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SOME CHARACTERIZATIONS OF 2-SYMMETRIC SUBMANIFOLDS
IN SPACES OF CONSTANT CURVATURE*

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INTRODUCTION

The notion of a (locally) symmetric submanifold M of the Euclidean space \mathbb{R}^n was given by Ferus [F] as a submanifold (locally) mapped into itself, for each $x \in M$, by the reflection of \mathbb{R}^n with respect to the (affine) normal space to M at x . Strübing studied locally symmetric submanifolds of any Riemannian manifold [St].

In [K-K] Kowalski and Kulich gave the notion of a k -symmetric submanifold of \mathbb{R}^n , $k \geq 2$, using suitable isometries of \mathbb{R}^n . They observed that a 2-symmetric submanifold M is invariant under the reflections of \mathbb{R}^n with respect to subspaces of the normal spaces of M ; so the 2-symmetric submanifolds appear as a generalization of the symmetric submanifolds. It is easy to check that a totally geodesic submanifold of a symmetric submanifold is a 2-symmetric submanifold but generally is not a symmetric submanifold.

In this paper we give a local extension of the notion of a 2-symmetric submanifold M of \mathbb{R}^n considering submanifolds of a space \bar{M} of constant curvature (Definition 2.1).

We remark that Definition 2.1, if $\bar{M} \equiv \mathbb{R}^n$, does not agree with that of Kowalski and Kulich but its local modification. For such 2-symmetric submanifolds there exists a local involutive isometry σ_x of \bar{M} , at $x \in M$, which maps M into itself. We prove that if \bar{M} is a standard space form (\bar{M} is a complete, simply connected Riemannian manifold of constant curvature) then the local isometry σ_x is in fact the restriction to a neighbourhood of a geodesic reflection of \bar{M} , with respect to a complete totally geodesic submanifold F_x of \bar{M} orthogonally meeting M at x (Proposition 2.2). Since each geodesic reflection of a space form with respect to a totally geodesic submanifold

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is an isometry (Theorem 1.2), we have an alternative definition for 2-symmetric submanifolds in terms of geodesic reflections (Corollary 2.8).

Moreover, for nicely curved submanifolds (Definition 1.3) we prove the following results. Let M be an essential 2-symmetric submanifold (Definition 1.11) of a space form \bar{M} . Then we prove that the totally geodesic submanifold F_x coincides with the image of the direct sum of the normal spaces of odd order under the exponential map; otherwise F_x has to contain this image (Corollary 3.5). For 2-symmetric submanifolds of a space of constant curvature we prove (Theorem 3.7) that the derivative of the k -fundamental form s^k of M , ∇s^k , is equal to zero.

In our main result (Theorem 4.23) we characterize a 2-symmetric submanifold of a space of zero curvature by means of the derivatives of s^k . Theorem 4.23 is a corollary of Theorem 4.21, which characterizes a 2-symmetric submanifold M of a Euclidean space both in terms of $\nabla s^k = 0$ and in terms of a totally geodesic map ν of M in a suitable Grassmannian. Theorem 4.21 is an extension of the analogous theorem for an N -symmetric submanifold of Euclidean spaces ([R]).

In Section 5 we extend the results of Section 4 to the case of a space of constant positive curvature.

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1. PRELIMINARIES

We now recall some definitions and results which will be useful in the sequel.

1.1. Theorem. *Let \bar{M} be a standard space form. For each point $x \in \bar{M}$ and each subspace $V_x \subset T_x \bar{M}$, $L_x = \exp_x(V_x)$ is the only complete totally geodesic submanifold through x with V_x as the tangent space at x .*

For the proof see [S].

Let M be a topologically embedded submanifold of a Riemannian manifold \bar{M} . The geodesic reflection ϱ_M with respect to M is defined as follows. For $z = \exp_y(X_y^\perp)$, where $y \in M$, $X_y^\perp \in (T_y M)^\perp$, let

$$\varrho_M(z) = \exp_y(-X_y^\perp).$$

It is possible to prove that ϱ_M is a well defined local diffeomorphism of a tubular neighbourhood of M ([G]).

1.2. Theorem. *If L is a totally geodesic submanifold topologically embedded in a space of constant curvature \bar{M} , then the geodesic reflection ϱ_L is an isometry.*

For the proof see [C-V].

Because each local isometry of a standard space form is the restriction of a global one, if L is a complete totally geodesic submanifold of a standard space form \bar{M} , then by Theorem 1.2 we deduce that ϱ_L is an isometry of the whole space \bar{M} .

Let now \bar{M} be a Riemannian manifold with a metric \bar{g} and the Levi-Civita connection $\bar{\nabla}$. If M is a submanifold of \bar{M} , we shall denote by g the induced metric and by ∇ the Levi-Civita connection of g . The k -normal space of M at $x \in M$ will be denoted by $N_x^k M$. By s, A we shall denote, respectively, the k -fundamental form and the k -Weingarten map of M . For these definitions see [S] (Volume 4, Chapter 7) or [CG-R].

1.3. Definition. A submanifold M of a Riemannian manifold \bar{M} is called nicely curved in \bar{M} if $\dim N_x^k M$ does not depend on the point $x \in M$ for each $k = 1, \dots$

We assume $N_x^0 M = T_x M$. If M is nicely curved in \bar{M} it is known that there exists an integer l , independent of the point $x \in M$, such that $N_x^l M \neq 0$ and $N_x^{l+1} M = N_x^{l+2} M = \dots = 0$. In the present paper l shall always have this meaning.

Let M be a nicely curved submanifold of \bar{M} .

We recall that, for each $x \in M$,

$$s : T_x M \times N_x^k M \rightarrow N_x^{k+1} M, \quad 0 \leq k \leq l-1$$

and

$$A : T_x M \times N_x^k M \rightarrow N_x^{k-1} M, \quad 1 \leq k \leq l$$

are bilinear maps.

If $\xi \in \Gamma(N^k M)$ is an extension of $\xi_x \in N_x^k M$, the connection $\bar{\nabla}$ induces a connection ∇^k on the vector bundle $N^k M$ by the formula

$$\nabla_{X_x}^k \xi = \bar{\Pi}(\bar{\nabla}_{X_x} \xi),$$

where $X_x \in T_x M$ and $\bar{\Pi}$ is the orthogonal projection of $T_x \bar{M}$ onto $N_x^k M$. Then we have the Frenet equations of M :

$$(1.4) \quad \nabla_{X_x}^k \xi = -A(X_x, \xi_x) + \nabla_{X_x}^k \xi + s(X_x, \xi_x).$$

The derivatives $\nabla^k s$, $\nabla^k A$ are defined by the formulas

$$(1.5) \quad (\nabla^k s)(X_x, Y_x, \xi_x) = \nabla^{k+1}_{X_x} s(Y, \xi) - s(\nabla_{X_x} Y, \xi_x) - s(Y_x, \nabla_{X_x} \xi),$$

$$(1.6) \quad (\nabla^k A)(X_x, Y_x, \xi_x) = \nabla^{k-1}_{X_x} A(Y, \xi) - A(\nabla_{X_x} Y, \xi_x) - A(Y_x, \nabla_{X_x} \xi),$$

where $Y \in \Gamma(TM)$ and $\xi \in \Gamma(NM)$ extend respectively X_x and ξ_x .

For each $x \in M$,

$$\begin{aligned} \nabla^k s &: T_x M \times T_x M \times N_x M \rightarrow N_x M, \\ \nabla^k A &: T_x M \times T_x M \times N_x M \rightarrow N_x M \end{aligned}$$

are multilinear forms which are symmetric with respect to the first two arguments ([S]).

$\nabla^k A$ and $\nabla^{k-1} s$ are "dual" in the following sense:

$$\bar{g}(\nabla^k A(X, \xi), \nabla^{k-1} s(X, \eta)) = \bar{g}(\nabla^{k-1} s(X, \eta), \nabla^k A(X, \xi)),$$

([S]). We deduce

$$\bar{g}(\nabla^k A(X, Y, \xi), \nabla^{k-1} s(X, \eta)) = \bar{g}(\nabla^{k-1} s(X, Y, \eta), \nabla^k A(X, \xi)).$$

We now have the following theorem.

