\xi')\Big)(\xi_k - \xi_k') \geqslant 0$ a.e. on $T \times Z$, for every $x \in \mathbb{R}$ and every $\xi, \xi' \in \mathbb{R}^N$;

(v) $\sum\limits_{k=1}^{N} a_k(t,z,\xi)\xi_k \geqslant c\|\xi\|^p$ a.e. on $T \times Z$ for every $\xi \in \mathbb{R}^N$ and with $c > 0$.

$\mathbf{H_0^1}$: There exist an upper solution $\varphi \in \widehat{W}_{pq}(T)$ and a lower solution $\psi \in \widehat{W}_{pq}(T)$ for problem (25) such that $\psi(t,z) \leqslant \varphi(t,z)$ for all $t \in T$ and almost all $z \in Z$.

$\mathbf{H(f)_1}$: $f\colon T \times Z \times \mathbb{R} \to \mathbb{R}$, is a function such that

(i) $f(\cdot,\cdot,\varphi(\cdot,\cdot)), f(\cdot,\cdot,\psi(\cdot,\cdot)) \in L^q(T \times Z)$;

(ii) there exists $M \geqslant 0$ such that for almost all $(t,z) \in T \times Z$ $x \to f(t,z,x) + Mx$ is strictly increasing on the interval $[\psi(t,z), \varphi(t,z)]$;

(iii) if $x \in C(T, L^2(Z))$ and for all $t \in T$ and almost all $z \in Z$, $\psi(t,z) \leqslant x(t,z) \leqslant \varphi(t,z)$, then $(t,z) \to f(t,z,x(t,z))$ is measurable.

Remark. If $f(\cdot,\cdot,\cdot)$ is a jointly Borel measurable function or more generally a Shragin function (see Appell-Zabrejko [1]), then hypothesis $H(f)_1$ (iii) is satisfied. This includes the case where $f(t,z,x)$ is a Caratheodory function; i.e. measurable in (t,z) and continuous in x. Moreover, by virtue of hypothesis $H(f)_1$ (ii) and Theorem 1.9 of Appell-Zabrejko [1], we see that $H(f)_1$ (iii) holds if and only if f is equivalent to a Borel function; i.e. there exists a Borel function $f_1\colon T \times Z \times \mathbb{R} \to R$ such that $f(t,z,x) = f_1(t,z,x)$ for all $(t,z) \in (T \times Z) \setminus N$ and all $x \in \mathbb{R}$, with N being a Lebesgue-null subset of $T \times Z$.

Theorem 9. *If hypotheses $H(a)_1, H_0^1$ and $H(f)_1$ hold, then problem (25) has extremal solutions in the order interval $K = [\psi, \varphi] = \{x \in C(T, L^2(Z))\colon \psi(t,z) \leqslant x(t,z) \leqslant \varphi(t,z)$ for all $t \in T$ and almost all $z \in Z\}$.*

Proof. As in the proof of Theorem 8, let $X = W_0^{1,p}(Z)$, $H = L^2(Z)$ and $X^* = W^{-1,q}(Z)$. Let $K_0 = [\psi(0,\cdot), \varphi(0,\cdot)] = \{y_0 \in H\colon \psi(0,z) \leqslant y_0(z) \leqslant \varphi(0,z)$ a.e. on $Z\}$. Given $(y, y_0) \in K \times K_0$, we consider the following nonlinear parabolic initial-boundary value problem:

$$\frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t,z,Dx) + Mx(t,z) = f(t,z,y(t,z)) + My(t,z) \quad \text{in } T \times Z \tag{26}$$

$$x(0,z) = y_0(z) \quad \text{a.e. on } Z, x|_{T\times\Gamma} = 0.$$

If $A\colon T \times X \to X^*$ is defined by $\langle A(t,x), y\rangle = \int_Z \sum\limits_{k=1}^{N} a_k(t,z,Dx)y(z)\,\mathrm{d}z$, then we can easily verify that $t \to A(t,x)$ is measurable, $x \to A(t,x)$ is demicontinuous and monotone (see hypothesis $H(a)$ (iv)), $\|A(t,x)\|_* \leqslant \widehat{a}(t) + \widehat{c}\|x\|^{p-1}$ a.e. on T, with $\widehat{a} \in L^q(T)$, $\widehat{c} > 0$ and $\langle A(t,x), x\rangle \geqslant c_0\|x\|^{p-1}$ for some $c_0 > 0$. Thus from a well-known existence theorem for evolution equations (see for example Zeidler [38], Theorem 30.A, p. 771), we infer that problem (26) has a unique solution $x = S(y, y_0) \in W_{pq}(T)$.

We claim that $S(K, K_0) \subseteq K$. To this end let $(y, y_0) \in K \times K_0$ and let $x = S(y, y_0)$. Because ψ is a lower solution of (25), by using $(\psi - x)_+ \in W_{pq}(T) \cap L^p(T \times Z)_+$ as our test function, we obtain

$$\begin{aligned} &-\int_0^b \left\langle \frac{\partial\psi}{\partial t}, (\psi - x)_+ \right\rangle \mathrm{d}t - \int_0^b\int_Z \sum_{k=1}^N a_k(t,z,D\psi)D_k(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \\ &\quad\leqslant -\int_0^b\int_Z f(t,z,\psi(t,z))(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t, \\ &\qquad \psi(0,z) \leqslant \psi(b,z) \quad \text{a.e. on } Z, \quad \psi|_{T\times\Gamma} \leqslant 0. \end{aligned} \tag{27}$$

