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STERNBERG'S STRUCTURE FUNCTION OF DIFFERENTIAL SYSTEMS IN DIMENSION FIVE

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INTRODUCTION

Let M be a differentiable manifold of finite dimension n which we are going to assume to be connected and of class C^{∞}. It is known that a differential system on M is a G-structure, B_G, where G is the group of all linear transformations leaving invariant a subspace V_1 of a n-dimensional vector space V. The Lie algebra of G is identified with a subspace of $\mathrm{Hom}(V, V)$. Such G-structure is interpreted as the set of all the adapted frames to a differential distribution, which, usually, is determined by a system in total differentials of Pfaffian system. The integral submanifolds of this distribution constitute the dynamical system of the G-structure.

Studying such G-structures the first situation that cannot be completely analyzed on the basis of Frobenius' and Darboux's theorems occurs when V is 5-dimensional and V_1 is 2 or 3-dimensional, even though it is known that the problem of classifying generic 3-dimensional and 2-dimensional differential systems on a five manifold are totally equivalent, when completely integrable systems and flag systems are excluded.

Then, let us consider a Pfaffian system of constant rank 3 on a 5-dimensional manifold, i.e., $\dim V = 5$, $\dim V_1 = 2$. In this case the group G is the group of all non-singular linear transformations of V leaving the subspace V_1 invariant. If we choose a basis $\{e_1, e_2, e_3, e_4, e_5\}$ of V such that $\{e_4, e_5\}$ span V_1 then G can be identified with the subgroup of the general linear group, $GL(5)$, of the matrices of

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the form

$$\left[\begin{array}{ccc|cc} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ \hline * & * & * & * & * \\ * & * & * & * & * \end{array}\right]$$

i.e., with zeros in the upper right-hand block and arbitrary entries elsewhere. This group is 19-dimensional and we will call it G_{19}.

By means of the effective application of the reduction technique for G-structures designed by Sternberg, we prove the following result:

Theorem. *Let $B_{G_{19}}$ be the G_{19}-structure associated to a Pfaffian System S, of constant rank 3 on a 5-dimensional manifold. If S is neither completely integrable nor a flag system, then:*

(i) *$B_{G_{19}}$ is reduced to a G-structure $B_{G_{16}}$, which has 16-dimensional group structure.*

(ii) *$B_{G_{16}}$ is reduced to a G-structure $B_{G_{12}}$, which has 12-dimensional group structure.*

(iii) *No reduction is obtained on $B_{G_{12}}$ by reiterating the Sternberg's procedure.*

Remark. The calculations that carry to the reduction in (ii) do not agree with those indicated by Sternberg.

STERNBERG'S STRUCTURE FUNCTION

In order to apply this technique, let us briefly describe the (first-order) structure function of a G-structure B_G, which we will call c (for details see [S. 1964]).

Let us first consider the differential form defined on the bundle of frames $\mathfrak{F}(M)$ and V-valued given by $\omega(X) = p^{-1}(\pi_*(X))$ for each $p \in \mathfrak{F}(M)$ and each $X \in T_p(\mathfrak{F}(M))$, with $\pi(p) = x$. We can restrict ω so as to obtain a V-valued form on B_G, which we will continue to call ω. As ω gives an isomorphism of any horizontal subspace H at p onto V, identifying $\Lambda^2(H)$ with $\Lambda^2(V)$ we obtain a map

$$c_H : \Lambda^2(V) \to V$$

defined as follows:

$$c_H(u \wedge v) = d\omega(X \wedge Y) \qquad \forall u, v \in V$$

where X and Y are the elements of H such that $\omega(X) = u$, $\omega(Y) = v$.

If we denote by ϱ the projection of $\mathrm{Hom}(V \wedge V, V)$ onto the quotient space

$$\mathrm{Hom}(V \wedge V, V)/\mathscr{A}(\mathrm{Hom}(V, \mathfrak{g})),$$

we have a well defined function, c, on B_G

$$c\colon B_G \to \mathrm{Hom}(V \wedge V, V)/\mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}))$$

given by

$$c(p) = \varrho(c_H)$$

which does not depend on the choise of the horizontal subspace H at p.

This structure function is very useful for studying the equivalence of G-structures. In fact, we have the next result ([S. 1964]): *Let B_G^1 and B_G^2 be G-structures over M_1 and M_2. If φ is an isomorphism of M_1 onto M_2, then $c^2 \circ \varphi_* = c^1$, where c^1 is the structure function of B_G^1 and c^2 the structure function of B_G^2*; where by an isomorphism we mean a diffeomorphism $\varphi\colon M_1 \to M_2$ such that $\varphi_*(B_G^1) = B_G^2$.

