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ON OSCILLATORY SOLUTIONS OF FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The paper deals with the oscillation of a differential equation $L_4y + P(t)L_2y + Q(t)y \equiv 0$ as well as with the structure of its fundamental system of solutions.

Keywords: linear differential equation, quasi-derivative, monotone solution, Kneser solution, oscillatory solution

MSC 2000: 34C10; Secondary 34D05

1. Introduction

Consider the linear differential equation of the fourth order with quasi-derivatives

$$L(y) \equiv L_4y + P(t)L_2y + Q(t)y = 0, \tag{L}$$

where

$$\begin{aligned}
L_0y(t) &= y(t),\\
L_1y(t) &= p_1(t)y'(t) = p_1(t)\,\mathrm{d}y(t)/\,\mathrm{d}t,\\
L_2y(t) &= p_2(t)(p_1(t)y'(t))' = p_2(t)(L_1y(t))',\\
L_3y(t) &= p_3(t)(p_2(t)(p_1(t)y'(t))')' = p_3(t)(L_2y(t))',\\
L_4y(t) &= (p_3(t)(p_2(t)(p_1(t)y'(t))')')' = (L_3y(t))',
\end{aligned}$$

$P(t), Q(t), p_i(t),\ i = 1, 2, 3$, are real-valued continuous functions on an interval $I = [a, \infty)$, $-\infty < a < \infty$. It is assumed throughout that

$$\begin{aligned}
&P(t) \leqslant 0,\ Q(t) \leqslant 0,\ p_i(t) > 0,\ i = 1, 2, 3,\ t \in I \text{ and}\\
&Q(t) \text{ is not identically zero in any subinterval of } I.
\end{aligned} \tag{A}$$

This paper is a continuation of [TP] where monotone (See Definitions 1, 6.) as well as Kneser (See Definition 6.) solutions of (L) have been studied. The main results of this article are presented in three theorems.

Theorems 1, 2 give sufficient conditions for (L) to be oscillatory. (See Definitions 4, 5.) Theorem 3 deals with sufficient conditions for the fundamental system of solutions of (L) on I to consist of two oscillatory solutions, one monotone solution which tends to infinity for $t \to \infty$, and one Kneser solution which converges to zero for $t \to \infty$.

Similar problems for n-th order ($n = 3, 4$) ordinary differential equations have been studied, for example, in [G], [Gr], [H], [LN], [Ro], [S], [Š] and [Šv].

In the end of this part we note that some results mentioned above are generalizations of those in [R], where J. Regenda considered the equation (L), $p_i(t) \equiv 1$, $i = 1, 2, 3$. (See Remarks 1, 2, 3.)

2. Definitions and preliminary results

Definition 1. A solution $y(t)$ of (L) on I is called positively (negatively) non-oscillatory iff there exists $t_0 \geqslant a$ such that $y(t) > 0$ ($y(t) < 0$), $t \geqslant t_0$.

Definition 2. A solution $y(t)$ of (L) on I is called non-oscillatory iff $y(t)$ is positively or negatively non-oscillatory.

Definition 3. The equation (L) is called non-oscillatory iff every non-trivial solution of (L) on I is non-oscillatory.

Definition 4. A non-trivial solution $y(t)$ of (L) on I is called oscillatory on I iff its set of all zeros on I is not bounded from above.

Definition 5. The equation (L) is called oscillatory iff there exists at least one oscillatory solution of (L) on I.

Definition 6. A positively non-oscillatory solution $y(t)$ of (L) on I such that $y(t) > 0$ for $t \geqslant t_0 \geqslant a$ is called monotone (Kneser) solution on $[t_0, \infty)$ iff $L_k y(t) > 0$ ($(-1)^k L_k y(t) > 0$), $k = 0, 1, 2, 3$, $t \geqslant t_0$.

Lemma 1. [H, Lemma 2.2] *Let $f(t)$ be a real valued function defined in $[t_0, \infty)$ for some real number $t_0 \geqslant 0$. Suppose that $f(t) > 0$ and that $f'(t)$ and $f''(t)$ exist for $t \geqslant t_0$. Suppose also that if $f'(t) \geqslant 0$ eventually, then $\lim\limits_{t\to\infty} f(t) = A < \infty$. Then*

$$\liminf_{t\to\infty} |t^{\alpha} f''(t) - \alpha t^{\alpha-1} f'(t)| = 0$$

for any $\alpha \leqslant 2$.