1.7. Theorem. *If M is a nicely curved submanifold of a Riemannian manifold \bar{M} , then the condition $\nabla^k s = 0$ for each $k = 0, 1, \dots, l-1$ is equivalent to the condition $\nabla^k A = 0$ for each $k = 1, 2, \dots, l$.*

Moreover, under the hypothesis of the previous theorem we have

1.8. Theorem. *If $\sigma: \bar{M} \rightarrow \bar{M}$ is an isometry such that $\sigma(M) \subset M$, then $d\sigma(N_x M) = N_{\sigma(x)} M$ for each $x \in M$.*

Finally, if $G(p, \bar{m})$ is the Grassmannian of the p -planes of $\mathbb{R}^{\bar{m}}$, we recall ([V] and [CG-R]) that there exists a canonical isomorphism $\varphi: T_\alpha G(p, \bar{m}) \rightarrow \text{Hom}(\alpha, \alpha^\perp)$, where α^\perp is the $(\bar{m} - p)$ -plane of $\mathbb{R}^{\bar{m}}$ through the origin, orthogonal to the p -plane α .

Let $\psi: M \rightarrow G(p, \bar{m})$ be a map and $d\psi$ its differential. We denote $\varphi(d\psi(X_x))(v_x)$ by $\psi_*(X_x, v_x)$ for $X_x \in T_x M$ and $v_x \in \psi(x)$.

Then, if we denote $\psi(x)$ by V_x , we have

$$(1.9) \quad \psi_*(X_x, v_x) = \Pi_{V_x^\perp} \bar{\nabla} X_x v,$$

where $\Pi_{V_x^\perp}$ is the orthogonal projection of $\mathbb{R}^{\bar{m}}$ onto V_x^\perp and v is a vector field of $\mathbb{R}^{\bar{m}}$, defined along a curve $c = c(t)$ of M through $x = c(0)$, with the tangent vector X_x at x , which extends v_x and such that $v_{c(t)} \in V_{c(t)} = \psi(c(t))$.

As $\psi_*: T_x M \times V_x \rightarrow V_x^\perp$ is a multilinear form for each $x \in M$, we can consider the derivative $\nabla \psi_*: T_x M \times T_x M \times V_x \rightarrow V_x^\perp$ defined by

$$(1.10) \quad (\nabla \psi_*)(X_x, Y_x, v_x) = \overset{V^\perp}{\nabla}_{X_x} \psi_*(Y, v) - \psi_*(\nabla_{X_x} Y, v_x) - \psi_*(Y_x, \overset{V}{\nabla}_{X_x} v),$$

where $\overset{V}{\nabla}$ and $\overset{V^\perp}{\nabla}$ are the connections on the vector bundles V and V^\perp , respectively, induced by the connection $\bar{\nabla}$ of $\mathbb{R}^{\bar{m}}$.

We conclude this section by recalling the following definition.

1.11. Definition. A submanifold M of a space form \bar{M} is called an essential submanifold if it is not contained in any proper totally geodesic submanifold of \bar{M} .

1.12. Proposition. *If M is a connected nicely curved submanifold of a space of constant curvature \bar{M} then M is an essential submanifold of a totally geodesic submanifold of \bar{M} of dimension equal to $\dim(\bigoplus_{k=0}^l N_x^k M)$. In particular, the normal space $\perp_x M$ of M at x is given by*

$$\perp_x M = \bigoplus_{k=1}^l N_x^k M$$

([S], Vol. 4, Prop. 67).

2. 2-SYMMETRIC SUBMANIFOLDS OF A SPACE FORM

Let $\bar{M} = \bar{M}(c)$ be a space of constant curvature c and of dimension \bar{m} . Let M be a submanifold of \bar{M} of dimension m .

2.1. Definition. M is a 2-symmetric submanifold of \bar{M} if, for each $x \in M$, there exist a neighbourhood \bar{U}_x of x in \bar{M} , a neighbourhood U_x of x in M , and an isometry $\sigma_x: \bar{U}_x \rightarrow \bar{U}_x$ such that

- (i) $\sigma_x^2 = \text{Id}_{\bar{U}_x}$,
- (ii) $U_x \subset \bar{U}_x$ and $\sigma_x(U_x) \subset U_x$,
- (iii) x is an isolated fixed point of σ_x in U_x .

2.2. Proposition. *Let M be a submanifold of a standard space form \bar{M} and $x \in M$. If there exists a neighbourhood \bar{U}_x of x in \bar{M} and an isometry $\sigma_x: \bar{U}_x \rightarrow \bar{U}_x$ satisfying the conditions (i), (ii), (iii) of Definition 2.1, then σ_x can be extended to a geodesic reflection of \bar{M} with respect to a complete totally geodesic submanifold F_x of \bar{M} meeting M orthogonally at x .*

Proof. For the sake of brevity, we shall denote the global extension of σ_x to \bar{M} by the same symbol σ_x . Because of (i) and $\sigma_x(x) = x$, the differential $d\sigma_x$ of σ_x is an operator of $T_x\bar{M}$ such that $(d\sigma_x)^2 = \text{Id}_{T_x\bar{M}}$. Then $T_x\bar{M} = V_1(T_x\bar{M}) \oplus V_{-1}(T_x\bar{M})$, where $V_1(T_x\bar{M})$ and $V_{-1}(T_x\bar{M})$ are, respectively, the eigenspaces of 1 and -1 .

Condition (ii) implies that T_xM is an invariant subspace. As before, $(d\sigma_x|_{T_xM})^2 = \text{Id}_{T_xM}$ and $T_xM = V_1(T_xM) \oplus V_{-1}(T_xM)$.

If $X_x \in V_1(T_xM) \subset T_xM$, then because σ_x an isometry, we have

$$\sigma_x(\exp_x(tX_x)) = \exp_{\sigma_x(x)}(d\sigma_x(tX_x)) = \exp_x(tX_x).$$

Then $X_x = 0$, otherwise x would not be an isolated fixed point of $U_x = M \cap \bar{U}_x$, as required in condition (iii). We conclude that $V_1(T_xM) = 0$ and hence

$$(2.3) \quad T_xM \subset V_{-1}(T_x\bar{M}).$$

In particular, we have

$$(2.4) \quad V_1(T_x\bar{M}) \perp T_xM.$$

Let now F_x be the totally geodesic submanifold of \bar{M} given by

$$(2.5) \quad F_x = \exp_x(V_1(T_x\bar{M})).$$

Due to (2.4), F_x meets orthogonally M at x .

We observe that all points of F_x are fixed by σ_x ; in fact, if $y \in F_x$, there exists a vector $X_x \in T_xF_x = V_1(T_x\bar{M})$ such that $y = \exp_x(X_x)$ and then

$$\sigma_x(y) = \sigma_x(\exp_x(X_x)) = \exp_{\sigma_x(x)}(X_x) = \exp_x(X_x) = y.$$

Moreover, in \bar{M} there are no fixed points different from the points of F_x .

In fact, if $y \in \bar{M}$, we can take a vector $X_x \in T_x\bar{M}$ such that $y = \exp_x(X_x)$ and, if we suppose $\sigma_x(y) = y$, then

$$\exp_x(X_x) = y = \sigma_x(y) = \sigma_x(\exp_x(X_x)) = \exp_{\sigma_x(x)}(d\sigma_x(X_x)) = \exp_x(d\sigma_x(X_x))$$

and the local bijectivity of \exp_x implies that $d\sigma_x(X_x) = X_x$, that is $X_x \in V_1(T_x\bar{M})$, and $y \in F_x$.

Then for $y \in F_x$ we have

$$(2.6) \quad T_y F_x = V_1(T_y\bar{M})$$

and

$$(2.7) \quad (T_y F_x)^\perp = V_{-1}(T_y\bar{M}).$$

By the definition of the geodesic reflection (see Theorem 1.2) we have, for each $y \in F_x$, $\varrho_{F_x}(y) = y$ and thus $\varrho_{F_x}|_{F_x} = \sigma_x|_{F_x}$.

