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ON IDEALS AND CONGRUENCES IN BCC-ALGEBRAS

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Abstract. We introduce a new concept of ideals in BCC-algebras and describe connections between such ideals and congruences.

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1. Introduction

By an algebra $\mathbf{G} = (G, \cdot, 0)$ we mean a non-empty set G together with a binary multiplication and a distinguished element 0. In the sequel a multiplication will be denoted by juxtaposition. Dots we use only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written as $(xy \cdot zy) \cdot xz = 0$.

Definition. An algebra $(G, \cdot, 0)$ is called *a BCC-algebra* if it satisfies the following axioms:

(1) $$(xy \cdot zy) \cdot xz = 0,$$

(2) $$xx = 0,$$

(3) $$0x = 0,$$

(4) $$x0 = x,$$

(5) $$xy = yx = 0 \text{ implies } x = y.$$

The above definition is a dual form of the ordinary definition (cf. [1], [6], [7]). In our convention *any BCK-algebra is a BCC-algebra,* but *there are BCC-algebras which are not BCK-algebras* (cf. [2]). Such BCC-algebras are called *proper*. Some methods of construction of BCC-algebras from BCK-algebras are given in [3]. Note that (cf. [2]) *a BCC-algebra is a BCK-algebra iff it satisfies*

$$xy \cdot z = xz \cdot y. \tag{6}$$

2. Ideals

As is well-known (cf. for example [4], [5]) a non-empty subset A of a BCK-algebra $(G, \cdot, 0)$ is called *an ideal* if

(i) $0 \in A$,

(ii) $xy \in A$ and $y \in A$ imply $x \in A$.

In the sequel this ideal will be called *a BCK-ideal* and will be considered also in BCC-algebras.

If A is a BCK-ideal of a BCK-algebra $\mathbf{G}$ then the relation $\sim$ defined on $\mathbf{G}$ by

$$x \sim y \ \text{ iff } \ xy,\ yx \in A \tag{7}$$

is a congruence (cf. [4]). We say that this relation is defined by the ideal A.

This result is not true for BCC-algebras.

Example 2.1. Let $G = \{0, 1, 2, 3, 4\}$ and let the multiplication be defined by the table

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

First we prove that this algebra is a BCC-algebra. It is clear that such algebra satisfies (2), (3), (4) and (5). We prove (1). If x, y, z are not different, then obviously (1) holds. For different x, y, z we verify only the case when one of elements x, y, z is equal to 4, because $S = \{0, 1, 2, 3\}$ is a BCC-algebra (cf. Table 14 in [2]). Since $xy \in S$, $4y \in \{3, 4\}$ and $u3 = u4 = 0$ for all $x, y, u \in S$, then (1) holds for $z = 4$. For $y = 4$ it holds, too. For $x = 4$ the left hand side of (1) has the form $(4y \cdot zy) \cdot 4z$, which for $y = 1$ and $y = 3$ is equal to 0 since $4y \cdot zy = 3 \cdot zy \in S$ and $u3 = u4 = 0$ for $u \in S$. The case $y = 0$ is obvious. If $y = 2$ then $(42 \cdot z2) \cdot 4z = (4 \cdot z2) \cdot 4z$, which for $z = 0$ trivially gives 0. For $z \in \{1, 3\}$ we obtain $(4 \cdot z2) \cdot 4z = 41 \cdot 3 = 0$. This completes the proof that G is a BCC-algebra.

It is not difficult to verify that $A = \{0, 1\}$ is a BCK-ideal of this BCC-algebra, but the relation $\sim$ defined by this ideal is not a congruence. Indeed, $4 \sim 4$, $2 \sim 3$ but *not* $(42 \sim 43)$ since $42 \cdot 43 = 3 \notin A$.

In connection with this fact we introduce a new concept of ideals.

Definition. A non-empty subset A of a BCC-algebra $\mathbf{G}$ is called *a BCC-ideal*, if

(8) $\quad 0 \in A,$

(9) $\quad xy \cdot z \in A$ and $y \in A$ imply $xz \in A$.