Lemma 2. [TP, Lemma 3] *Let* (A) *and* $\int^{\infty}(1/p_1(t))\,\mathrm{d}t = \infty$ *hold. Then for every non-oscillatory solution* $y(t)$ *of* (L) *there exists a number* $t_0 \geqslant a$ *such that*

$$(y(t)L_1y(t) > 0,\ y(t)L_2y(t) > 0) \text{ or } (y(t)L_1y(t) < 0,\ y(t)L_2y(t) > 0) \text{ or}$$
$$(y(t)L_1y(t) > 0,\ y(t)L_2y(t) < 0) \text{ for all } t \geqslant t_0.$$

Lemma 3. [TP, Lemma 4] *Suppose that* (A) *holds and let* $y(t)$ *be a non-trivial solution of* (L) *satisfying the initial conditions*

$$y(t_0) = y_0 \geqslant 0,\ L_1y(t_0) = y_0' \geqslant 0,$$
$$L_2y(t_0) = y_0'' \geqslant 0,\ L_3y(t_0) = y_0''' \geqslant 0$$

($t_0 \in I$ *arbitrary and* $y_0 + y_0' + y_0'' + y_0''' \neq 0$). *Then*

$$y(t) > 0,\ L_1y(t) > 0,\ L_2y(t) > 0,\ L_3y(t) > 0 \text{ for all } t > t_0.$$

Lemma 4. [TP, Lemma 5] *Suppose that* (A) *holds and let* $y(t)$ *be a non-trivial solution of* (L) *satisfying the initial conditions*

$$y(t_0) = y_0 \geqslant 0,\ L_1y(t_0) = y_0' \leqslant 0,\ L_2y(t_0) = y_0'' \geqslant 0,\ L_3y(t_0) = y_0''' \leqslant 0,$$

($t_0 \in I$ *arbitrary,* $y_0^2 + y_0'^2 + y_0''^2 + y_0'''^2 > 0$). *Then*

$$y(t) > 0,\ L_1y(t) < 0,\ L_2y(t) > 0,\ L_3y(t) < 0 \quad \text{for all } t \in [a, t_0).$$

Lemma 5. [TP, Theorem 2] *Suppose that* (A) *holds. Then there exists a solution* $y(t)$ *of* (L) *such that*

$$y(t) > 0,\ L_1y(t) < 0,\ L_2y(t) > 0,\ L_3y(t) < 0 \quad \text{for all} \quad t \in I = [a, \infty).$$

3. Results

Lemma 6. *Let* (A) *hold. If every positively non-oscillatory solution of* (L) *on* I *is either monotone or Kneser, then* (L) *is oscillatory.*

Proof. We construct two oscillatory solutions $u^+(t)$ and $v^+(t)$ similar to what was done in [S], Theorem 3. Since there are some differences in proving their oscillation, we go through the whole proof.

Let functions $z_k(t)$, $k = 0, 1, 2, 3$ form the fundamental system of solutions of (L) on I such that $L_k z_m(a) = \delta_{km}$, $k, m = 0, 1, 2, 3$ where δ_{km} is the Kronecker symbol. It is obvious that there exist real numbers b_{0n}, b_{3n}, c_{2n} and c_{3n} such that

$$\begin{aligned} b_{0n}^2 + b_{3n}^2 = c_{2n}^2 + c_{3n}^2 = 1, \\ b_{0n} z_0(n) + b_{3n} z_3(n) = 0, \\ c_{2n} z_2(n) + c_{3n} z_3(n) = 0 \end{aligned}$$

for all natural numbers $n > a$. Let us put for $n > a$

$$\begin{aligned} u_n^+(t) &= b_{0n} z_0(t) + b_{3n} z_3(t), \\ v_n^+(t) &= c_{2n} z_2(t) + c_{3n} z_3(t). \end{aligned}$$

Because of the boundedness of b_{0n}, b_{3n}, c_{2n} and c_{3n}, there exist real numbers b_0, b_3, c_2 and c_3 such that

$$\begin{gathered} b_{0n_k} \to b_0,\ b_{3n_k} \to b_3,\ c_{2n_k} \to c_2 \text{ and } c_{3n_k} \to c_3 \text{ for } k \to \infty, \\ b_0^2 + b_3^2 = c_2^2 + c_3^2 = 1. \end{gathered}$$

If we put

$$\begin{aligned} u^+(t) &= b_0 z_0(t) + b_3 z_3(t), \\ v^+(t) &= c_2 z_2(t) + c_3 z_3(t), \end{aligned}$$

it is obvious that $u^+(t)$ and $v^+(t)$ are non-trivial solutions of (L) on I. Now we prove their oscillation.