If $z \notin F_x$ but z is near F_x , we can write $z = \exp_y(X_y^\perp)$, where $y \in F_x$ and $X_y^\perp \in (T_y F_x)^\perp$. Using (2.7), we have

$$\sigma_x(z) = \sigma_x(\exp_y(X_y^\perp)) = \exp_y(d\sigma_x(X_y^\perp)) = \exp_y(-X_y^\perp)$$

and hence, from the definition of ϱ_{F_x} , we obtain $\varrho_{F_x}(z) = \sigma_x(z)$. □

As an immediate consequence of Proposition 2.2 we have the following corollary.

2.8. Corollary. *M is a 2-symmetric submanifold of a standard space form \bar{M} if and only if, for each $x \in M$, there exists a complete totally geodesic submanifold F_x of \bar{M} , meeting M at x orthogonally, such that the geodesic reflection ϱ_{F_x} with respect to F_x locally maps M into itself.*

With reference to this corollary, we shall say that M is symmetric with respect to F_x and F_x will be called a submanifold of symmetry for M .

We conclude this section with the following definition, which is equivalent to that of Ferus if $\bar{M} = \bar{\mathbb{R}}^m$.

2.9. Definition. M is called a (locally) symmetric submanifold of \bar{M} if, for each $x \in M$, M is (locally) mapped into itself by the geodesic reflection of \bar{M} with respect to $\exp_x(\perp_x M)$.

2.10. Remark. By Corollary 2.8, a locally symmetric submanifold M of \bar{M} is just a 2-symmetric submanifold with the submanifold of symmetry $F_x = \exp_x(\perp_x M)$, $x \in M$.

3. PROPERTIES OF 2-SYMMETRIC SUBMANIFOLDS

Henceforth we will consider only nicely curved submanifolds.

3.1. Lemma. *If M is a submanifold of a Riemannian manifold \bar{M} and $\sigma: \bar{M} \rightarrow \bar{M}$ is an isometry such that $\sigma(M) \subset M$, then, for $x \in M$, $X_x, Y_x \in T_x M$ and $\xi_x \in N_x M$, we have*

- (i) $d\sigma(\overset{k}{s}(X_x, \xi_x)) = \overset{k}{s}(d\sigma(X_x), d\sigma(\xi_x))$,
- (ii) $d\sigma[(\nabla^k)(X_x, Y_x, \xi_x)] = (\nabla^k)(d\sigma(X_x), d\sigma(Y_x), d\sigma(\xi_x))$.

Proof. These properties are obvious because σ is an isometry and $\sigma(M)$ is an open part of M . □

The following proposition gives a more precise property than property (2.4).

3.2. Proposition. *Let M be a submanifold of a space of constant curvature \bar{M} and $x \in M$. If there exist two neighbourhoods \bar{U}_x, U_x of x in \bar{M}, M , respectively, and an isometry $\sigma_x: \bar{U}_x \rightarrow U_x$ satisfying the conditions (i), (ii), (iii) of Definition 2.1, we have*

$$(a) \quad V_1(T_x \bar{M}) \supset \bigoplus_{h=0}^q N_x^{2h+1} M, \quad V_{-1}(T_x \bar{M}) \supset \bigoplus_{h=0}^p N_x^{2h} M,$$

where $q = \lfloor \frac{l-1}{2} \rfloor$ and $p = \lfloor \frac{l}{2} \rfloor$.

In particular, if M is also an essential submanifold, then

$$(b) \quad V_1(T_x \bar{M}) = \bigoplus_{h=0}^q N_x^{2h+1} M, \quad V_{-1}(T_x \bar{M}) = \bigoplus_{h=0}^p N_x^{2h} M.$$

Proof. By (2.3) and Lemma 3.1 we have $d\sigma_x(\overset{0}{s}(X_x, Y_x)) = \overset{0}{s}(d\sigma_x(X_x), d\sigma_x(Y_x)) = \overset{0}{s}(X_x, Y_x)$, for $X_x, Y_x \in T_x M$. Then $\overset{0}{s}(X_x, Y_x) \in V_1(T_x \bar{M})$, but $N_x^1 M$ is spanned by $\{\overset{0}{s}(X_x, Y_x) \mid X_x, Y_x \in T_x M\}$ and so

$$(3.3) \quad N_x^1 M \subset V_1(T_x \bar{M}).$$

Take $X_x \in T_x M$, $\xi_x \in N_x^1 M$; by (2.3), Lemma 3.1 and (3.3) we have $d\sigma_x(\overset{1}{s}(X_x, \xi_x)) = \overset{1}{s}(d\sigma_x(X_x), d\sigma_x(\xi_x)) = \overset{1}{s}(-X_x, \xi_x) = -\overset{1}{s}(X_x, \xi_x)$. This implies $\overset{1}{s}(X_x, \xi_x) \in V_{-1}(T_x \bar{M})$, but $N_x^2 M$ is spanned by $\{\overset{1}{s}(X_x, \xi_x) \mid X_x \in T_x M, \xi_x \in N_x^1 M\}$ and so

$$(3.4) \quad N_x^2 M \subset V_{-1}(T_x \bar{M}).$$

Proceeding by induction, we obtain the statement (a).

If M is also an essential submanifold of \bar{M} , then $T_x \bar{M} = \left(\bigoplus_{h=0}^q N_x^{2h+1} M \right) \oplus \left(\bigoplus_{h=0}^p N_x^{2h} M \right)$. However, $T_x \bar{M} = V_1(T_x \bar{M}) \oplus V_{-1}(T_x \bar{M})$, so the inclusions $V_1(T_x \bar{M}) \supset \bigoplus_{h=0}^q N_x^{2h+1} M$ and $V_{-1}(T_x \bar{M}) \supset \bigoplus_{h=0}^p N_x^{2h} M$ give the statement (b). \square

Let now \bar{M} be a standard space form. Since the connected component of M through x is contained in $\exp_x \left(\bigoplus_{k=0}^l N_x^k M \right)$ ([S]), then putting $\bar{B}_x = \left(\bigoplus_{k=0}^l N_x^k M \right)^\perp$ we have

3.5. Corollary. *If M is a 2-symmetric submanifold of a standard space form \bar{M} , then, for each $x \in M$, a totally geodesic submanifold F_x is a submanifold of symmetry for M through x if and only if it is of the type*

$$F_x = \exp_x \left(\bigoplus_{h=0}^q N_x^{2h+1} M \oplus B_x \right),$$

where B_x is an arbitrary subspace of \bar{B}_x . In particular, if M is also essential, $F_x = \exp_x \left(\bigoplus_{h=0}^q N_x^{2h+1} M \right)$.

3.6. Remark. If M is a non-essential 2-symmetric submanifold of \bar{M} , then we see from Corollary 3.5 that, for each $x \in M$, submanifold of symmetry for M through x is not unique, but each of them induces the same geodesic reflection on the totally geodesic submanifold $\tilde{B} = \exp_x \left(\bigoplus_{k=0}^l N_x^k M \right) \supset M$.

The main result we want to prove in this section is the following theorem.

3.7. Theorem. *If M is a 2-symmetric submanifold of a space of constant curvature \bar{M} , then $\nabla \tilde{s}^k = 0$ for each $k = 0, 1, \dots, l-1$ (and hence, by Theorem 1.7, $\nabla^k A = 0$ for each $k = 1, 2, \dots, l$).*

Proof. We first recall that $(\nabla^k \tilde{s})(X_x, Y_x, \xi_x) \in N_x^{k+1} M$. If k is even, then $k+1$ is odd and, by Proposition 3.2, we have

$$(\nabla^k \tilde{s})(X_x, Y_x, \xi_x) \in N_x^{k+1} M \subset V_1(T_x \bar{M}).$$

Then for $\sigma = \sigma_x$

$$(3.8) \quad d\sigma \left((\nabla^k \tilde{s})(X_x, Y_x, \xi_x) \right) = (\nabla^k \tilde{s})(X_x, Y_x, \xi_x),$$

but by Lemma 3.1 and Proposition 3.2 this yields

$$\begin{aligned} d\sigma [(\nabla^k \dot{s})(X_x, Y_x, \xi_x)] &= (\nabla^k \dot{s})(d\sigma(X_x), d\sigma(Y_x), d\sigma(\xi_x)) \\ &= (\nabla^k \dot{s})(-X_x, -Y_x, -\xi_x) = -(\nabla^k \dot{s})(X_x, Y_x, \xi_x). \end{aligned}$$

Hence, we conclude that $\nabla^k \dot{s} = 0$ for k even.