Lemma 2.2. *In a BCC-algebra any BCC-ideal is a BCK-ideal.*

Indeed, putting $z = 0$ in (9) we obtain (ii).

On the other hand, using (6) we have

Lemma 2.3. *In a BCK-algebra any BCK-ideal is a BCC-ideal.*

Lemma 2.4. *In a BCC-algebra any BCK-ideal is a BCC-subalgebra.*

P r o o f. Let A be a BCK-ideal. Then $0 \in A$ and $xy \cdot x = 0$ for all $x, y \in G$ (cf. [2]). Thus for $x, y \in A$ we have $xy \cdot x \in A$, which implies $xy \in A$. □

Corollary 2.5. *Any BCC-ideal of a BCC-algebra is a BCC-subalgebra.*

The following example shows that a BCC-ideal is not a BCK-subalgebra, in general.

Example 2.6. Let $G = \{0, 1, 2, 3, 4, 5\}$ and let the multiplication be defined by the table

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Since $S = \{0, 1, 2, 3, 4\}$ is a BCC-algebra (cf. Table 2 in [1]), then $\mathbf{G}$ is a BCC-algebra by Proposition 4 in [2] (cf. also Construction 3 in [3]). It is easy to see that S is a BCC-ideal of $\mathbf{G}$. It is not a BCK-algebra since $21 \cdot 4 \neq 24 \cdot 1$.

On the other hand, in Example 2.1 $S = \{0, 1, 2, 3\}$ is a BCC-subalgebra which is not a BCK-ideal, because $43 = 3 \in S$ but $4 \notin S$. Similarly, $A = \{0, 1, 2\}$ is a BCK-subalgebra which is not a BCK-ideal since $32 \in A$ but $3 \notin A$. Thus in BCC-algebras BCC-ideals, BCK-ideals and BCK-subalgebras are independent concepts.

Proposition 2.7. *Let* **G** *be a BCC-algebra. Then a BCC-subalgebra* **A** *of* **G** *is a BCC-ideal iff* $x \in A$, $yz \notin A$ *imply* $yx \cdot z \notin A$.

P r o o f. If a BCC-subalgebra A is a BCC-ideal, then $x \in A$, $yz \notin A$ imply $yx \cdot z \notin A$. If not, then $yx \cdot z \in A$, $x \in A$ imply $yz \in A$, which is a contradiction.

Conversely, let A be a BCC-subalgebra in which $x \in A$, $yz \notin A$ imply $yx \cdot z \notin A$. Then obviously $0 \in A$. Moreover, $x \in A$, $yx \cdot z \in A$ gives $yz \in A$, because for $yz \notin A$ we have (by assumption) $yx \cdot z \notin A$. Hence A is a BCC-ideal. □

Putting $z = 0$ in the above Proposition we obtain

Proposition 2.8. *Let* **G** *be a BCC-algebra. Then a BCC-subalgebra* **A** *of* **G** *is a BCK-ideal iff* $x \in A$, $y \notin A$ *imply* $yx \notin A$.

Proposition 2.9. *Let* A *be a BCK-ideal of a BCC-algebra* **G**. *If* B *is a BCK-ideal of* A, *then it is a BCK-ideal of* **G**.

P r o o f. Since B is a BCK-ideal of A, then $0 \in B$. Let $y, xy \in B$ for some $x \in G$. Then $y, xy \in A$ and $x \in A$ because $B \subset A$ and A is a BCK-ideal of **G**. Thus $x \in A$ and $xy, y \in B$ imply $x \in B$. This completes the proof. □

Corollary 2.10. *Let* A *be a BCC-ideal of a BCC-algebra* **G**. *If* B *is a BCK-ideal of* A, *then* B *is a BCK-ideal of* **G**.

Remark 2.11. If a BCC-ideal A is a BCK-subalgebra of **G**, then any of its sub-BCK-ideals is a BCC-ideal, but in general it is not a BCC-ideal of **G**.