Let, for example, $u^+(t)$ be non-oscillatory. Without loss of generality, we can assume $u^+(t)$ is positively non-oscillatory. (If it were not so, then $u^+(t)$ would be negatively non-oscillatory, and to obtain a contradiction, we should take into account the function $-u^+(t)$.) Then $u^+(t)$ is either monotone or Kneser. If it is monotone, then there exists $t_0 \geqslant a$ such that $L_k u^+(t) > 0$ on $[t_0, \infty)$, $k = 0, 1, 2, 3$. Let us take any fixed $\tau > t_0$. Then there exists an integer positive number $n_0 > a$ such that $L_i u^+_{n_k}(\tau) > 0$ for $n_k > n_0$, $i = 0, 1, 2, 3$. If n_k is any fixed number satisfying the condition $n_k > \max\{n_0, \tau\}$, then Lemma 3 yields $u^+_{n_k}(n_k) > 0$. However, this is a contradiction because $u^+_{n_k}(n_k) = 0$.

If $u^+(t)$ is a Kneser solution of (L) on I, then there exists $t_1 > a$ such that $(-1)^k L_k u^+(t) > 0$ for $t \geqslant t_1$, $k = 0, 1, 2, 3$. Then Lemma 4 implies $(-1)^k L_k u^+(t) > 0$ on $[a, t_1)$. In particular, $L_1 u^+(a) < 0$. But $L_1 u^+(a) = b_0 L_1 z_0(a) + b_3 L_1 z_3(a) = 0$, which is a contradiction.

In the case of $v^+(t)$ the proof is practically the same, hence it will be omitted. The lemma is proved. □

Later, in Theorem 3, we will show linear independence of $u^+(t)$ and $v^+(t)$ on I.

Lemma 7. *Let* (A) *hold, let $p_1(t)$ be non-increasing on $[b, \infty)$, $b \in I$, $p_3'(t) \leqslant 0$ on $[b, \infty)$, $\int^\infty (1/p_2(t))\,\mathrm{d}t = \int^\infty -t^2 Q(t)\,\mathrm{d}t = \infty$. Then for every positively non-oscillatory solution $y(t)$ of* (L) *on I there exists $c \geqslant b$ such that $y(t)$ is monotone on $[c, \infty)$ or $y(t)$ is Kneser on $[c, \infty)$ or $y(t) > 0$, $L_1 y(t) > 0$, $L_2 y(t) < 0$ on $[c, \infty)$.*

Proof. We have $\int^\infty (1/p_1(t))\,\mathrm{d}t = \infty$ because $p_1(t)$ is non-increasing on $[b, \infty)$, $b \in I$. Let $y(t)$ (in accordance with the first (or the second) part of the assertion of Lemma 2) be a positively non-oscillatory solution of (L) on I. Then $y(t) > 0$, $L_2 y(t) > 0$ on $[t_0, \infty)$, $t_0 \geqslant b$. It follows from (A) that $L_4 y(t) \equiv -P(t) L_2 y(t) - Q(t) y(t) \geqslant 0$ and $L_4 y(t) = 0$ at isolated points only, i.e. $L_3 y(t)$ is an increasing function on $[t_0, \infty)$. So only the following five cases (involving the third part of the assertion of Lemma 2) may occur:

a) $y(t) > 0$, $L_1 y(t) > 0$, $L_2 y(t) > 0$, $L_3 y(t) > 0$ on $[t_1, \infty)$, $t_1 \geqslant t_0$,
b) $y(t) > 0$, $L_1 y(t) > 0$, $L_2 y(t) > 0$, $L_3 y(t) < 0$ on $[t_0, \infty)$,
c) $y(t) > 0$, $L_1 y(t) < 0$, $L_2 y(t) > 0$, $L_3 y(t) < 0$ on $[t_0, \infty)$,
d) $y(t) > 0, L_1 y(t) < 0$, $L_2 y(t) > 0$, $L_3 y(t) > 0$ on $[t_2, \infty)$, $t_2 \geqslant t_0$,
e) $y(t) > 0, L_1 y(t) > 0, L_2 y(t) < 0$, on $[t_0, \infty)$.