Also since $x \in W_{pq}(T)$ is a solution of (26), we have

$$\begin{aligned} &\int_0^b \left\langle \frac{\partial x}{\partial t}, (\psi - x)_+ \right\rangle \mathrm{d}t + \int_0^b\int_Z \sum_{k=1}^N a_k(t,z,Dx)D_k(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \\ &\qquad + M\int_0^b\int_Z x(t,z)(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \\ &\quad = -\int_0^b\int_Z f(t,z,y(t,z))(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \\ &\qquad + M\int_0^b\int_Z y(t,z)(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t. \end{aligned} \tag{28}$$

Adding (27) and (28), we obtain

$$\begin{aligned} &\int_0^b \left\langle \frac{\partial(x - \psi)}{\partial t}, (\psi - x)_+ \right\rangle \mathrm{d}t \\ &\qquad + \int_0^b\int_Z \sum_{k=1}^N (a_k(t,z,Dx) - a_k(t,z,D\psi))D_k(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \\ &\qquad + M\int_0^b\int_Z x(t,z)(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \\ &\quad \geqslant \int_0^b\int_Z (f(t,z,y(t,z)) - f(t,z,\psi(t,z)))(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \\ &\qquad + M\int_0^b\int_Z y(t,z)(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t. \end{aligned} \tag{29}$$

Note that

$$\int_0^b \left\langle \frac{\partial(x-\psi)}{\partial t}, (\psi - x)_+ \right\rangle \mathrm{d}t = -\frac{1}{2}\|(\psi(b,\cdot) - x(b,\cdot))_+\|^2_{L^2(Z)} \leqslant 0. \tag{30}$$

Also because of hypothesis $H(a)_1$ (iv), we have

$$\int_0^b \int_Z \sum_{k=1}^N (a_k(t,z,Dx) - a_k(t,z,D\psi)) D_k(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \leqslant 0. \tag{31}$$

Finally hypothesis $H(f)_1$ (ii) implies that

$$\begin{aligned} &\int_0^b \int_Z (f(t,z,y(t,z)) + My(t,z) - f(t,z,\psi(t,z)) \\ &- Mx(t,z))(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \\ \geqslant &\int_0^b \int_Z (f(t,z,y(t,z)) + My(t,z) - f(t,z,\psi(t,z)) \\ &- M\psi(t,z))(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t \geqslant 0. \end{aligned} \tag{32}$$

Combining (29) $\rightarrow$ (32), we obtain

$$\begin{aligned} &\int_0^b \int_Z (f(t,z,y(t,z)) + My(t,z) - f(t,z,\psi(t,z)) \\ &- Mx(t,z))(\psi - x)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t = 0 \\ \Rightarrow &\iint_{\{\psi \geqslant x\}} (f(t,z,y(t,z)) + My(t,z) - f(t,z,\psi(t,z)) \\ &- Mx(t,z))(\psi - x)(t,z)\,\mathrm{d}z\,\mathrm{d}t = 0. \end{aligned}$$

This last equality in conjunction with hypothesis $H(f)$ (ii), implies that $T_+ = \{t \in T\colon \psi(t,\cdot) > x(t,\cdot) \text{ in } L^2(Z)\}$ is Lebesgue-null. So because $t \rightarrow x(t,\cdot)$, $t \rightarrow \psi(t,\cdot) \in C(T, L^2(Z))$, we infer that $x(t,z) \geqslant \psi(t,z)$ for all $t \in T$ and almost all $z \in Z$. Similarly we show that $x(t,z) \leqslant \varphi(t,z)$ for all $t \in T$ and almost all $z \in Z$. Hence $x \in K$ and so $S(K, K_0) \subseteq K$.

Next we will show that $S(\cdot,\cdot)$ is a monotone increasing operator from $K \times K_0$ into K. To this end let $(y_1, y_0^1), (y_2, y_0^2) \in K \times K_0$ and assume that $y_1(t,\cdot) \leqslant y_2(t,\cdot)$ in $L^2(Z)$ for all $t \in T$ and $y_0^1 \leqslant y_0^2$ in $L^2(Z)$. Let $x_1 = S(y_1, y_0^1)$ and $x_2 = S(y_2, y_0^2)$.

We have

$$(33)\quad \int_0^b \left\langle \frac{\partial x_1}{\partial t}, (x_1 - x_2)_+ \right\rangle \mathrm{d}t + \int_0^b \int_Z \sum_{k=1}^N a_k(t, z, Dx_1) D_k (x_1 - x_2)_+(t, z) \,\mathrm{d}z \,\mathrm{d}t$$
$$+ M \int_0^b \int_Z x_1(t, z)(x_1 - x_2)_+(t, z) \,\mathrm{d}z \,\mathrm{d}t$$
$$= \int_0^b \int_Z f(t, z, y_1(t, z))(x_1 - x_2)_+(t, z) \,\mathrm{d}z \,\mathrm{d}t$$
$$+ M \int_0^b \int_Z y_1(t, z)(x_1 - x_2)_+(t, z) \,\mathrm{d}z \,\mathrm{d}t.$$