Remark 2.12. On any BCC-algebra $(G, \cdot, 0)$ one can define (cf. [2]) the so-called *natural order* by putting

$$x \leqslant y \ \text{ iff } \ xy = 0.$$

As in the case of BCK-algebras, this order is partial and 0 is its smallest element. Thus any BCC-algebra may be viewed as a groupoid $(G, \cdot, 0)$ with the natural order satisfying conditions $xy \cdot zy \leqslant xz$, $0 \leqslant x$, $x0 = x$, $x \leqslant y \leqslant x$ imply $x = y$ (cf. Theorem 2 from [2]). But, in general, BCC-algebras with the same partial order are not isomorphic as groupoids (cf. [2]).

Remark 2.13. The above ideals are ideals in the sense of ordered structures. Indeed, if A is a BCC-ideal (or a BCK-ideal), then $y \in A$ and $x \leqslant y$ imply $x \in A$.

In this section we describe congruences on BCC-algebras. We start with the following

Theorem 3.1. *If A is a BCC-ideal of a BCC-algebra* $\mathbf{G}$*, then the relation* $\sim$ *defined by* (7) *is a congruence on* $\mathbf{G}$*.*

P r o o f. It is clear that this relation is reflexive and symmetric. It is also transitive, because $x \sim y$ and $y \sim z$ imply $xy, yx, yz, zy \in A$ and $(xz \cdot yz) \cdot xy = 0 \in A$, which by Lemma 2.2 gives $xz \in A$. Similarly $(zx \cdot yx) \cdot zy = 0 \in A$ gives $zx \in A$. Thus $x \sim z$ and $\sim$ is an equivalence relation.

If $x \sim u$ and $y \sim v$, then $(xy \cdot uy) \cdot xu = 0 \in A$ and $xu \in A$, which by Lemma 2.2 gives $xy \cdot uy \in A$. Similarly $uy \cdot xy \in A$. Hence $xy \sim uy$. On the other hand $(uy \cdot vy) \cdot uv = 0 \in A$ and $vy \in A$ imply $uy \cdot uv \in A$. In the same manner from $(uv \cdot yv) \cdot uy = 0 \in A$ and $yv \in A$ we obtain $uv \cdot uy \in A$. Thus $uy \sim uv$. Since $\sim$ is transitive, then $xy \sim uv$, which proves that $\sim$ is a congruence. □

Lemma 3.2. *If $\sim$ is a congruence on a BCC-algebra* $\mathbf{G}$*, then*

$$C_0 = \{x \in G\colon\ x \sim 0\}$$

is a BCC-ideal.

P r o o f. Obviously $0 \in C_0 = \{x \in G\colon\ x \sim 0\}$. If $xy \cdot z, y \in C_0$, then $xy \cdot z \sim 0$ and $y \sim 0$. But $x \sim x$ and $z \sim z$ imply $xy \cdot z \sim x0 \cdot z = xz$. Thus $xz \sim 0$, which completes the proof. □

Since $C_0 = A$ for any congruence defined by (7), then as a consequence of the above results we obtain

Corollary 3.3. *Any BCC-ideal is determined by some congruence.*

Corollary 3.4. *The lattice of all congruences of a BCC-algebra is complete. The least congruence is defined by the BCC-ideal* $\{0\}$*, the greatest by* $A = G$*.*

Let $\sim$ be a congruence relation on $\mathbf{G}$ and let $C_x = \{y \in G\colon\ y \sim x\}$. Then the family $\{C_x\colon\ x \in G\}$ gives a partition of G which is denoted by $G/_{\sim}$. For $x, y \in G$ we define $C_x * C_y = C_{xy}$. Since $\sim$ has the substitution property, the operation $*$ is well-defined. As is easily seen, $(G/_{\sim}, *, C_0)$ satisfies all axioms of a BCC-algebra except (5). This axiom is not satisfied also in the case of BCK-algebras (cf. [5] and

[8]). If (5) holds for all classes $C_x \in G/_\sim$, i.e. if $(G/_\sim, *, C_0)$ is a BCC-algebra, then the congruence $\sim$ is called *regular*.

For a congruence defined by (7) we put $G/_\sim = G/A$ and $C_0 = A$.