Let b) be valid. Then $y'(t)$ is a positive and non-decreasing function on $[t_0, \infty)$ because $L_1 y(t) = p_1(t) y'(t)$ is increasing and $p_1(t)$ is non-increasing. So

$$y(t) = y(t_0) + \int_{t_0}^t y'(s)\,\mathrm{d}s \geqslant y(t_0) + y'(t_0) \int_{t_0}^t \mathrm{d}s = y(t_0) + y'(t_0)(t - t_0) \quad \text{on } [t_0, \infty).$$

From (L) it follows that

$$\begin{aligned}\int_{t_0}^t s L_4 y(s)\,\mathrm{d}s &= \int_{t_0}^t -s P(s) L_2 y(s)\,\mathrm{d}s + \int_{t_0}^t -s Q(s) y(s)\,\mathrm{d}s \geqslant \int_{t_0}^t -s Q(s) y(s)\,\mathrm{d}s \\ &\geqslant y(t_0) \int_{t_0}^t -s Q(s)\,\mathrm{d}s + y'(t_0) \int_{t_0}^t -s Q(s)(s - t_0)\,\mathrm{d}s \to \infty \text{ for } t \to \infty.\end{aligned}$$

Integration of $sL_4y(s)$ by parts over $[t_0, t]$ yields

$$\int_{t_0}^{t} L_3y(s)\,\mathrm{d}s = tL_3y(t) - t_0L_3y(t_0) - \int_{t_0}^{t} sL_4y(s)\,\mathrm{d}s \to -\infty \text{ for } t \to \infty.$$

However,

$$\begin{aligned}\int_{t_0}^{t} L_3y(s)\,\mathrm{d}s &= \int_{t_0}^{t} p_3(s)(L_2y(s))'\,\mathrm{d}s \\ &= p_3(t)L_2y(t) - p_3(t_0)L_2y(t_0) + \int_{t_0}^{t} -p_3'(s)L_2y(s)\,\mathrm{d}s \\ &\geqslant -p_3(t_0)L_2y(t_0) = \text{const.} > -\infty, \quad t > t_0.\end{aligned}$$

This contradiction proves the impossibility of the case b).

So let d) be valid. Then $(t \geqslant t_2)$

$$L_2y(t) = L_2y(t_2) + \int_{t_2}^{t} \frac{L_3y(s)}{p_3(s)}\,\mathrm{d}s \geqslant L_2y(t_2).$$

Consequently

$$L_1y(t) = L_1y(t_2) + \int_{t_2}^{t} \frac{L_2y(s)}{p_2(s)}\,\mathrm{d}s \geqslant L_1y(t_2) + L_2y(t_2)\int_{t_2}^{t} \frac{\mathrm{d}s}{p_2(s)} \to \infty \quad \text{for } t \to \infty,$$

which contradicts $L_1y(t) < 0$ on $[t_2, \infty)$. The lemma is proved. □

Lemma 8. *Let* (A) *and* $\int^{\infty}(1/p_1(t))\,\mathrm{d}t = \int^{\infty}(1/p_2(t))\,\mathrm{d}t = \int^{\infty} -Q(t)\,\mathrm{d}t = \infty$ *hold. Then for every positively non-oscillatory solution* $y(t)$ *of* (L) *on* I *there exists* $t_0 \geqslant a$ *such that* $y(t)$ *is monotone on* $[t_0, \infty)$ *or* $y(t)$ *is Kneser on* $[t_0, \infty)$ *or* $y(t) > 0$, $L_1y(t) > 0$, $L_2y(t) < 0$ *on* $[t_0, \infty)$.

Proof. It is obvious that for every positively non-oscillatory solution $y(t)$ of (L) on I, only the cases a), b), c), d) and e) (mentioned in the proof of Lemma 7) can occur.

Let b) be valid. Then from (L) we have $(t \geqslant t_0)$

$$\begin{aligned}L_3y(t) &= L_3y(t_0) + \int_{t_0}^{t} L_4y(s)\,\mathrm{d}s = L_3y(t_0) + \int_{t_0}^{t} -P(s)L_2y(s)\,\mathrm{d}s + \int_{t_0}^{t} -Q(s)y(s)\,\mathrm{d}s \\ &\geqslant L_3y(t_0) + y(t_0)\int_{t_0}^{t} -Q(s)\,\mathrm{d}s \to \infty \quad \text{for } t \to \infty\end{aligned}$$

because $y(t)$ is an increasing function. This contradicts $L_3y(t) < 0$ on $[t_0, \infty)$. So the case b) is not valid.