The practical procedure of using this result is based on the searching of the isotropy group of a certain action of G on $\mathrm{Hom}(V \wedge V, V)/\mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}))$. Specifically, the action induced by

$$\sigma(a)L(u \wedge v) = aL(a^{-1}u \wedge a^{-1}v) \tag{1}$$

for each $a \in G$ and each $L \in \mathrm{Hom}(V \wedge V, V)$.

If the orbits of this action are of maximal dimension, discrete, and finite in number, it has sense to restrict the local equivalence problem for G-structures to *generic* points; i.e., those $x \in M$ such that $c(p) = z$ lies in an orbit of maximal dimension.

Let z be some fixed point of this orbit and let G_1 be the isotropy subgroup of z. By using this method we reduce the G-structure equivalence problem to a G_1-structure equivalence problem. This G_1-structure can be reduced itself by means of its own structure function and the procedure can be reiterated until the corresponding structure function is constant.

In order to prove the Theorem, let us now apply this reduction procedure to the case of differential systems. Let $x \in M$ be a fixed generic point, $p \in B_{G_{19}}$, with $\pi(p) = x$, and $z = c^{19}(p) \in \mathrm{Hom}(V \wedge V, V)/\mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}_{19}))$, where c^{19} denotes the Sternberg's structure function on $B_{G_{19}}$. We look for the isotropy group of z, that is, the elements $a \in G_{19}$ such that

$$az(a^{-1}u \wedge a^{-1}v) = z(u \wedge v) \qquad \forall u, v \in V$$

or equivalently, the elements $a \in G$ such that

$$z(au \wedge av) = az(u \wedge v) \qquad \forall u, v \in V. \tag{2}$$

Let us consider a horizontal subspace H_p at p. Since $\omega_p : H_p \longrightarrow V$ is an isomorphism, for each u in V there exists an unique vector X_p^u of H_p, such that $\omega_p(X_p^u) = u$. We consider a connection $\mathscr{H} = \{H_q / q \in B_{G_{19}}\}$ such that the horizontal subspace at p is the initial H_p, for having horizontal differentiable vector fields in a neighborhood of p. (Remark: it is not necessary to take a connection; it is sufficient to consider a family $\mathscr{H}$ of horizontal subspaces). When we fix a basis $\{e_4, e_5\}$ of V_1 we get $X = X^{e_4}$, $Y = X^{e_5}$ horizontal vector fields such that $\omega_q(X_q) = e_4$ and $\omega_q(Y_q) = e_5$ for each $q \in B_{G_{19}}$. Then

$$\begin{gathered} c_{H_q}^{19}(q)(e_4 \wedge e_5) = d\omega_q(X \wedge Y) \\ = X_q(\omega(Y)) - Y_q(\omega(X)) - \omega_q([X, Y]_q) = -\omega_q([X, Y]_q). \end{gathered}$$

Let $e_3 = -c_{H_p}^{19}(p)(e_4 \wedge e_5) = \omega_q([X, Y]_q)$.

Excluding the integrable differential systems, $\{e_3, e_4, e_5\}$ are linearly independent. Therefore X, Y and the horizontal projection of $[X, Y]$ are linearly independent in a neighborhood of p. In general, $[X, Y]_q$ will not be a vector of H_q, but changing, if necessary, the horizontal distribution, we can get subspaces H_q such that $X_q, Y_q, [X, Y]_q \in H_q$ for each q in a neighborhood of p. (This horizontal distribution may not verify the connection relations: $R_{a*}(H_q) = H_{qa}$, for all $a \in G_{19}$. A discussion of conditions on this subspaces to be a connection can be found in [S. 1964]).

Let G' be the subgroup of isotropy of z, and let $B_{G'}$ be the G'-structure consisting of those q in $B_{G_{19}}$ such that $c(q) = z$. Since

$$c_{H_q}^{19}(q)(e_4 \wedge e_5) = c_{H_p}^{19}(e_4 \wedge e_5) = -e_3 \qquad \forall q \in B_{G'} \subset B_{G_{19}}$$

the horizontal vector field $[X, Y] = X^{e_3}$ is a horizontal lift, i.e., for each q in a neighborhood of p $[X, Y]_q = X_q^{e_3}$.