Theorem 3.5. *A congruence is regular iff it is defined by some BCC-ideal.*

P r o o f. Let ϱ_A be a congruence defined by a BCC-ideal A. Then $A_0 = A$ and $A_x * A_y = A_0 = A_y * A_x$ imply $xy, yx \in A$, which shows that $x \sim y$ and $A_x = A_y$. Hence a congruence defined by a BCC-ideal is regular.

Now let ϱ be an arbitrary regular congruence. If $x\varrho y$, then $xy\varrho 0$ and $yx\varrho 0$ since ϱ is reflexive and has the substitution property. Therefore $C_{xy} = C_0 = C_{yx}$, $xy, yx \in C_0$ and $A = C_0$ is a BCC-ideal (by Lemma 3.2). Hence $\varrho \leqslant \varrho_A$.

Conversely, if $x\varrho_A y$, then $xy, yx \in A = C_0$ and $C_x * C_y = C_0 = C_y * C_x$, which implies $C_x = C_y$ because ϱ_A is regular. Thus $x\varrho y$ and $\varrho_A \leqslant \varrho$. Hence $\varrho = \varrho_A$. The proof is complete. □

Corollary 3.6. *All congruences of a finite BCC-algebra are regular.*

If $\mathbf{G/A}$ is a BCC-algebra, then the canonical mapping $f\colon \mathbf{G} \mapsto \mathbf{G/A}$ defined by $f(x) = A_x$ is an epimorphism. Since the kernel $ker f = f^{-1}(0)$ of any BCC-homomorphism is a BCC-ideal, then in the same manner as in [5] we can prove the following results:

Theorem 3.7. *If f is an epimorphism from a BCC-algebra* $\mathbf{G}$ *onto a BCC-algebra* $\mathbf{H}$*, then the quotient BCC-algebra* $\mathbf{G/ker(}f\mathbf{)}$ *is isomorphic to* $\mathbf{H}$.

Theorem 3.8. *Let* $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ *be BCC-algebras, let $h\colon \mathbf{X} \mapsto \mathbf{Y}$ be an epimorphism, and let $g\colon \mathbf{X} \mapsto \mathbf{Z}$ be a homomorphism. If $ker(h) \subset ker(g)$, then there exists a unique homomorphism $f\colon \mathbf{X} \mapsto \mathbf{Z}$ such that $f \circ h = g$.*

Corollary 3.9. *Let ϱ be a regular congruence on a BCC-algebra* $\mathbf{X}$ *defined by a BCC-ideal A, and let h be a canonical mapping from* $\mathbf{X}$ *onto* $\mathbf{Y} = \mathbf{X/A}$*. If $A_0 \subset ker(g)$, then there exists a unique homomorphism $f\colon \mathbf{X} \mapsto \mathbf{Z}$ such that $f \circ h = g$.*

Corollary 3.10. *Let h be a homomorphism from a BCC-algebra* $\mathbf{G}$ *onto a BCC-algebra* $\mathbf{H}$*. Then the inverse image of a BCC-ideal, a BCC-subalgebra and a BCK-subalgebra of* $\mathbf{H}$ *is a BCC-ideal, a BCC-subalgebra and a BCK-subalgebra of* $\mathbf{G}$*, respectively.*

The theory of universal algebras yields immediately

Theorem 3.11. *The composition $\varrho \circ \sigma$ of two congruences on a BCC-algebra* **G** *is a congruence on* **G** *iff these congruences commute, i.e. iff $\varrho \circ \sigma = \sigma \circ \varrho$.*

Corollary 3.12. *Let A and B be BCC-ideals. If congruences ϱ_A and ϱ_B commute, then*

$$\bigcup_{a \in A} B_a = \bigcup_{b \in B} A_b$$

is a BCC-ideal.