If k is odd, then $k + 1$ is even and, exactly as in the case of k even, we deduce $\nabla^k \dot{s} = 0$. □

4. CHARACTERIZATION OF 2-SYMMETRIC SUBMANIFOLDS OF A SPACE FORM OF ZERO CURVATURE

Let M be a submanifold of the Euclidean space $\mathbb{R}^{\bar{m}}$. We denote by V the vector bundle on M whose fiber at $x \in M$ is

$$(4.1) \quad V_x = \bigoplus_{h=0}^q N_x^{2h+1} M, \quad q = \left[\frac{l-1}{2} \right].$$

Let $p = \dim V$ and let $G(p, \bar{m})$ be the Grassmannian of the p -planes of $\mathbb{R}^{\bar{m}}$ through the origin. We define a map $v: M \rightarrow G(p, \bar{m})$ by

$$(4.2) \quad v(x) = V_x, \quad x \in M.$$

The map v is said to be totally geodesic if $\nabla v_* = 0$ (see (1.10)).

The following proposition holds.

4.3. Proposition. *If M is a submanifold of $\mathbb{R}^{\bar{m}}$ such that $\nabla^k \dot{s} = 0$ for each $k = 0, 1, \dots, l-1$, then the map v is totally geodesic.*

Proof. Let $\gamma = \gamma(s)$ be the geodesic through $x = \gamma(0) \in M$, with X_x as the tangent vector at x . The tangent vector of γ at the point $\gamma(s)$ will be denoted by $X(s)$.

Let $Y \in \Gamma(TM)$ and $\xi \in \Gamma(V)$ be parallel along γ in TM and V , respectively, i.e.

$$(4.4) \quad \nabla_{X(s)} Y = 0, \quad \nabla_{X(s)}^V \xi = 0.$$

Since $\xi_{\gamma(s)} \in V_{\gamma(s)} = \bigoplus_{h=0}^q N_{\gamma(s)}^{2h+1}$, we have $\xi_{\gamma(s)} = \sum_{h=0}^q \xi_{\gamma(s)}^{2h+1}$ with $\xi_{\gamma(s)}^{2h+1} \in N_{\gamma(s)}^{2h+1} M$.

It is easy to check that $\overset{V}{\nabla}_{X(s)} \xi = 0$ implies

$$(4.5) \quad \overset{2h+1}{\nabla}_{X(s)} \overset{2h+1}{\xi} = 0, \quad h = 0, 1, \dots, q.$$

Then, by (1.10) for $\psi_* = v_*$ and $v_x = \xi_x$ and by (4.4), (4.5) we obtain

$$(\nabla v_*)(X_x, Y_x, \xi_x) = \overset{V^\perp}{\nabla}_{X_x} v_*(Y, \xi) = \Pi_{V^\perp} (\bar{\nabla}_{X_x} v_*(Y, \xi)).$$

From (1.9) we have

$$v_*(Y, \xi) = \Pi_{V^\perp} (\bar{\nabla}_Y \xi) = \sum_{h=0}^q \Pi_{V^\perp} (\bar{\nabla}_Y \overset{2h+1}{\xi})$$

and, using the Frenet equations of M , we obtain

$$v_*(Y, \xi) = \sum_{h=0}^q \Pi_{V^\perp} \left[-\overset{2h+1}{A} (Y, \overset{2h+1}{\xi}) + \overset{2h+1}{\nabla}_Y \overset{2h+1}{\xi} + \overset{2h+1}{s} (Y, \overset{2h+1}{\xi}) \right].$$

But $\overset{2h+1}{A} (Y, \overset{2h+1}{\xi}) \in \overset{2h}{N} M \subset V^\perp$, $\overset{2h+1}{\nabla}_Y \overset{2h+1}{\xi} \in \overset{2h+1}{N} M \subset V$, $\overset{2h+1}{s} (Y, \overset{2h+1}{\xi}) \in \overset{2h+1}{N} M \subset V^\perp$, so we conclude

$$v_*(Y, \xi) = \sum_{h=0}^q \left[-\overset{2h+1}{A} (Y, \overset{2h+1}{\xi}) + \overset{2h+1}{s} (Y, \overset{2h+1}{\xi}) \right]$$

and hence

$$(\nabla v_*)(X_x, Y_x, \xi_x) = \sum_{h=0}^q \Pi_{V^\perp} \left[\bar{\nabla}_{X_x} \left(-\overset{2h+1}{A} (Y, \overset{2h+1}{\xi}) + \overset{2h+1}{s} (Y, \overset{2h+1}{\xi}) \right) \right].$$

Applying again the Frenet equations, we have

$$\begin{aligned} (\nabla v_*)(X_x, Y_x, \xi_x) &= \Pi_{V^\perp} \left(\bar{\nabla}_{X_x} \left(-\overset{1}{A} (Y, \overset{1}{\xi}) \right) + \overset{0}{s} (X_x, -\overset{1}{A} (Y, \overset{1}{\xi})) \right) \\ &+ \sum_{h=1}^q \Pi_{V^\perp} \left(\overset{2h}{A} (X_x, \overset{2h+1}{A} (Y_x, \overset{2h+1}{\xi_x})) - \overset{2h}{\nabla}_{X_x} \left(\overset{2h+1}{A} (Y, \overset{2h+1}{\xi}) \right) \right. \\ &- \overset{2h}{s} (X_x, \overset{2h+1}{A} (Y_x, \overset{2h+1}{\xi_x})) \left. \right) + \sum_{h=0}^q \Pi_{V^\perp} \left(-\overset{2h+2}{A} (X_x, \overset{2h+1}{s} (Y_x, \overset{2h+1}{\xi_x})) \right. \\ &+ \overset{2h+2}{\nabla}_{X_x} \left(\overset{2h+1}{s} (Y, \overset{2h+2}{\xi}) \right) + \overset{2h+2}{s} (X_x, \overset{2h+1}{s} (Y_x, \overset{2h+1}{\xi_x})) \left. \right) \\ &= \nabla_{X_x} \left(-\overset{1}{A} (Y, \overset{1}{\xi}) \right) + \sum_{h=1}^q \left[-\overset{2h}{\nabla} \left(\overset{2h+1}{A} (Y, \overset{2h+1}{\xi}) \right) + \overset{2h+2}{\nabla}_{X_x} \left(\overset{2h+1}{s} (Y, \overset{2h+1}{\xi}) \right) \right] = 0. \end{aligned}$$

□

Our next goal is to prove the following proposition, which is a generalization of a result obtained for $\overset{1}{N}$ -symmetric submanifold of Euclidean spaces ([R]).

4.6. Proposition. *Let M be a submanifold of $\bar{M} = \mathbb{R}^{\bar{m}}$. If there exists a totally geodesic map $v: M \rightarrow G(p, \bar{m})$ such that, for each $x \in M$, $\overset{1}{N}_x M \subset v(x) \subset \perp_x M$, then M is a 2-symmetric submanifold of $\mathbb{R}^{\bar{m}}$.*

Before proving this proposition, we need some lemmas. Let V be a vector bundle on M , whose fiber V_x at $x \in M$ is such that

$$\overset{1}{N}_x M \subset V_x \subset \perp_x M.$$

We denote by W the vector bundle on M whose fiber W_x is given by $W_x = (T_x M \oplus V_x)^\perp$.