By using ω_p, we can define for $u, v \in V$, the bracket

$$[u, v] = \omega_p([X^u, X^v]_p).$$

This bracket depends on the choice of the horizontal subspaces. However this depedence does no affect the isotropy of z because, if we change the subspaces, the corresponding brackets differ in an element of $\mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}_{19}))$. So equation (2) is equivalent to

$$(2') \qquad [au, av] = a[u, v] + \theta(u, v) \qquad \forall u, v \in V$$

with $\theta \in \mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}_{19}))$. To characterize the elements of $\mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}_{19}))$ let us call θ_{ij}^k the coordinates of θ with respect to a natural basis of $\mathrm{Hom}(V \wedge V, V)$. We have $\theta_{ij}^k = -\theta_{ji}^k$. Furthemore

$$\varrho(\theta) = 0 \quad \text{iff} \quad \theta_{ij}^k = a_{ij}^k - a_{ji}^k$$

with $A_i = (a_{ij}^k) \in \mathfrak{g}_{19}$, $i = 1, 2, 3, 4, 5$. Since given $A_i = (a_{ij}^k) \in \mathfrak{g}_{19}$,

$$a_{ij}^k = 0 \quad \text{if} \quad j = 4, 5, \ k = 1, 2, 3, \ i = 1, 2, 3, 4, 5$$

we obtain

$$\theta_{45}^k = \theta_{54}^k = 0 \qquad k = 1, 2, 3$$

and hence if $\varrho(c_{H_1}) = z_1$, $\varrho(c_{H_2}) = z_2$

$$z_1 = z_2 \quad \text{iff} \quad (c_{H_1})_{54}^k = (c_{H_1})_{45}^k = (c_{H_2})_{45}^k = (c_{H_2})_{54}^k, \qquad \text{for } k = 1, 2, 3$$

or, equivalently

$$z_1 = z_2 \quad \text{iff} \quad z_1(e_4 \wedge e_5) + V_1 = z_2(e_4 \wedge e_5) + V_1$$

and therefore (2′) is equivalent to

$$(2'') \qquad [ae_4, ae_5] + V_1 = ae_3 + V_1.$$

Suppose

$$a = \left[\begin{array}{ccc|cc} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ \hline * & * & * & \alpha & \beta \\ * & * & * & \gamma & \delta \end{array}\right]$$

and let $\Delta = \alpha\delta - \beta\gamma$. Then

$$[ae_4, ae_5] = [\alpha e_4 + \gamma e_5, \beta e_4 + \delta e_5] = \Delta[e_4, e_5] = \Delta e_3.$$

Thus $a \in G_{19}$ is an element of the group of isotropy of z if and only if a is of the form

$$a = \left[\begin{array}{ccc|cc} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \Delta & 0 & 0 \\ \hline * & * & * & \alpha & \beta \\ * & * & * & \gamma & \delta \end{array}\right]$$

where $\Delta = \alpha\delta - \beta\gamma$.

Therefore we have reduced the G_{19}-structure to a structure $B_{G_{16}}$ whose structure group, G_{16}, is 16-dimensional.

We have to reiterate the procedure on this new G-structure. Let now $z = c^{16}(p)$. In the same way that before we consider

$$c_{H_p}(e_4 \wedge e_5) = -e_3, \quad c_{H_p}(e_3 \wedge e_4) = -e_1, \quad c_{H_p}(e_3 \wedge e_5) = -e_2$$

i.e., in a neighborhood of p

$$[X^{e_4}, X^{e_5}] = X^{e_3}, \quad [X^{e_3}, X^{e_4}] = X^{e_1}, \quad [X^{e_3}, X^{e_5}] = X^{e_2}.$$

Identifying by ω_p we have

$$[e_4, e_5] = e_3, \quad [e_3, e_4] = e_1, \quad [e_3, e_5] = e_2$$

where $\{e_1, e_2, e_3, e_4, e_5\}$ is a basis of V in the general case that the system is neither completely integrable nor a flag system.

Now we look for those elements $a \in G_{16}$ such that

$$(3) \qquad [au, av] = a[u, v] + \theta(u, v) \qquad \forall u, v \in V$$

where $\theta \in \mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}_{16}))$. Since $\mathfrak{g}_{16}$ is the algebra of the matrices of the form

$$A = \left[\begin{array}{ccc|cc} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & a_1 + a_4 & 0 & 0 \\ \hline * & * & * & a_1 & a_2 \\ * & * & * & a_3 & a_4 \end{array}\right]$$

given $A_i = (a^k_{ij}) \in \mathfrak{g}_{16}$, $i = 1,2,3,4,5$, we have

$$\begin{array}{lllll} a^k_{ij} = 0 & \text{if } j = 4,5, & k = 1,2,3, & i = 1,2,3,4,5, \\ a^k_{ij} = 0 & \text{if } j = 3, & k = 1,2, & i = 1,2,3,4,5, \\ a^3_{i3} = a^4_{i4} + a^5_{i5} & \text{if} & & i = 1,2,3,4,5, \end{array}$$