The impossibility of d) is proved in the same way as in Lemma 7. The lemma is established.

□

Now the main results will be introduced.

Theorem 1. *Let* (A) *hold, let $p_1(t)$ be non-increasing on $[b,\infty)$, $p_2'(t)\leqslant 0$ on $[b,\infty)$, $p_3'(t)\leqslant 0$ on $[b,\infty)$, $(tp_3(t))'\geqslant 0$ on $[b,\infty)$, $t^2P(t)\geqslant -M$ on $[b,\infty)$, where M is a real positive constant, $b\geqslant \max\{0,a\}$, $\int^\infty -t^2Q(t)\,\mathrm{d}t=\infty$. Then* (L) *is oscillatory.*

Proof. The assumption $p_2'(t)\leqslant 0$ implies $\int^\infty (1/p_2(t))\,\mathrm{d}t=\infty$. Lemma 7 yields the following three possibilities for every positively non-oscillatory solution $y(t)$:

a) $y(t)$ is monotone on $[t_0,\infty)$, $t_0\geqslant b$,

b) $y(t)$ is Kneser on $[t_0,\infty)$, $t_0\geqslant b$,

c) $y(t)>0$, $L_1y(t)>0$, $L_2y(t)<0$ on $[t_0,\infty)$, $t_0\geqslant b$.

Now we prove the impossibility of c). Let us assume for a while that c) is valid. Then

$$L_4y(t)+P(t)L_2y(t)+Q(t)y(t)=0 \quad \text{on } [t_0,\infty). \tag{1}$$

Multiplying (1) by t^2 and integrating (1) over $[t_0,t]$, $t\geqslant t_0$, we obtain by a little rearrangement of (1)

$$\begin{aligned}t^2L_3y(t)&-t_0^2L_3y(t_0)-2tp_3(t)L_2y(t)+2t_0p_3(t_0)L_2y(t_0)\\&+\int_{t_0}^t s^2P(s)L_2y(s)\,\mathrm{d}s+\int_{t_0}^t (2sp_3(s))'L_2y(s)\,\mathrm{d}s+\int_{t_0}^t s^2Q(s)y(s)\,\mathrm{d}s\\&=0 \quad \text{on } [t_0,\infty).\end{aligned} \tag{2}$$

Now we present (2) in the form

$$\begin{aligned}&A(t)+B(t)+C(t)+D(t)+E(t)=0 \text{ on } [t_0,\infty), \text{ where}\\ &A(t)=t^2L_3y(t)-2tp_3(t)L_2y(t),\\ &B(t)=-t_0^2L_3y(t_0)+2t_0p_3(t_0)L_2y(t_0),\\ &C(t)=\int_{t_0}^t s^2P(s)L_2y(s)\,\mathrm{d}s,\\ &D(t)=\int_{t_0}^t (2sp_3(s))'L_2y(s)\,\mathrm{d}s,\\ &E(t)=\int_{t_0}^t s^2Q(s)y(s)\,\mathrm{d}s.\end{aligned}$$

We have $A(t)=p_3(t)[t^2f''(t)-2tf'(t)]$, where $f'(t)=L_2y(t)$ on $[t_0,\infty)$. The function $f(t)$ can be expressed in the following way:

$$f(t)=f(t_0)+\int_{t_0}^t L_2y(s)\,\mathrm{d}s=f(t_0)-p_2(t_0)L_1y(t_0)+p_2(t)L_1y(t)-\int_{t_0}^t p_2'(s)L_1y(s)\,\mathrm{d}s.$$

It is obvious that we can choose $f(t_0)$ such that $f(t) > 0$ on $[t_0, \infty)$. Because of $p_3(t) \leqslant p_3(t_0)$ on $[t_0, \infty)$, Lemma 1 yields $\liminf_{t\to\infty} |A(t)| = 0$. So $B(t) = B = \text{const.}$, and

$$C(t) = \int_{t_0}^{t} s^2 P(s) L_2 y(s)\,\mathrm{d}s \leqslant -M \int_{t_0}^{t} L_2 y(s)\,\mathrm{d}s = -M \int_{t_0}^{t} f'(s)\,\mathrm{d}s$$
$$= M[f(t_0) - f(t)] \leqslant M f(t_0) < \infty$$