Let \tilde{M} denote a submanifold of $\mathbb{R}^{\bar{m}}$ which is a tubular neighbourhood of M in the set $\{x + w \mid x \in M, w \in W_x\}$. Then the following lemma holds.

4.7. Lemma. *M is totally geodesic submanifold of \tilde{M} .*

PROOF. Obviously $T_x \tilde{M} = T_x M \oplus W_x$, $x \in M$. Let $\tilde{\nabla}$ denote the Levi-Civita connection induced on \tilde{M} by the standard connection $\bar{\nabla}$ of $\mathbb{R}^{\bar{m}}$. Then for $X_x \in T_x M$, $Y \in \Gamma(TM)$ we have

$$\tilde{\nabla}_{X_x} Y = \Pi_{T_x \tilde{M}}(\bar{\nabla}_{X_x} Y) = \Pi_{T_x M \oplus W_x}(\nabla_{X_x} Y + \overset{0}{s}(X_x, Y_x)) = \nabla_{X_x} Y.$$

□

Now we consider the Gauss map $g_{\tilde{M}}: \tilde{M} \rightarrow G(\tilde{m}, \bar{m})$ and the normal map $\gamma_{\tilde{M}}: \tilde{M} \rightarrow G(\bar{m} - \tilde{m}, \bar{m})$ of \tilde{M} , where $\tilde{m} = \dim \tilde{M}$. They are defined by

$$(4.8) \quad g_{\tilde{M}}(x) = T_x \tilde{M} \text{ and } \gamma_{\tilde{M}}(x) = \perp_x \tilde{M}.$$

Let $s_{\tilde{M}}^0$ denote the second fundamental form of \tilde{M} (as a submanifold of $\mathbb{R}^{\bar{m}}$), ([V]). Then for $\tilde{X}_x, \tilde{Y}_x \in T_x \tilde{M}$ and $\tilde{Y} \in \Gamma(T\tilde{M})$ the extension of \tilde{Y}_x , by (1.9) we have [V]

$$(4.9) \quad (g_{\tilde{M}})_*(\tilde{X}_x, \tilde{Y}_x) = s_{\tilde{M}}^0(\tilde{X}_x, \tilde{Y}_x).$$

Moreover, if $\xi \in \Gamma(\perp \tilde{M})$ is an extension of $\xi_x \in \perp_x \tilde{M}$, then

$$(4.10) \quad (\gamma_{\tilde{M}})_*(\tilde{X}_x, \xi_x) = \Pi_{T_x \tilde{M}}(\bar{\nabla}_{\tilde{X}_x} \xi).$$

Let $v: M \rightarrow G(p, \bar{m})$ be a map satisfying the condition

$$(4.11) \quad \overset{1}{N}_x M \subset v(x) \subset \perp_x M, \quad x \in M.$$

If we define the vector bundle V on M taking $V_x = v(x)$, we see that M is a totally geodesic submanifold of \tilde{M} (see Lemma 4.7) and the normal map $\gamma_{\tilde{M}}$ coincides with v for all points of M :

$$\gamma_{\tilde{M}}(x) = v(x), \quad x \in M.$$

Moreover, for $x \in M$, $X_x \in T_x M$ and $\xi_x \in V_x$ we have, by (1.9) and (4.10),

$$(4.12) \quad (\gamma_{\tilde{M}})_*(X_x, \xi_x) = v_*(X_x, \xi_x).$$

4.13. Lemma. *Let M be a submanifold of $\mathbb{R}^{\bar{m}}$. If the map $v: M \rightarrow G(p, \bar{m})$ satisfies (4.11) and is totally geodesic, then $\nabla_{X_x}(g_{\tilde{M}})_*(Y_x, -) = 0$ for each $x \in M$ and $X_x, Y_x \in T_x M$.*

Proof. Let $Y \in \Gamma(TM)$ and $\xi \in \Gamma(V)$ be extensions of Y_x, ξ_x , respectively. Then, by (1.10) and (4.12),

$$(\nabla v_*)(X_x, Y_x, \xi_x) = \overset{V^\perp}{\nabla}_{X_x}(\gamma_{\tilde{M}})_*(Y, \xi) - (\gamma_{\tilde{M}})_*(\nabla_{X_x} Y, \xi_x) - (\gamma_{\tilde{M}})_*(Y_x, \overset{V}{\nabla}_{X_x} \xi).$$

Since the image of $(\gamma_{\tilde{M}})_*$ is $T_x \tilde{M} = V^\perp$ we obtain by Lemma 4.7

$$(4.14) \quad (\nabla v_*)(X_x, Y_x, \xi_x) = \nabla(\gamma_{\tilde{M}})_*(X_x, Y_x, \xi_x).$$

Let $\mu: G(\bar{m} - p, \bar{m}) \rightarrow G(p, \bar{m})$ be the isometry which associates to an $(\bar{m} - p)$ -plane through the origin of $\mathbb{R}^{\bar{m}}$ its orthogonal complement. The normal map $\gamma_{\tilde{M}}$ and the Gauss map $g_{\tilde{M}}$ are related by $\gamma_{\tilde{M}} = \mu \circ g_{\tilde{M}}$ and hence

$$(4.15) \quad d\gamma_{\tilde{M}} = d\mu \circ dg_{\tilde{M}}.$$

Since the canonical isomorphism φ (see Section 1) commutes with ∇ ($[V]$) and μ is an isometry, applying (4.14) and (4.15) we can write

$$\begin{aligned} (\nabla v_*)(X_x, Y_x, \xi_x) &= (\nabla(\gamma_{\tilde{M}})_*)(X_x, Y_x, \xi_x) = [(\nabla_{X_x}(\varphi \circ d\gamma_{\tilde{M}}))(Y_x)](\xi_x) \\ &= [\varphi \circ d\mu((\nabla_{X_x} dg_{\tilde{M}})(Y_x))](\xi_x) \\ &= [\varphi \circ d\mu \circ \varphi^{-1}((\nabla_{X_x}(\varphi \circ dg_{\tilde{M}}))(Y_x))](\xi_x). \end{aligned}$$

Then, if $\nabla v_* = 0$, we obtain that $\varphi \circ d\mu \circ \varphi^{-1}((\nabla_{X_x}(\varphi \circ dg_{\tilde{M}}))(Y_x))$ is the zero morphism of $\text{Hom}(V_x, V_x^\perp)$. But $\varphi \circ d\mu \circ \varphi^{-1}$ is an isomorphism, so we conclude that $(\nabla_{X_x}(\varphi \circ dg_{\tilde{M}}))(Y_x)$ is the zero morphism of $\text{Hom}(V_x^\perp, V_x)$ and hence $(\nabla_{X_x}(\varphi \circ dg_{\tilde{M}}))(Y_x) = (\nabla(g_{\tilde{M}})_*)(X_x, Y_x, -)$. \square

If we consider the second fundamental form of \tilde{M} , $s_{\tilde{M}}^0$, with values in $\perp\tilde{M}$, we shall denote by $\overset{\perp}{\nabla}s_{\tilde{M}}^0$ its derivative. Then if $\tilde{X}_x, \tilde{Y}_x, \tilde{Z}_x \in T_x\tilde{M}$ and $\tilde{Y}, \tilde{Z} \in \Gamma(T\tilde{M})$ extend \tilde{Y}_x, \tilde{Z}_x , we have

$$(4.16) \quad (\overset{\perp}{\nabla}s_{\tilde{M}}^0)(\tilde{X}_x, \tilde{Y}_x, \tilde{Z}_x) = \overset{\perp}{\nabla}_{\tilde{X}_x}^{\tilde{M}}(s_{\tilde{M}}^0(\tilde{Y}, \tilde{Z})) - s_{\tilde{M}}^0(\overset{\perp}{\nabla}_{\tilde{X}_x}\tilde{Y}, \tilde{Z}_x) - s_{\tilde{M}}^0(\tilde{Y}_x, \overset{\perp}{\nabla}_{\tilde{X}_x}\tilde{Z}).$$