P r o o f. Let $\varrho_A \circ \varrho_B = \varrho$. Then by Lemma 3.2

$$\begin{aligned} \bigcup_{a \in A} B_a &= \{x \in B\colon\ x \varrho_B a \text{ for some } a \in A\} \\ &= \{x \in G\colon\ x \varrho_B a \text{ and } a \varrho_A 0\} = \{x \in G\colon\ x \varrho 0\} = C_0 \end{aligned}$$

is a BCC-ideal. Since $\varrho_A \circ \varrho_B = \varrho_B \circ \varrho_A$ then

$$\bigcup_{a \in A} B_a = \bigcup_{b \in B} A_b.$$

□

4. Maximal ideals

A proper ideal is called *maximal* iff it is not properly contained in any proper ideal of the same type. A BCC-algebra without proper BCC-ideals (BCK-ideals) is called *BCC-simple (BCK-simple).* Obviously any BCK-simple BCC-algebra is BCC-simple. The converse is not true. A BCC-algebra **G** given in our Example 2.1 is BCC-simple, but it is not BCK-simple because it has two maximal BCK-ideals $A = \{0, 1\}$ and $B = \{0, 2\}$.

A BCC-simple BCC-algebra has only two regular congruences.

Theorem 4.1. *Let A be a proper BCC-ideal of a BCC-algebra* **G**. *Then A is a maximal BCC-ideal of* **G** *iff* **G/A** *is a BCC-simple BCC-algebra.*

P r o o f. Let **G/A** be a BCC-simple BCC-algebra. If A is not a maximal BCC-ideal, then there exists a proper BCC-ideal B such that $A \subset B \subset G$. Obviously B/A is properly contained in G/A and has at least two elements, because $x \in A_x \subset B/A$ for all $x \in B - A$. Obviously $A = A_0 \in B/A$. Moreover, if $A_y \in B/A$ and $A_{xy \cdot z} = (A_x * A_y) * A_z \in B/A$, then $y, xy \cdot z \in B$, which implies $xz \in B$. Therefore

$A_x * A_z \in B/A$. Thus B/A is a proper BCC-ideal of $\mathbf{G/A}$, i.e. $\mathbf{G/A}$ is not simple. This contradiction proves that A is a maximal BCC-ideal.

Conversely, if A is a maximal BCC-ideal of $\mathbf{G}$ and $\mathbf{G/A}$ is not BCC-simple, then there exists a proper BCC-ideal D of $\mathbf{G/A}$. Then $\varphi^{-1}(D)$, where $\varphi(x) = A_x$ is the canonical homomorphism from $\mathbf{G}$ onto $\mathbf{G/A}$, is a proper BCC-ideal of $\mathbf{G}$. Moreover $A = A_0 \subset \varphi^{-1}(D)$ and $A \neq \varphi^{-1}(D)$, which contradicts our hypothesis. Hence $\mathbf{G/A}$ is simple. The proof is complete. □

Theorem 4.2. *Any BCC-algebra may be viewed as a maximal BCC-ideal of some BCC-algebra.*

Proof. Corollary 3 in [2] (cf. also Construction 5 in [3]) implies that if S is a (proper) BCC-algebra and $e \notin S$, then $G = S \cup \{e\}$ with the multiplication defined by

$$x * y = \begin{cases} xy & \text{for } x, y \in S, \\ 0 & \text{for } y = e, \\ e & \text{for } x = e, y \neq e \end{cases}$$

is a (proper) BCC-algebra and e is the greatest element of G. It is not difficult to verify that S is a maximal BCC-ideal of $\mathbf{G}$. □

Corollary 4.3. *If a BCC-algebra* $\mathbf{G}$ *has an element* e *such that* $xy = e$ *iff* $x = e$, $y \neq e$, *then* $G - \{e\}$ *is the maximal BCC-ideal of* $\mathbf{G}$.

Proof. Assume $xy \cdot z \neq e$ for some $y \neq e$. Then $xz \neq e$. If not, then $xz = e$, by the assumption, implies $x = e$, $z \neq e$. Hence $xy \cdot z = ey \cdot z = ez = e$, which is impossible. □

Corollary 4.4. *If a BCC-algebra* $\mathbf{G}$ *has an element* e *such that* $G \setminus \{e\}$ *is a BCC-ideal (BCK-ideal), then* $ey = e$ *for all* $y \neq e$ *and* e *is the maximal element of* $\mathbf{G}$.

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