because $f(t) > 0$ on $[t_0, \infty)$, $D(t) \leqslant 0$ on $[t_0, \infty)$, $E(t) \leqslant y(t_0) \int_{t_0}^{t} s^2 Q(s)\,\mathrm{d}s$ because $y(t)$ is increasing on $[t_0, \infty)$. Hence $\lim_{t\to\infty} E(t) = -\infty$. We have

$$\begin{aligned} 0 = \liminf_{t\to\infty} 0 &= \liminf_{t\to\infty} (A(t) + B(t) + C(t) + D(t) + E(t)) \\ &\leqslant \liminf_{t\to\infty} (|A(t)| + B + M f(t_0) + 0 + E(t)) \\ &= B + M f(t_0) + \liminf_{t\to\infty} (|A(t)| + E(t)) = -\infty, \end{aligned}$$

which is a contradiction. Then Lemma 6 yields the assertion of the theorem. □

Remark 1. Theorem 1.5 in [R] is a special case of the previous theorem for $p_k(t) \equiv 1$, $k = 1, 2, 3$.

Theorem 2. *Let* (A), $\int^{\infty} (1/p_k(t))\,\mathrm{d}t = \int^{\infty} -Q(t)\,\mathrm{d}t = \infty$, $k = 1, 2, 3$, $p_2(t) \leqslant m$ *on* $[t_0, \infty)$, $t_0 \geqslant a$, $-m \leqslant P(t)$ *on* $[t_0, \infty)$ *hold, where* m *is a positive real constant. Then* (L) *is oscillatory.*

P r o o f. Let us assume (L) to be non-oscillatory for a while. Then Lemma 6 yields the existence of a positively non-oscillatory solution $y(t)$ such that $y(t)$ is neither monotone nor Kneser on any $[t_1, \infty)$, $t_1 \geqslant a$. Lemma 8 implies the existence of $t_0 \geqslant a$ such that $y(t) > 0$, $L_1 y(t) > 0$, $L_2 y(t) < 0$ on $[t_0, \infty)$. So we have

$$\begin{aligned} L_4 y(t) + P(t) L_2 y(t) &= (L_3 y(t))' + P(t) p_2(t) (L_1 y(t))' \\ &\leqslant (L_3 y(t))' + P(t) m (L_1 y(t))' \leqslant (L_3 y(t))' - m^2 (L_1 y(t))' \\ &= (L_3 y(t) - m^2 L_1 y(t))' \quad \text{for } t \geqslant t_0. \end{aligned}$$

Hence
$$(L_3 y(t) - m^2 L_1 y(t))' + Q(t) y(t) \geqslant 0 \quad \text{for } t \geqslant t_0.$$

Integration of the last expression over $[t_0, t]$, $t > t_0$ yields

$$L_3 y(t) \geqslant m^2 L_1 y(t) + L_3 y(t_0) - m^2 L_1 y(t_0) - \int_{t_0}^{t} Q(s) y(s)\,\mathrm{d}s \to \infty \text{ for } t \to \infty$$

because $L_1y(t) > 0$ on $[t_0, \infty)$, $y(t)$ is increasing on $[t_0, \infty)$. Hence

$$L_2y(t) = L_2y(t_0) + \int_{t_0}^{t} \frac{L_3y(s)}{p_3(s)}\,\mathrm{d}s \to \infty \quad \text{for } t \to \infty.$$

This fact is a contradiction with $L_2y(t) < 0$ on $[t_0, \infty)$. The theorem is established. $\square$

Remark 2. Theorem 1.6 in [R] is a special case of the previous theorem for $p_k(t) \equiv 1$, $k = 1, 2, 3$.

Theorem 3. *Let the assumptions of Theorem* 1 *or Theorem* 2 *be valid. Then the fundamental system of solutions of* (L) *on* I *consists of two oscillatory solutions, one monotone and one Kneser solution on* I*. The monotone solution tends to infinity for* $t \to \infty$ *and the Kneser solution converges to zero for* $t \to \infty$*.*