Also we have

$$(34)\quad - \int_0^b \left\langle \frac{\partial x_2}{\partial t}, (x_1 - x_2)_+ \right\rangle \mathrm{d}t - \int_0^b \int_Z \sum_{k=1}^N a_k(t, z, Dx_2) D_k (x_1 - x_2)_+(t, z) \,\mathrm{d}z \,\mathrm{d}t$$
$$- M \int_0^b \int_Z x_2(t, z)(x_1 - x_2)_+(t, z) \,\mathrm{d}z \,\mathrm{d}t$$
$$= - \int_0^b \int_Z f(t, z, y_2(t, z))(x_1 - x_2)_+(t, z) \,\mathrm{d}z \,\mathrm{d}t$$
$$- M \int_0^b \int_Z y_2(t, z)(x_1 - x_2)_+(t, z) \,\mathrm{d}z \,\mathrm{d}t.$$

Adding (33) and (34), we obtain

$$(35)\quad \int_0^b \left\langle \frac{\partial (x_1 - x_2)}{\partial t}, (x_1 - x_2)_+ \right\rangle \mathrm{d}t + \int_0^b \int_Z \sum_{k=1}^N (a_k(t, z, Dx_1)$$
$$- a_k(t, z, Dx_2)) D_k (x_1 - x_2)_+(t, z) \,\mathrm{d}z \,\mathrm{d}t$$
$$+ M \int_0^b \int_Z (x_1 - x_2)(x_1 - x_2)_+ \,\mathrm{d}z \,\mathrm{d}t$$
$$= \int_0^b \int_Z \left(f(t, z, y_1(t, z)) - f(t, z, y_2(t, z)) \right) (x_1 - x_2)_+(t, z) \,\mathrm{d}z \,\mathrm{d}t$$
$$+ M \int_0^b \int_Z (y_1 - y_2)(x_1 - x_2)_+ \,\mathrm{d}z \,\mathrm{d}t.$$

Observe that $(x_1(0, \cdot) - x_2(0, \cdot))_+ = 0$, because $x_1(0, \cdot) = y_0^1 \leqslant y_0^2 = x_2(0, \cdot)$ in $L^2(Z)$. So

$$(36)\quad \int_0^b \left\langle \frac{\partial (x_1 - x_2)}{\partial t}, (x_1 - x_2)_+ \right\rangle \mathrm{d}t = \frac{1}{2} \|x_1(b, \cdot) - x_2(b, \cdot)\|^2_{L^2(Z)} \geqslant 0.$$

Also hypothesis $H(a)_1$ (ii) implies that

$$\int_0^b \int_Z \sum_{k=1}^N (a_k(t,z,Dx_1) - a_k(t,z,Dx_2)) D_k(x_1 - x_2)(t,z)\,\mathrm{d}z\,\mathrm{d}t \geqslant 0. \tag{37}$$

Furthermore, note that

$$M \int_0^b \int_Z (x_1 - x_2)(x_1 - x_2)_+\,\mathrm{d}z\,\mathrm{d}t \geqslant 0. \tag{38}$$

Using (36) → (38) in (35) and exploiting the monotonicity of $x \to f(t,z,x) + Mx$ (see hypothesis $H(f)_1$ (ii)), we obtain

$$\begin{aligned} \int_0^b \int_Z \big(f(t,z,y_1(t,z)) + My_1(t,z) - f(t,z,y_2(t,z)) \\ - My_2(t,z)\big)(x_1 - x_2)_+(t,z)\,\mathrm{d}z\,\mathrm{d}t = 0 \\ \Rightarrow x_1(t,\cdot) \leqslant x_2(t,\cdot) \ \text{ in } L^2(Z) \ \text{ for every } t \in T. \end{aligned}$$

So $S(\cdot,\cdot)$ is monotone increasing from $K \times K_0$ into K.

Next let $\widehat{e}_b\colon C(T,H) \to C(T,H) \times H$ be defined by $\widehat{e}_b(x) = (x, x(b))$. Then define $R\colon K \times K_0 \to K \times K_0$ by $R = \widehat{e}_b \circ S$. Evidently $R(\cdot,\cdot)$ is a nondecreasing map on $K \times K_0$. Let $\{(y_n, y_0^n)\}_{n\geqslant 1}$ be a monotone sequence in $K \times K_0$. From the monotone convergence theorem, we have $y_0^n \to y_0$ in H as $n \to \infty$. Let $\widehat{f}(t,y)(\cdot) = f(t,\cdot,y(\cdot))$ for every $y\colon Z \to \mathbb{R}$ measurable (i.e. the Nemitsky operator corresponding to $f(t,z,x)$). Then if $u_n(\cdot) = \widehat{f}(\cdot, y_n(\cdot)) + My_n$, $n \geqslant 1$, by hypotheses $H(f)_1$ we see that $\{u_n\}_{n\geqslant 1}$ is bounded in $L^q(T,H)$. So by passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in $L^q(T,H)$. If $x_n = S(y_n, y_0^n)$, $n \geqslant 1$, we have

$$\begin{aligned} \dot{x}_n(t) + A(t, x_n(t)) = u_n(t) \text{ a.e. on } T \\ x_n(0) = y_0^n. \end{aligned}$$

Invoking Proposition 4, we have $x_n \to x$ in $C(T,H)$ as $n \to \infty$, with $x \in W_{pq}(T)$ being the unique solution of

$$\begin{aligned} \dot{x}(t) + A(t, x(t)) = u(t) \text{ a.e. on } T \\ x(0) = y_0. \end{aligned}$$