4.17. Lemma. *Let M be a submanifold of $\mathbb{R}^{\bar{m}}$. If $\nu: M \rightarrow G(p, \bar{m})$ is totally geodesic, then $\overset{\perp}{\nabla}s_M^0 = 0$.*

Proof. We fix $x \in M$ and choose $X_x, Y_x \in T_xM$, $\tilde{Z}_x \in T_x\tilde{M}$; by (4.9) ([V]) we obtain

$$(\overset{\perp}{\nabla}s_M^0)(X_x, Y_x, \tilde{Z}_x) = (\nabla(g_{\tilde{M}})_*) (X_x, Y_x, \tilde{Z}_x),$$

and since $\nabla\nu_* = 0$, Lemma 4.13, implies

$$(4.18) \quad (\overset{\perp}{\nabla}s_M^0)(X_x, Y_x, \tilde{Z}_x) = 0.$$

But s_M^0 is symmetric and hence also

$$(4.19) \quad (\overset{\perp}{\nabla}s_M^0)(X_x, \tilde{Z}_x, Y_x) = 0.$$

Consider now the following decomposition of the vectors $\tilde{X}_x, \tilde{Y}_x, \tilde{Z}_x \in T_x\tilde{M}$: $\tilde{X}_x = X_x + \eta_x$, $\tilde{Y}_x = Y_x + \vartheta_x$, $\tilde{Z}_x = Z_x + \zeta_x$, where $X_x, Y_x, Z_x \in T_xM$, and $\eta_x, \vartheta_x, \zeta_x \in W_x$. From (4.18), (4.19) we obtain $(\overset{\perp}{\nabla}s_M^0)(X_x, \tilde{Y}_x, \tilde{Z}_x) = (\overset{\perp}{\nabla}s_M^0)(X_x, Y_x, Z_x) + (\overset{\perp}{\nabla}s_M^0)(X_x, \vartheta_x, Z_x) + (\overset{\perp}{\nabla}s_M^0)(X_x, Y_x, \zeta_x) + (\overset{\perp}{\nabla}s_M^0)(X_x, \vartheta_x, \zeta_x) = (\overset{\perp}{\nabla}s_M^0)(X_x, \vartheta_x, \zeta_x)$.

Let γ be a curve of M through x with the tangent vector X_x and suppose that ϑ, ζ are obtained by the parallelism on \tilde{M} along γ from ϑ_x, ζ_x . Then

$$\begin{aligned} (\overset{\perp}{\nabla}s_M^0)(X_x, \vartheta_x, \zeta_x) &= \overset{\perp}{\nabla}_{X_x}^{\tilde{M}}(s_M^0(\vartheta, \zeta)) - s_M^0(\overset{\perp}{\nabla}_{X_x}\vartheta, \zeta_x) - s_M^0(\vartheta_x, \overset{\perp}{\nabla}_{X_x}\zeta) \\ &= \overset{\perp}{\nabla}_{X_x}^{\tilde{M}}(s_M^0(\vartheta, \zeta)). \end{aligned}$$

But for $x' \in \gamma$ we have $\vartheta_{x'}, \zeta_{x'} \in W_{x'}$. In fact $\vartheta_{x'}, \zeta_{x'}$ are obtained by the parallelism in \tilde{M} along a curve γ contained in M from the vectors $\vartheta_x, \zeta_x \in W_x$ orthogonal to T_xM . Because M is a totally geodesic submanifold of \tilde{M} , the parallel transport on \tilde{M} along a curve of M of any vector of T_xM gives also a vector of TM and hence we have $\vartheta_{x'}, \zeta_{x'} \in \perp_{x'}M \cap T_{x'}\tilde{M} (\subset W_{x'})$, for $x' \in \gamma$.

Then we can extend $\zeta_{x'}$ along the straight line $\{x' + t\vartheta_{x'} \mid t \in \mathbb{R}\}$ of \tilde{M} as a constant vector, and this vector is always tangent to \tilde{M} . So we have $s_{\tilde{M}}^0(\vartheta_{x'}, \zeta_{x'}) = 0$ and $\nabla_{X_x}^{\perp \tilde{M}}(s_{\tilde{M}}^0(\vartheta, \zeta)) = 0$ and, consequently,

$$(4.20) \quad (\nabla_{X_x}^{\perp \tilde{M}})(\tilde{X}_x, \tilde{Y}_x, \tilde{Z}_x) = 0.$$

Now, if we use (4.18), (4.19), (4.20) and the symmetry of $s_{\tilde{M}}^0$ and $\nabla_{\tilde{M}}^{\perp}$, we obtain

$$(\nabla_{\tilde{M}}^{\perp} s_{\tilde{M}}^0)(\tilde{X}_x, \tilde{Y}_x, \tilde{Z}_x) = (\nabla_{\tilde{M}}^{\perp} s_{\tilde{M}}^0)(\eta_x, \vartheta_x, \zeta_x).$$

However,

$$(\nabla_{\tilde{M}}^{\perp} s_{\tilde{M}}^0)(\eta_x, \vartheta_x, \zeta_x) = \nabla_{\eta_x}^{\perp \tilde{M}}(s_{\tilde{M}}^0(\vartheta, \zeta)) - s_{\tilde{M}}^0(\tilde{\nabla}_{\eta_x} \vartheta, \zeta_x) - s_{\tilde{M}}^0(\vartheta_x, \tilde{\nabla}_{\eta_x} \zeta)$$

and, as above, we can assume ϑ, ζ constant along the straight line $\{x + t\eta_x \mid t \in \mathbb{R}\}$ of \tilde{M} and tangent to \tilde{M} . Hence

$$(\nabla_{\tilde{M}}^{\perp} s_{\tilde{M}}^0)(\eta_x, \vartheta_x, \zeta_x) = 0.$$

□

We can now prove Proposition 4.6.

Due to Lemma 4.17, $\nabla_{\tilde{M}}^{\perp} s_{\tilde{M}}^0 = 0$ at the points $x \in M$. Then $s_{\tilde{M}}^0$ is a parallel form along each geodesic of M and so all curvatures of each geodesic of M are constant ([St], Theorem I). Moreover, the Frenet vectors of even orders of each geodesic of M are in $\perp \tilde{M} = V$, while those of odd orders are in $T\tilde{M}$. We conclude that, for $x \in M$, the reflection of $\mathbb{R}^{\bar{m}}$ with respect to the affine normal V_x maps each geodesic of M through x , into itself (see [St], Theorem I and Lemma I). Then the above reflection maps M into itself and M is a 2-symmetric submanifold of $\mathbb{R}^{\bar{m}}$.

The results obtained in Theorem 3.7, Proposition 4.3, Proposition 4.6 give the following theorem:

4.21. Theorem. *If M is a submanifold of $\mathbb{R}^{\bar{m}}$, the following properties are equivalent:*

- (i) M is a 2-symmetric submanifold,
- (ii) $\nabla s^k = 0$, for each $k = 0, 1, \dots, l - 1$,
- (iii) there exists a totally geodesic map $v: M \rightarrow G(p, \bar{m})$ such that $N_x^1 M \subset v(x) \subset \perp_x M$.

Proof. In fact, if M is a 2-symmetric submanifold of $\mathbb{R}^{\bar{m}}$, Theorem 3.7 implies $\nabla^k s = 0$ for each $k = 0, 1, \dots, l-1$; then it follows from Proposition 4.3 that the map $v: M \rightarrow G(p, \bar{m})$ defined by (4.1) and (4.2) satisfies the conditions $N_x^1 M \subset v(x) \subset \perp_x M$ and $\nabla v_* = 0$.