Proof. Let the functions $z_k(t)$, $k = 0, 1, 2, 3$ be the same as in the proof of Lemma 6. Then Lemma 5 ensures the existence of a Kneser solution $k(t)$ on I. Lemma 3 yields that $m(t)$, where $L_im(a) = 1$, $i = 0, 1, 2$, $L_3m(a) = L_3k(a)/L_1k(a)$, is monotone on $I = [a, \infty)$. According to Theorems 1,2 and Lemma 6, there exist two oscillatory solutions $u^+(t) = b_0z_0(t) + b_3z_3(t)$, $v^+(t) = c_2z_2(t) + c_3z_3(t)$ on I. Let $W(m(t), k(t), u^+(t), v^+(t))$ denote the Wronski determinant of the functions $m(t)$, $k(t)$, $u^+(t)$, $v^+(t)$. So

$$W(m(t), k(t), u^+(t), v^+(t)) = \begin{vmatrix} m(t), & k(t), & u^+(t), & v^+(t) \\ L_1m(t), & L_1k(t), & L_1u^+(t), & L_1v^+(t) \\ L_2m(t), & L_2k(t), & L_2u^+(t), & L_2v^+(t) \\ L_3m(t), & L_3k(t), & L_3u^+(t), & L_3v^+(t) \end{vmatrix}.$$

Thus

$$\begin{aligned} W(m(a), k(a), u^+(a), v^+(a)) &= \begin{vmatrix} 1, & k(a), & b_0, & 0 \\ 1, & L_1k(a), & 0, & 0 \\ 1, & L_2k(a), & 0, & c_2 \\ L_3k(a)/L_1k(a), & L_3k(a), & b_3, & c_3 \end{vmatrix} \\ &= b_0c_3[L_2k(a) - L_1k(a)] + b_3c_2[k(a) - L_1k(a)]. \end{aligned}$$

We want to prove $W(m(a), k(a), u^+(a), v^+(a)) \neq 0$. Because of non-triviality of $u^+(t)$ on I, we have that at least one of the numbers b_0, b_3 is not equal to zero. If $b_0 = 0$, $b_3 \neq 0$ ($b_0 \neq 0$, $b_3 = 0$), then $u^+(t) = b_3z_3(t)$ ($u^+(t) = b_0z_0(t)$) is non-oscillatory according to Lemma 3, which is impossible. So $b_0 \neq 0 \neq b_3$. Similarly it can be proved that $c_2 \neq 0 \neq c_3$. It is obvious that $b_0b_3 < 0$, $c_2c_3 < 0$. If not so, then

$b_0b_3 > 0$, $c_2c_3 > 0$ and the lastmentioned lemma yields that $u^+(t) = b_0z_0(t) + b_3z_3(t), v^+(t) = c_2z_2(t) + c_3z_3(t)$ are non-oscillatory. Without loss of generality we can assume $b_0 > 0$, $b_3 < 0$, $c_2 < 0$, $c_3 > 0$. Then $W(m(a), k(a), u^+(a), v^+(a)) \neq 0$ because $L_2k(a) - L_1k(a) > 0$, $k(a) - L_1k(a) > 0$. Therefore, $m(t)$, $k(t)$, $u^+(t)$, $v^+(t)$ are linearly independent on I.

From the assumptions of Theorem 1 ($p_1(t)$ is non-increasing on $[b, \infty)$, $b \geqslant \max\{0, a\}$) as well as from Theorem 2 we find that $\int^\infty (1/p_1(s))\,\mathrm{d}s = \infty$. We have $L_1m(t) > 0$ on I and $L_1m(t)$ is increasing on I because $L_2m(t) = p_2(t)(L_1m(t))' > 0$. So ($t > b$)

$$m(t) = m(b) + \int_b^t \frac{L_1m(s)}{p_1(s)}\,\mathrm{d}s \geqslant m(b) + L_1m(b)\int_b^t \frac{\mathrm{d}s}{p_1(s)} \to \infty \text{ for } t \to \infty,$$

which was to prove.

Now it is sufficient to show that $k(t) \to 0$, $t \to \infty$. Since $k(t) > 0, L_1k(t) = p_1(t)k'(t) < 0$ on I, there exists a real constant $c \geqslant 0$ such that $k(t) \to c$, $t \to \infty$. Let $c > 0$. It is obvious that $k(t) > c$ on I. There are the following two possibilities:

a) Let the assumptions of Theorem 1 be fulfilled. Multiplying the left-hand side of (L) by t^2, where $y(t) = k(t)$, integrating it over $[b, t]$, $t > b$ and rearranging a little we obtain

$$\begin{aligned} 0 &= t^2L_3k(t) - b^2L_3k(b) + \int_b^t -2sp_3(s)(L_2k(s))'\,\mathrm{d}s + \int_b^t s^2P(s)L_2k(s)\,\mathrm{d}s \\ &\quad + \int_b^t s^2Q(s)k(s)\,\mathrm{d}s \leqslant t^2L_3k(t) - b^2L_3k(b) + p_3(b)\int_b^t -2s(L_2k(s))'\,\mathrm{d}s \\ &\quad + \int_b^t s^2P(s)L_2k(s)\,\mathrm{d}s + \int_b^t s^2Q(s)k(s)\,\mathrm{d}s. \end{aligned}$$