Then $R(y_n, y_0^n) = (x_n, x_n(b)) \to (x, x(b)) = R(y, y_0)$ in $X(T,H)) \times H$ as $n \to \infty$. So we can apply Theorem 1.2.2, p. 23, of Heikkila-Lakshmikantham [18] and produce (x_*, x_0^1) and (x^*, x_0^2) the least and greatest fixed points in $K \times K$ of R (extremal fixed points). Evidently x_* and x^* are the extremal solutions of (25) in $K = [\psi, \varphi]$. □

In this section we focus our attention to nonlinear parabolic with discontinuities. So with T and $Z \subseteq \mathbb{R}^N$ as in the previous sections, we consider the following nonlinear initial-boundary value problem:

$$\begin{aligned} &\frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t,z,Dx) = f(x(t,z)) \text{ in } T \times Z \\ &\qquad\qquad x(0,z) = x_0(z) \text{ a.e. on } Z,\ z|_{T\times\Gamma} = 0. \end{aligned} \tag{39}$$

Here $f\colon \mathbb{R} \to \mathbb{R}$ is a locally bounded, measurable but in general discontinuous function. It is well-known that in the absence of continuity hypotheses on $f(\cdot)$, in general we can not expect to have solutions for (39). In this case it is advisable to consider instead a multivalued version of (39), for which an adequate existence theory can be established. This approach is developed in the book Filippov [15] (for ordinary differential equations), in the papers of Rauch [34] and Chang [7] (for semilinear elliptic equations) and in the paper of Feireisl [14] (for semilinear parabolic problems). Parabolic problems with discontinuities arise in various problems of mathematical physics and engineering.

To introduce a multivalued variant of (39), for which we will be able to prove an existence theorem, we define $F(r) = [f_1(r), f_2(r)]$, $r \in \mathbb{R}$, where $f_1(r) = \underline{\lim}_{t\to r} f(t)$ and $f_2(r) = \overline{\lim}_{t\to r} f(t)$. Let $\varrho\colon \mathbb{R} \to \mathbb{R}$ be defined by $\varrho(r) = \int_0^r f(t)\,\mathrm{d}t$. Then $\varrho(\cdot)$ is locally Lipschitz and se we can define its subdifferential $\partial\varrho(r)$ in the sense of Clarke [8]. Then $\partial\varrho(r) \subseteq F(r)$ and if the one sided limits $f(r^{\pm})$ exist at $r \in \mathbb{R}$, then $\partial\varrho(r) = F(r)$ (see Chang [7]). In a more applied language, this last equality implies that the multivalued law is characterized by the Clarke subdifferential of a nonsmooth potential $\varrho(\cdot)$. Then instead of (39), we consider the following multivalued nonlinear parabolic problem:

$$\begin{aligned} &\frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t,z,Dx) \in F(x(t,z)) \text{ in } T \times Z \\ &\qquad\qquad x(0,z) = x_0(z) \text{ a.e. on } Z, x|_{T\times\Gamma} = 0. \end{aligned} \tag{40}$$

Definition. A function $\varphi \in \widehat{W}_{pq}(T)$ is said to be an “upper solution” of (40), if $f_2(\varphi(\cdot,\cdot)) \in L^q(T \times Z)$,

$$\left(\left(\frac{\partial\varphi}{\partial t}, u\right)\right) + \int_0^b\int_Z \sum_{k=1}^{N} a_k(t,z,Dx) D_k u(t,z)\,\mathrm{d}z\,\mathrm{d}t \geqslant \int_0^b\int_Z f_2(\varphi(t,z))u(t,z)\,\mathrm{d}z\,\mathrm{d}t$$

for all $u \in L^p(T, W_0^{1,p}(Z)) \cap L^p(T\times Z)_+$, $\varphi(0,z) \geqslant x_0(z)$ a.e. on Z and $\varphi|_{T\times\Gamma} \geqslant 0$. Similarly a function $\psi \in \widehat{W}_{pq}(T)$ is a “lower solution” of (40), if $f_1(\psi(\cdot,\cdot)) \in L^q(T\times Z)$ and the inequalities in the previous definition are reversed and f_2 is replaced by f_1.

$\mathbf{H_0^2}$: There exist an upper solution φ and a lower solution ψ such that $\psi(t,z) \leqslant \varphi(t,z)$ for all $t \in T$ and almost all $z \in Z$.

Our hypotheses on the nonlinear discontinuity $f(t,z,x)$, are the following:

$\mathbf{H(f)_2}$: $f\colon \mathbb{R} \to \mathbb{R}$ belong in $L^\infty_{loc}(\mathbb{R})$ and for almost all $(t,z) \in T \times Z$ and all $r \in [\psi(t,z), \varphi(t,z)]$, we have that $|f(r)| \leqslant \beta_2(t,z) + c_2(t)|r|$, with $\beta_2 \in L^2(T \times Z)$ and $c_2 \in L^2(T)$.