If M is a submanifold of $\mathbb{R}^{\bar{m}}$ and there exists a totally geodesic map $v: M \rightarrow G(p, \bar{m})$ such that $N_x^1 M \subset v(x) \subset \perp_x M$, then, by Proposition 4.6, M is a 2-symmetric submanifold of $\mathbb{R}^{\bar{m}}$. \square

4.22. Remark. If M is a submanifold of $\mathbb{R}^{\bar{m}}$ and we suppose the existence of the totally geodesic map $v: M \rightarrow G(p, \bar{m})$ satisfying the conditions $N_x^1 M \subset v(x) \subset \perp_x M$, then in the proof of Proposition 4.6 we have seen that M is mapped into itself by the reflection of $\mathbb{R}^{\bar{m}}$ with respect to the affine space F_x through each point $x \in M$, parallel to $v(x)$. Then F_x is a submanifold of symmetry of M and, from Corollary 3.5, we conclude $v(x) \supset \bigoplus_{h=0}^q N_x^{2h+1} M$.

We have studied 2-symmetric submanifolds of $\mathbb{R}^{\bar{m}}$. It is now easy to obtain some results also in the case of a general space form \bar{M} of zero curvature. In fact we have the following theorem.

4.23. Theorem. *If M is a submanifold of a space form \bar{M} of zero curvature, then the following conditions are equivalent:*

- (i) M is a 2-symmetric submanifold of \bar{M} ,
- (ii) $\nabla^k s = 0$ for each $k = 0, 1, \dots, l-1$.

Proof. The implication (i) \Rightarrow (ii) is just Theorem 3.7. In order to have the implication (ii) \Rightarrow (i) we observe that the forms s^k and their derivatives $\nabla^k s$ are invariant under isometries of the ambient space \bar{M} . If \bar{M} is a space form of zero curvature, then for each $x \in M$ we can find a neighbourhood \bar{U}'_x of x in \bar{M} and a local isometry $\varphi: \bar{U}'_x \rightarrow \mathbb{R}^{\bar{m}}$. If we denote by s^k_φ the k -fundamental form of $\varphi(M \cap \bar{U}'_x)$ in $\mathbb{R}^{\bar{m}}$, then

$$\nabla s^k_\varphi = 0.$$

It follows from Theorem 4.21 that $\varphi(M \cap \bar{U}'_x)$ is a 2-symmetric submanifold of $\mathbb{R}^{\bar{m}}$. This implies that, for each $y \in \varphi(M \cap \bar{U}'_x)$, there exists a local isometry σ_y such that the conditions (i), (ii), (iii) of Definition 2.1 are satisfied for σ_y . However, then the same conditions follow for the local isometry $\varphi^{-1} \circ \sigma_y \circ \varphi$. \square

5. CHARACTERIZATION OF 2-SYMMETRIC SUBMANIFOLDS
OF A SPACE FORM OF POSITIVE CURVATURE

In this section we want to extend the previous results to the case of a submanifold M of a space of positive constant curvature: $\bar{M} = \bar{M}(c)$, $c > 0$. For the sake of simplicity we consider the case $c = 1$.

We start from $\bar{M} = S^{\bar{m}}(1) \subset \mathbb{R}^{\bar{m}+1}$.

To distinguish the derivative on $\mathbb{R}^{\bar{m}+1}$ from the induced derivative on $S^{\bar{m}}(1)$ we denote the former by $\hat{\nabla}$ and the latter by $\bar{\nabla}$.

The second fundamental form of $S^{\bar{m}}(1)$ in $\mathbb{R}^{\bar{m}+1}$ will be denoted by s_S^0 . The Gauss equations of $S^{\bar{m}}(1)$ are

$$\hat{\nabla}_{X_x} Y = \bar{\nabla}_{X_x} Y + s_S^0(X_x Y_x), \quad x \in S^{\bar{m}}(1), \quad X_x, Y_x \in T_x S^{\bar{m}}(1),$$

and, recalling that

$$(5.1) \quad s_S^0(X_x, Y_x) = -\bar{g}(X_x, Y_x)x,$$

where \bar{g} is the metric of $\bar{M} = S^{\bar{m}}(1)$ induced by the usual metric of $\mathbb{R}^{\bar{m}+1}$, we can write

$$(5.2) \quad \hat{\nabla}_{X_x} Y = \bar{\nabla}_{X_x} Y - \bar{g}(X_x, Y_x)x.$$

Suppose now that M is a submanifold of $\bar{M} = S^{\bar{m}}(1)$ satisfying the condition

$$(5.3) \quad \nabla s^k = 0, \quad k = 0, 1, \dots, l-1,$$

where we are using the symbol s^k to denote the k -fundamental form of M as a submanifold of $\bar{M} = S^{\bar{m}}(1)$.

Let $N_x^k M$ and $\perp_x M$, respectively, denote the k -normal space and the normal space at x of $M \subset \bar{M} = S^{\bar{m}}(1)$.

We define

$$(5.4) \quad V_x = \bigoplus_{h=0}^q N_x^{2h+1} M, \quad x \in M,$$

and

$$(5.5) \quad \hat{V}_x = V_x \oplus \langle x \rangle, \quad x \in M.$$

If we view M as a submanifold of $\mathbb{R}^{\bar{m}+1}$, we shall denote the k -normal space, the normal space and the k -fundamental form by $\hat{N}_x M$, $\hat{\perp}_x M$, \hat{s} , respectively. For $x \in M$, $X_x, Y_x \in T_x M$ and an extension $Y \in \Gamma(TM)$ of Y_x we have, using (5.2),

$$\hat{s}(X_x, Y_x) = \Pi_{\hat{\perp}_x M}(\hat{\nabla}_{X_x} Y) = \Pi_{\hat{\perp}_x M}(\bar{\nabla}_{X_x} Y - \bar{g}(X_x, Y_x)x).$$

Further,

$$(5.6) \quad \hat{\perp}_x M = \perp_x M \oplus \langle x \rangle,$$

and then

$$\hat{s}(X_x, Y_x) = \Pi_{\perp_x M}(\bar{\nabla}_{X_x} Y) - \Pi_{\langle x \rangle}(\bar{g}(X_x, Y_x)x) = \overset{0}{s}(X_x, Y_x) - \bar{g}(X_x, Y_x).$$

In particular, it follows that $\hat{s}(X_x, Y_x) \in \hat{N}_x M \oplus \langle x \rangle$ and hence $\hat{N}_x M \subset \overset{1}{N}_x M \oplus \langle x \rangle$; (5.4) and (5.5) imply that

$$\hat{V}_x \supset \overset{1}{N}_x M.$$

Moreover, by (5.6) we can see that

$$\hat{V}_x \subset \hat{\perp}_x M,$$

and then we have the inclusions

$$(5.7) \quad \overset{1}{N}_x M \subset \hat{V}_x \subset \hat{\perp}_x M.$$

Let $\hat{v}: M \rightarrow G(p+1, \bar{m}+1)$ ($p = \dim V$) be the map defined by

$$(5.8) \quad \hat{v}(x) = \hat{V}_x, \quad x \in M.$$

By (1.9) we have

$$(5.9) \quad \hat{v}_*(X_x, \hat{\xi}_x) = \Pi_{\hat{V}_x^\perp}(\hat{\nabla}_{X_x} \hat{\xi}),$$

where $x \in M$, $X_x \in T_x M$, $\hat{\xi}_x \in \hat{V}_x$ and $\hat{\xi} \in \Gamma(\hat{V})$ is an extension of $\hat{\xi}_x$ in the vector bundle \hat{V} on M with the fiber \hat{V}_x at $x \in M$.

We fix $x \in M$, $X_x, Y_x \in T_x M$, $\hat{\xi}_x \in \hat{V}_x$ and consider the geodesic $\gamma = \gamma(s)$ through $x = \gamma(0)$, with the tangent vector X_x at x . We denote by Y the parallel transport of Y_x along γ , and by $\hat{\xi} = \sum_{h=0}^q \xi^{2h+1} + a\gamma$ the parallel transport of $\hat{\xi}_x$ in the vector

bundle \hat{V} along γ ; the vectors $\xi_{\gamma(s)}^{2h+1} \in N_{\gamma(s)}^{2h+1} M$ are the component vectors of $\hat{\xi}_{\gamma(s)}$ in $N_{\gamma(s)}^{2h+1} M$, and a is a function defined on γ .