If we replace the term $p_3(b)\int_b^t -2s(L_2k(s))'\,\mathrm{d}s$ in the previous formula by the term (which is equal to the former)

$$\begin{aligned} &2bp_3(b)L_2k(b) - 2tp_3(b)L_2k(t) + 2p_3(b)p_2(t)L_1k(t) \\ &- 2p_3(b)p_2(b)L_1k(b) - 2p_3(b)\int_b^t p_2'(s)L_1k(s)\,\mathrm{d}s, \end{aligned}$$

we obtain (after little arrangement)

$$\begin{aligned}
t^2L_3k(t) \geqslant{} & b^2L_3k(b) - 2bp_3(b)L_2k(b) + 2tp_3(b)L_2k(t) \\
& - 2p_3(b)p_2(t)L_1k(t) + 2p_3(b)p_2(b)L_1k(b) + 2p_3(b)\int_b^t p_2'(s)L_1k(s)\,\mathrm{d}s \\
& - \int_b^t s^2P(s)L_2k(s)\,\mathrm{d}s - \int_b^t s^2Q(s)k(s)\,\mathrm{d}s \\
\geqslant{} & b^2L_3k(b) - 2bp_3(b)L_2k(b) + 2p_3(b)p_2(b)L_1k(b) \\
& - c\int_b^t s^2Q(s)\,\mathrm{d}s \to \infty \quad \text{for } t \to \infty,
\end{aligned}$$

which is a contradiction with $L_3k(t) < 0$ on I.

b) Let the assumptions of Theorem 2 be fulfilled. Then an integration of $L_4k(t) = -P(t)L_2k(t) - Q(t)k(t)$ over $[a,t]$, $t > a$ yields ($c > 0$)

$$\begin{aligned}
L_3k(t) &= L_3k(a) - \int_a^t P(s)L_2k(s)\,\mathrm{d}s - \int_a^t Q(s)k(s)\,\mathrm{d}s \\
&\geqslant L_3k(a) - \int_a^t P(s)L_2k(s)\,\mathrm{d}s - c\int_a^t Q(s)\,\mathrm{d}s \to \infty \text{ for } t \to \infty,
\end{aligned}$$

which is a contradiction with $L_3k(t) < 0$ on I. The theorem is proved. □

Remark 3. The hypotheses of Theorem 1.7 in [R] consist of the disjunction of the three assumptions. The second as well as the third of them is a special case of the previous theorem for $p_k(t) \equiv 1$, $k = 1, 2, 3$. In this case we note that the assertion of Theorem 1.7 in [R] is weaker than the analogous one in our Theorem 3.

Example 1. The equation

$$\left(\frac{1}{\sqrt{t}}\left(\frac{1}{t}\left((2+\mathrm{e}^{-t})y'\right)'\right)'\right)' - \frac{1}{t^2\sqrt{1+t^2}}\frac{1}{t}((2+\mathrm{e}^{-t})y')' - \frac{1}{t^2}y \equiv 0$$

is oscillatory ($I = [1,\infty)$) according to Theorem 1. We note that Theorem 2 cannot be used because $\int_1^\infty -Q(t)\,\mathrm{d}t = \int_1^\infty t^{-2}\,\mathrm{d}t < \infty$.

Example 2. The equation

$$\left(\sqrt[3]{t}\left(\frac{t+1}{t}(ty')'\right)'\right)' + (-\arctan t)\frac{t+1}{t}(ty')' - t^2y \equiv 0$$

is oscillatory ($I = [a,\infty)$, $a > 0$) according to Theorem 2. The assumptions of Theorem 1 are not fulfilled because $p_1(t) = t$, $p_1'(t) = 1 > 0$ on $[a,\infty)$.

Theorem 3 in both the examples yields that the fundamental system of solutions consists of two oscillatory solutions, one monotone solution which tends to infinity for $t \to \infty$, and one Kneser solution which converges to zero for $t \to \infty$.

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