Let Y be a separable Banach space and let $P_f(Y)$ (resp. $P_{fc}(Y)$) denote the family of nonempty, closed (resp. nonempty, closed, convex) subsets of Y. On $P_f(Y)$ we can define a generalized metric, known in the literature as "Hausdorff metric", by setting

$$h(A,C) = \max\left[\sup_{a \in A} d(a,C), \sup_{c \in C} d(c,A)\right]$$

for all $A, C \in P_f(Y)$. It is well-known that $(P_f(Y), h)$ is a complete metric space and $(P_{fc}(Y), h)$ is closed (hence complete) subspace of it (see for example Klein-Thompson [23]). Also let $h^*(A,C) = \sup[d(a,C)\colon \alpha \in A]$. If V is a Hausdorff topological space, a multifunction (set-valued function) $G\colon V \to 2^V \setminus \{\emptyset\}$ is said to be "h^*-upper semicontinuous (h^*-usc)", if for every $v \in V$ the function $v' \to h^*(G(v'), G(v))$ is continuous at v. Recall that if $G(\cdot)$ is upper-semicontinuous (i.e. for every $C \subseteq Y$ closed, $G^-(C) = \{v \in V\colon\ G(v) \cap C \neq \emptyset\}$ is closed in V), then $G(\cdot)$ is h^*-usc, while the converse is true if $G(\cdot)$ has compact values. Moreover, both notions imply that $GrG = \{(v,y) \in V \times Y\ y \in G(v)\}$ is closed in $V \times Y$. A multifunction $G\colon V \to P_f(Y)$ is said to be "h-continuous" (resp. h-Lipschitz), if it is continuous (resp. Lipschitz) as a function from V into $(P_f(Y), h)$. For details on these and related notions we refer to DeBlasi-Myjak [11]. Finally a multifunction $F\colon T \to P_f(Y)$ is said to be measurable if $GrF = \{(t,y) \in T \times Y\colon\ y \in F(t)\} \in \mathcal{L} \times B(Y)$, with $\mathcal{L}$ being the Lebesque σ-field of T and $B(Y)$ the Borel σ-field of Y.

The following lemma can be proved as Proposition 4.1 of DeBlasi [9], with minor obvious modifications to accomodate the presence of $t \in T$.

Lemma 10. *If $T = [0,b]$, Y is a seperable Banach space and $F\colon T \times Y \to P_{fc}(Y)$ is a multifunction which is measurable in $t \in T$, h^*-usc in $y \in Y$ and $|F(t,y)| = \sup[\|v\|_Y\colon\ v \in F(t,y)] \leqslant \theta(t)$ a.e. on T with $\theta \in L^2(T)$, then there exists a sequence of multifunctions $F_n\colon T \times Y \to P_{fc}(Y)$, $n \geqslant 1$, such that for every $y \in Y$ there exist $k(y) > 0$ and $\varepsilon > 0$ such that if $y_1, y_2 \in \overline{B_\varepsilon(y)} = \{y' \in Y\colon\ \|y' - y\| \leqslant \varepsilon\}$, then $h(F_n(t,y_1), F_n(t,y_2)) \leqslant k(y)\theta(t)\|y_1 - y_2\|_Y$ a.e. on T (i.e. $F_n(t,\cdot)$ is locally h-Lipschitz), $F(t,y) \subseteq \ldots \subseteq F_n(t,y) \subseteq F_{n+1}(t,y) \subseteq \ldots$, $|F_n(t,y)| = \sup[\|v\|_Y\colon\ v \in F_n(t,y)] \leqslant \theta(t)$ a.e. on T, $n \geqslant 1$, for every $[t,y] \in T \times Y$ $F_n(t,y) \xrightarrow{h} F(t,y)$ as $n \to \infty$ and there exists $u_n\colon T \times Y \to Y$, $n \geqslant 1$, measurable in t, locally Lipschitz in y and for every $[t,y] \in T \times Y$ $u_n(t,y) \in F_n(t,y)$, $n \geqslant 1$. Moreover, if $F(t,\cdot)$ is* *h-continuous, then $t \to F_n(t,x)$ is measurable (hence $(t,x) \to F_n(t,x)$ is measurable too; see Papageorgiou [29]).*

In this analysis we will be using the following truncation map $\widehat{\tau}\colon T \times L^2(Z) \to L^2(Z)$ and penalty map $B\colon T \times L(Z) \to L^q(Z)$, defined by

$$\widehat{\tau}(t,x)(z) = \begin{cases} \varphi(t,z) & \text{if } \varphi(t,z) \leqslant x(z) \\ x(z) & \text{if } \psi(t,z) \leqslant x(z) \leqslant \varphi(t,z) \\ \psi(t,z) & \text{if } x(z) \leqslant \psi(t,z) \end{cases}$$

and

$$B(t,x)(z) = \begin{cases} x(z) - \varphi(t,z) & \text{if } \varphi(t,z) \leqslant x(z) \\ 0 & \text{if } \psi(t,z) \leqslant x(z) \leqslant \varphi(t,z) \\ x(z) - \psi(t,z) & \text{if } x(z) \leqslant \psi(t,z) \end{cases}$$

it is straightforward to verify the validity of the following lemmas:

Lemma 11. *$\widehat{\tau}(t,x)$ is measurable in t and continuous in x.*

Remark. From Gilbarg-Trudinger [16] (p. 145), we know that for every $t \in T$ and every $x \in W^{1,p}(Z)$, $\widehat{\tau}(t,x) \in W^{1,p}(Z)$.

Lemma 12. *$B(t,x)$ is measurable in t, continuous in x and satisfies the following growth condition: there exist $a^* \in L^2(T)$ and $c^* > 0$ such that for almost all $t \in T$ and all $x \in L^2(Z)$*

$$\|B(t,x)\|_2 \leqslant a^*(t) + c^*\|x\|_2.$$

Now we are ready for a theorem, which not only establishes the existence of a solution in $K = [\psi, \varphi]$ for problem (4), but also provides information about the topological structure of this solution set. The set of solutions of (40) located in $K = [\psi, \varphi]$, will be denoted by $S(x_0)$.