It is easy to check that, if \hat{V}_x is defined by (5.5) and (5.4), the following lemma holds.

5.10. Lemma. *The condition $\hat{\nabla}_{X(s)} \hat{\xi} = 0$ implies $\hat{\nabla}_{X(s)} \xi^{2h+1} = 0$, $h = 0, 1, \dots, q$ and $a = \text{const}$, where $X(s)$ is the tangent vector of γ at $\gamma(s)$.*

Now we compute $(\nabla \hat{v}_*)(X_x, Y_x, \hat{\xi}_x)$ for $x \in M$, $X_x, Y_x \in T_x M$, $\hat{\xi}_x \in \hat{V}_x$. We obtain

$$\begin{aligned} (\nabla \hat{v}_*)(X_x, Y_x, \hat{\xi}_x) &= \hat{\nabla}_{X_x}^{\perp} (\hat{v}_*(Y, \hat{\xi})) - v_*(\nabla_{X_x} Y, \hat{\xi}_x) - \hat{v}_*(Y_x, \hat{\nabla}_{X_x} \hat{\xi}) \\ (5.11) \qquad \qquad \qquad &= \hat{\nabla}_{X_x}^{\perp} (\hat{v}_*(Y, \hat{\xi})) = \Pi_{\hat{V}_x^{\perp}} \left(\hat{\nabla}_{X_x} (\hat{v}_*(Y, \hat{\xi})) \right). \end{aligned}$$

However, due to (1.9), (5.2), (5.10) we have, along γ ,

$$\begin{aligned} \hat{v}_*(Y, \hat{\xi}) &= \Pi_{\hat{V}^{\perp}} \left(\hat{\nabla}_Y \left(\sum_{h=0}^q \xi^{2h+1} + a\gamma \right) \right) \\ &= \sum_{h=0}^q \Pi_{\hat{V}^{\perp}} \left(\hat{\nabla}_Y \xi^{2h+1} \right) + a \Pi_{\hat{V}^{\perp}} (\hat{\nabla}_Y \gamma) \\ &= \sum_{h=0}^q \Pi_{\hat{V}^{\perp}} \left[\bar{\nabla}_Y \xi^{2h+1} - \bar{g}(Y, \xi^{2h+1}) \gamma \right] + a \Pi_{\hat{V}^{\perp}} (Y). \end{aligned}$$

Now, because $Y \in TM$, we conclude that $Y \perp \xi^{2h+1}$ and $Y \in \hat{V}^{\perp}$. So applying the Frenet equations for M as a submanifold of $\bar{M} = S^{\bar{m}}(1)$, we have

$$\begin{aligned} \hat{v}_*(Y, \hat{\xi}) &= \sum_{h=0}^q \Pi_{\hat{V}^{\perp}} \left[\bar{\nabla}_Y \xi^{2h+1} \right] + aY \\ &= \sum_{h=0}^q \Pi_{\hat{V}^{\perp}} \left[-A^{2h+1} (Y, \xi^{2h+1}) + \nabla_Y^{2h+1} \xi^{2h+1} + s^{2h+1} (Y, \xi^{2h+1}) \right] + aY. \end{aligned}$$

If we recall that $A^{2h+1} (Y, \xi^{2h+1}) \in N M \perp \hat{V}$, $\nabla_Y^{2h+1} \xi^{2h+1} \in N M \perp \hat{V}$, $s^{2h+1} (Y, \xi^{2h+1}) \in N M \perp \hat{V}$, we can write

$$\hat{v}_*(Y, \hat{\xi}) = \sum_{h=0}^q \left(-A^{2h+1} (Y, \xi^{2h+1}) + s^{2h+1} (Y, \xi^{2h+1}) \right) + aY.$$

Coming back to Formula 5.11, we obtain, using also (5.2),

$$\begin{aligned}
 (\nabla \hat{v}_*)(X_x, Y_x, \hat{\xi}_x) &= \Pi_{V^\perp} \left\{ \hat{\nabla}_{X_x} \left[\sum_{h=0}^q \left(-\binom{2h+1}{A} \left(Y, \binom{2h+1}{\xi} \right) + \binom{2h+1}{s} \left(Y, \binom{2h+1}{\xi} \right) \right) + aY \right] \right\} \\
 &= \Pi_{V^\perp} \left\{ \bar{\nabla}_{X_x} \left[\sum_{h=0}^q \left(-\binom{2h+1}{A} \left(Y, \binom{2h+1}{\xi} \right) + \binom{2h+1}{s} \left(Y, \binom{2h+1}{\xi} \right) \right) + aY \right] \right. \\
 &\quad \left. - \bar{g} \left(X_x, \sum_{h=0}^q \left(-\binom{2h+1}{A} \left(Y_x, \binom{2h+1}{\xi_x} \right) + \binom{2h+1}{s} \left(Y_x, \binom{2h+1}{\xi_x} \right) \right) + aY_x \right) x \right\}.
 \end{aligned}$$

Now $x \in \hat{V}_x$. Moreover, since $\nabla^k s = 0$ for $k = 0, 1, \dots, l-1$ (and hence also $\nabla^k A = 0$ for $k = 1, \dots, l$) and $Y, \binom{2h+1}{\xi}$ are parallel along γ due to Lemma 5.10, we have $\nabla_{X_x} \left(\binom{2h}{A} \left(Y, \binom{2h+1}{\xi} \right) \right) = 0$ and $\nabla_{X_x} \left(\binom{2h+1}{s} \left(Y, \binom{2h+1}{\xi} \right) \right) = 0$. Then by the same computation as in the Proposition 4.3 we obtain $(\nabla \hat{v}_*)(X_x, Y_x, \hat{\xi}_x) = 0$ and we arrive at the following proposition:

5.12. Proposition. *If M is a submanifold of $S^{\bar{m}}(1)$ such that $\nabla^k s = 0$ for each $k = 0, 1, \dots, l-1$, then the map \hat{v} (defined in (5.4), (5.5), (5.8)) is totally geodesic.*

For submanifolds of $S^{\bar{m}}(1)$ we have the following theorem analogous to Theorem 4.21.

5.13. Theorem. *If M is a submanifold of $S^{\bar{m}}(1)$, the following properties are equivalent:*

- (i) M is a 2-symmetric submanifold of $S^{\bar{m}}(1)$,
- (ii) $\nabla^k s = 0$, for each $k = 0, 1, \dots, l-1$,
- (iii) the map \hat{v} (defined in (5.4), (5.5), (5.8)) is totally geodesic.

Proof. The implication (i) \Rightarrow (ii) is always true by Theorem 3.7. The implication (ii) \Rightarrow (iii) is just Proposition 5.12.

In order to verify implication (iii) \Rightarrow (i) we observe the following facts.

Because of the definition of \hat{v} and in particular of (5.7), it is possible to apply Proposition 4.6 to M as to a submanifold of $\mathbb{R}^{\bar{m}+1} (\supset S^{\bar{m}}(1))$. Then we can conclude that M is a 2-symmetric submanifold of $\mathbb{R}^{\bar{m}+1}$. By Remark 4.22, M is mapped into itself by the reflection of $\mathbb{R}^{\bar{m}+1}$ with respect to $\hat{v}(x)$ for each $x \in M$. Each such reflection induces a geodesic reflection on $S^{\bar{m}}(1)$ with respect to the totally geodesic submanifold $\hat{v}(x) \cap S^{\bar{m}}(1)$ of $S^{\bar{m}}(1)$, which maps M into itself, and hence M will be also a 2-symmetric submanifold of $S^{\bar{m}}(1)$. \square

A similar argument as in the proof of Theorem 4.23 can be used for proving the following theorem:

5.14. Theorem. *If M is a submanifold of a space form $\overline{M} = \overline{M}(c)$ of positive constant curvature c , then the following conditions are equivalent:*

- (i) M is a 2-symmetric submanifold of \overline{M} ,
- (ii) $\nabla^k \mathfrak{s} = 0$ for each $k = 0, 1, \dots, l - 1$.

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