Definition. By an "R_δ-set" we mean a set S in a metric space Y which is homeomorphic to the intersection of a decreasing sequence $\{S_n\}_{n \geqslant 1}$ of absolute retracts. If every S_n is compact, we say that S is a "compact R_δ-set".

Remark. Recall that a closed subset A of Y is said to be an "absolute retract" (AR), if every homeomorphic image of A in a metric space V, is retract of V. A subset C of V is said to be a "retract", if there exists a continuous mapping (retraction) $r\colon V \to C$ such that $r|_C$ coincides with the identity map (see Kuratowski [25]). Hyman's theorem [21] states that a subset A of a metric space is a compact R_δ-set if and only if it is the intersection of a decreasing sequence of contractible compact metric spaces. Observe that every compact R_δ-set is a continuous (nonempty, compact and connected), but, in contrast to contractible sets, need not be path-connected.

Theorem 13. *If hypotheses $H(a)_1$, H_0^2 and $H(f)_2$ hold, then $S(x_0)$ is a compact R_δ-set in $C(T, L^2(Z))$.*

P r o o f. As before we set $X = W_0^{1,p}(Z)$, $H = L^2(Z)$ and $X^* = W^{-1,q}(Z)$. Let $\widehat{F}\colon T \times H \to P_{fc}(H)$ be defined by

$$\widehat{F}(t,x) = \{h \in H\colon\ f_1(\tau(t,x)(z)) \leqslant h(z) \leqslant f_2(\tau(t,x)(z)) \ \text{ a.e. on } Z\}.$$

We claim that for every $x \in H$, $t \to \widehat{F}(t,x)$ is measurable. To this end note that

$$\begin{aligned} Gr\widehat{F}(\cdot,x) = \ &\{(t,y) \in T \times H\colon\ f_1(\tau(t,x)(z)) \leqslant y(z) \leqslant f_2(\tau(t,x)(z)) \ \text{ a.e. on } Z\} \\ = &\left\{(t,y) \in T \times H\colon\ \int_C f_1(\tau(t,x)(z))\,\mathrm{d}z \leqslant \int_C y(z) \leqslant \int_C f_2(\tau(t,x)(z))\,\mathrm{d}z,\right. \\ &\left. C \in B(Z)\right\}. \end{aligned}$$

Here by $B(Z)$ we denote the Borel σ-field of Z. Recall that $B(Z)$ is countably generated. So there exists a countable field $\{C_n\}_{n\geqslant 1}$ such that $B(Z) = \sigma(\{C_n\}_{n\geqslant 1})$. Hence we can write

$$\begin{aligned} Gr\widehat{F}(\cdot,x) = \bigcap_{n\geqslant 1} &\left\{(t,y) \in T \times H\colon\right. \\ &\left. \int_{C_n} f_1(\tau(t,x)(z))\,\mathrm{d}z \leqslant \int_{C_n} y(z)\,\mathrm{d}z \leqslant \int_{C_n} f_2(\tau(t,x)(z))\,\mathrm{d}z\right\}. \end{aligned}$$

But $f_1(\cdot)$ is lower semicontinuous and $f_2(\cdot)$ is upper semicontinuous, hence measurable (see for example Rauch [34] or Chang [7]). Then by lemma 11 and Fubini's theorem, it follows that $t \to \int_{C_n} f_1(\tau(t,x)(z))\,\mathrm{d}z$ $t \to \int_{C_n} f_2(\tau(t,x)(z))\,\mathrm{d}z$, $n \geqslant 1$, are measurable. Thus

$$\begin{aligned} Gr\widehat{F}(\cdot,x) = \bigcap_{n\geqslant 1} &\left\{(t,y) \in T \times H\colon\right. \\ &\left. \int_{C_n} f_1(\tau(t,x)(z))\,\mathrm{d}z \leqslant \int_{C_n} y(z)\,\mathrm{d}z \leqslant \int_{C_n} f_2(\tau(t,x)(z))\,\mathrm{d}z\right\} \\ &\in \mathcal{L} \times B(H). \end{aligned}$$

where we recall that $\mathcal{L}$ denotes the Lebesque σ-field of T and $B(H)$ the Borel σ-field of H. Moreover, from Papageorgiou [30], we know that

$$h^*(\widehat{F}(t,x), \widehat{F}(t,y)) = \int_Z h^*(F(\tau(t,x)(z)), F(\tau(t,y)(z)))\,\mathrm{d}z.$$

But $F(\cdot)$ is h^*-usc (since $f_1(\cdot)$ is lower semicontinuous, $f_2(\cdot)$ is upper semicontinuous and $F(r) = [f_1(r), f_2(r)]$ for all $r \in \mathbb{R}$; see Klein-Thompson [23]). Therefore, via Fatou's Lemma we check at once for every $t \in T$, $\widehat{F}(t, \cdot)$ is h^*-usc. In addition, because of hypotheses $H(f)_2$, we have

$$|\widehat{F}(t,x)| = \sup\{|v| \colon\ v \in \widehat{F}(t,x)\} \leqslant \widehat{\beta}_2(t) + c_2(t)|x| \quad \text{a.e. on } T$$

with $\widehat{\beta}_2(t) = \|\beta_2(t,\cdot)\|_{L^2(Z)}$ and $c_2 \in L^2(T)$ as in $H(f)_2$. Set

$$\widehat{F}_1(t,x) = \widehat{F}(t,x) - B(t,x).$$

Evidently by virtue of Lemma 12, we see that $\widehat{F}_1(t,x)$ satisfies the same measurability, continuity and growth properties as $\widehat{F}(t,x)$.

Let $A\colon\ T \times X^*$ be defined by

$$\langle A(t,x), y\rangle = \int_Z \sum_{k=1}^N a_k(t,z,Dx) D_k y(z)\,\mathrm{d}z$$

for all $y \in X$. We know from the proof of Theorem 9, that $t \to A(t,x)$ is measurable, $x \to A(t,x)$ is demicontinuous, monotone, $\|A(t,x)\|_* \leqslant \widehat{a}(t) + \widehat{c}\|x\|^{p-1}$ a.e. on T with $\widehat{a} \in L^q(T)$, $\widehat{c} > 0$ and $\langle A(t,x)\rangle \geqslant c_0\|x\|^{p-1}$ for some $c_0 > 0$. Then consider the following evolution inclusion:

$$\begin{aligned} &\dot{x}(t) + A(t,x(t)) \in \widehat{F}_1(t,x(t)) \quad \text{a.e. on } T \\ &x(0) = x_0. \end{aligned} \tag{41}$$

By standard a priori estimation, we may assume without any loss of generality that $|\widehat{F}_1(t,x)| \leqslant \theta(t)$ a.e. on T with $\theta \in L^2(T)$ (see Papageorgiou-Shahzad [32]).

Now let $\widehat{F}_{1n}\colon\ T \times H \to P_{fc}(H)$, $n \geqslant 1$, be a sequence of multifunctions postulated by Lemma 10. For every $n \geqslant 1$, consider the following Cauchy problem

$$\begin{aligned} &\dot{x}(t) + A(t,x(t)) \in \widehat{F}_{1n}(t,x(t)) \quad \text{a.e. on } T \\ &x(0) = x_0. \end{aligned} \tag{42}$$

Let $\widehat{S}(x_0)$ and $\widehat{S}_n(x_0)$ be the solution sets of (41) and (42) respectively. They are subsets of $W_{pq}(T) \subseteq C(T,H)$ and by virtue of Proposition 4, they are compact sets in $C(T,H)$ (see also Papageorgiou-Shahzad [32]). We will show that for every $n \geqslant 1$, $\widehat{S}_n(x_0)$ is contractible. Let $u_n(t,x)$ be the Caratheodory (in fact locally Lipschitz in x) selector of $\widehat{F}_{1n}(t,x)$, postulated by Lemma 10. For every $r \in [0,b)$ and $x \in \widehat{S}_n(x_0)$, let $w(t,x)(\cdot) \in W_{pq}(T)$ be the unique solution of $\dot{w}(t) + A(t,w(t)) =$

$u_n(t, w(t))$ a.e. on $[r, b]$, $w(r) = x(r)$. For $r = b$, we set $w(b, x)(b) = x(b)$. Define $h_n \colon T \times \widehat{S}_n(x_0) \to \widehat{S}_n(x_0)$ by

$$h_n(r,x)(t) = \begin{cases} x(t) & \text{if } 0 \leqslant t \leqslant r \\ w(r,x)(t) & \text{if } r \leqslant t \leqslant b. \end{cases}$$

Evidently $h_n(0, x) = w(0, x)$ and $h_n(b, x) = x$ for every $x \in \widehat{S}_n(x_0)$. It remains to show that $h(\cdot, \cdot)$ is continuous in $C(T, H)$. To this end let $[r_m, x_m] \to [r, x]$ in $T \times \widehat{S}_n(x_0) \subseteq T \times C(T, H)$. We consider two distinct cases:

Case I: $r_m \geqslant r$ for every $m \geqslant 1$.

Let $v_m(t) = h_n(r_m, x_m)(t)$, $t \in T$. Evidently $v_m \in \widehat{S}_n(x_0)$ for all $m \geqslant 1$ and so by passing to a subsequence if necessary, we may assume that $v_m \to v$ in $C(T, H)$ as $m \to \infty$. Clearly $v(t) = x(t)$ for $0 \leqslant t \leqslant r$. Also let $y \in W_{pq}(T)$ be the unique solution of $\dot{y}(t) + A(t, y(t)) = u_n(t, v(t))$ a.e. on $[r, b]$, $y(r) = v(r)$. Let $N \geqslant 1$. Then for $m \geqslant N$ large enough, $v_m(\cdot)$ satisfies $\dot{v}_m(t) + A(t, v_m(t)) = u_n(t, v_m(t))$ a.e. on $[r_N, b]$. Because $A(t, \cdot)$ is monotone, we have

$$\begin{aligned} &\langle \dot{y}(t) - \dot{v}_m(t), y(t) - v_m(t) \rangle \leqslant (u_n(t, v(t)) - u_n(t, v_m(t)), y(t) - v_m(t)) \\ &\text{a.e on } [r_N, b] \\ \Rightarrow\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|y(t) - v_m(t)|^2 \leqslant |u_n(t, v(t)) - u_n(t, v_m(t))| \cdot |y(t) - v_m(t)| \\ &\text{a.e on } [r_N, b] \\ \Rightarrow\ &\frac{1}{2}|y(t) - v_m(t)|^2 \leqslant \frac{1}{2}|y(r_N) - v_m(r_N)|^2 \\ &\qquad\qquad + \int_{r_N}^{t} |u_n(s, v(s)) - u_n(s, v_m(s))| \cdot |y(s) - v_m(s)| \,\mathrm{d}s. \end{aligned}$$

Invoking Lemma A.5, p. 157 of Brezis [5], we obtain

$$|y(t) - v_m(t)| \leqslant |y(r_N) - v_m(r_N)| + \int_{r_N}^{t} |u_n(s, v(s)) - u_n(s, v_m(s))| \,\mathrm{d}s.$$

Passing to the limit as $m \to \infty$, we obtain

$$|y(t) - v(t)| \leqslant |y(r_N) - v(r_N)| \quad \text{for } r_N \leqslant t \leqslant b.$$

Since $y(r_N) \to x(r)$ and $v(r_N) \to v(r) = x(r)$ in H as $N \to \infty$, in the limit we have

$$|y(t) - x(t)| = 0$$

for $r \leqslant t \leqslant b$. Hence $\dot{v}(t) + A(t, v(t)) = u_n(t, v(t))$ a.e. on $[r, b]$, $v(r) = x(r)$ and so $v = h(r, x)$. Therefore $h(r_m, x_m) \to h(r, x)$ in $C(T, H)$

Case II: $r_m \leqslant r$ for every $m \geqslant 1$.

Keeping the notation introduced in the analysis of Case I, we have that $v(t) = x(t)$ for $0 \leqslant t \leqslant r$. Also via the same argument as in Case I, we have

$$\begin{aligned} &|y(t) - v_m(t)| \leqslant |y(r) - v_m(r)| + \int_r^t |u_n(s, v(s)) - u_n(s, v_m(s))| \, \mathrm{d}s \\ &\quad \text{for } t \in [r, b] \\ &\Rightarrow |y(t) - v(t)| \leqslant |y(r) - v(r)| \quad \text{for } t \in [r, b] \end{aligned}$$

But $y(r) = x(r) = v(r)$. So $y(t) = v(t)$ for $t \in [r, b]$. Hence $v = h(r, x)$, which implies that $h(r_m, x_m) \to h(r, x)$ as $m \to \infty$ in $C(T, H)$. Therefore $\widehat{S}_n(x_0)$ is compact and contractible in $C(T, H)$.

In general we can always find a subsequence $\{r_m\}_{m\geqslant 1}$ satisfying Case I or Case II. Thus we have established the continuity of $h_n(\cdot, \cdot)$, $n \geqslant 1$. So for every $n \geqslant 1$, the solution set $S_n(x_0) \subseteq C(T, H)$ is compact and contractible.

Next we claim that $\widehat{S}(x_0) = \bigcap_{n\geqslant 1} \widehat{S}_n(x_0)$. Clearly $\widehat{S}(x_0) \subseteq \bigcap_{n\geqslant 1} \widehat{S}_n(x_0)$. Let $x \in \bigcap_{n\geqslant 1} S_n(x_0)$. Then $x = \widehat{p}(f_n, x_0)$ for some $f_n \in L^2(T, H)$ such that $f_n(t) \in \widehat{F}_1(t, x_n(t))$ a.e. on T. But $\{f_n\}_{n\geqslant 1}$ is bounded in $L^2(T, H)$, so by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^2(T, H)$. Then $f(t) \in F(t, x(t))$ (see Papageorgiou [28], Theorem 4.5). Also by Proposition 4 $x = \widehat{p}(f, x_0)$. So $\widehat{S}(x_0) = \bigcap_{n\geqslant 1} \widehat{S}_n(x_0)$. Now the theorem of Hyman [21] implies that $\widehat{S}(x_0)$ is compact R_δ-set in $C(T, H)$. Moreover, from the proof of Theorem 8, we know that $\widehat{S}(x_0) \subseteq K = [\psi, \varphi]$. So $\widehat{S}(x_0) = S(x_0)$. Therefore, $S(x_0)$ is a compact R_δ-set in $C(T, H)$. □

An immediate consequence of this theorem is the following Kneser-type result for problem (40).

Corollary 14. *If hypotheses $H(a)_1, H_0^2$ and $H(f)_2$ hold, then for every $t \in T$, $\{x(t, \cdot) \in H\colon\ x \in S(x_0)\}$ is nonempty, compact and connected (i.e. a continuum) in $H = L^2(Z)$.*

Remark. Analogous structural results for the solution set of differential inclusions in $\mathbb{R}^N$, were established by DeBlasi-Myjak [10] and Hu-Papageorgiou [20]. In addition, Corollary 14 above extends the results of Ballotti [4] and Kikuchi [22], who consider semilinear parabolic problems with a continuous perturbation term.

Another consequence of Theorem 13, is the following corollary:

Corollary 15. *If hypotheses $H(a)_1, H_0^2, H(f)_2$ hold and there is a compact, convex set $C \subseteq [\psi(0, \cdot), \varphi(0, \cdot)] \subseteq L^2(Z)$ such that $S(x_0)(b) = \{x(b, \cdot) \in L^2(Z)\colon\ x \in S(x_0)\} \subseteq C$, then problem (40) has a periodic solution.*

P r o o f. Let $R\colon C \to P_k(K)$ be defined by $R(y_0) = e_b(S(y_0))$. Recalling that a compact R_δ-set is acyclic, we see that $R(\cdot)$ is pseudo-acyclic in the sense of Lasry-Robert [26] and so Theorem 8 of [26], gives a $y_0 \in R(y_0)$. Let $x \in S(y_0)$. Then $x(0,\cdot) = x(b,\cdot) = y_0(\cdot)$, i.e. x is the desired periodic solution. □

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