Hence we deduce for $\theta \in \mathrm{Hom}(V \wedge V, V)$

$$\varrho(\theta) = 0 \quad \text{iff} \quad \theta^k_{ij} = a^k_{ij} - a^k_{ji}, \qquad (a^k_{ij}) \in \mathfrak{g}_{16}$$

i.e., if and only if

$$\begin{aligned} \theta^k_{45} = \theta^k_{54} = 0 \qquad & k = 1,2,3, \\ \theta^k_{34} = \theta^k_{43} = 0 \qquad & k = 1,2, \\ \theta^k_{35} = \theta^k_{53} = 0 \qquad & k = 1,2. \end{aligned}$$

Therefore, given two horizontal subspaces at p such that $\varrho(c_{H_1}) = z_1$, $\varrho(C_{H_2}) = z_2$, $z_1 = z_2$ if and only if

$$\begin{aligned} (c_{H_1})^k_{54} = (c_{H_1})^k_{45} = (c_{H_2})^k_{45} = (c_{H_2})^k_{54} \qquad & k = 1,2,3, \\ (c_{H_1})^k_{43} = (c_{H_1})^k_{34} = (c_{H_2})^k_{34} = (c_{H_2})^k_{43} \qquad & k = 1,2, \\ (c_{H_1})^k_{53} = (c_{H_1})^k_{35} = (c_{H_2})^k_{35} = (c_{H_2})^k_{53} \qquad & k = 1,2. \end{aligned}$$

That is, $z_1 = z_2$ iff

$$\begin{aligned} z_1(e_4 \wedge e_5) + V_1 &= z_2(e_4 \wedge e_5) + V_1, \\ z_1(e_3 \wedge e_4) + V_2 &= z_2(e_3 \wedge e_4) + V_2, \\ z_1(e_3 \wedge e_5) + V_2 &= z_2(e_3 \wedge e_5) + V_2, \end{aligned}$$

where $V_2 = \mathrm{span}\{e_3, e_4, e_5\}$. Therefore, given $a \in G_{16}$

$$a = \begin{bmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \Delta & 0 & 0 \\ * & * & \varepsilon & \alpha & \beta \\ * & * & \mu & \gamma & \delta \end{bmatrix}$$

where $\Delta = \alpha\delta - \beta\gamma$, since

$$\begin{aligned} [ae_4, ae_5] &= \Delta e_3, \\ [ae_3, ae_4] &= \Delta\alpha e_1 + \Delta\gamma e_2 + (\varepsilon\gamma - \alpha\mu)e_3, \\ [ae_3, ae_5] &= \Delta\beta e_1 + \Delta\delta e_2 + (\varepsilon\delta - \beta\mu)e_3, \end{aligned}$$

we deduce that the isotropy subgroup of z is the 12-dimensional group which we will call G_{12}, consisting of all the matrices of the form

$$\left[\begin{array}{cc|c|cc} \Delta\alpha & \Delta\beta & 0 & 0 & 0 \\ \Delta\gamma & \Delta\delta & 0 & 0 & 0 \\ \hline * & * & \Delta & 0 & 0 \\ \cline{3-5} * & * & * & \alpha & \beta \\ * & * & * & \gamma & \delta \end{array}\right]$$

with $\Delta = \alpha\delta - \beta\gamma$.

The Lie algebra of this group is

$$\mathfrak{g}_{12}: \left[\begin{array}{cc|c|cc} 2a_1 + a_4 & a_2 & 0 & 0 & 0 \\ a_3 & a_1 + 2a_4 & 0 & 0 & 0 \\ \hline * & * & a_1 + a_4 & 0 & 0 \\ \cline{3-5} * & * & * & a_1 & a_2 \\ * & * & * & a_3 & a_4 \end{array}\right]$$

For applying again the same procedure on this G_{12}-structure, $B_{G_{12}}$, we must consider the projection

$$\varrho\colon \operatorname{Hom}(V \wedge V, V) \to \operatorname{Hom}(V \wedge V, V)/\mathscr{A}(\operatorname{Hom}(V, \mathfrak{g}_{12}))$$

and characterize previously the elements of $\mathscr{A}(\operatorname{Hom}(V, \mathfrak{g}_{12}))$.

Given $\theta \in \operatorname{Hom}(V \wedge V, V)$, $\varrho(\theta) = 0$ iff $\theta^k_{ij} = a^k_{ij} - a^k_{ji}$, $(a^k_{ij}) \in \mathfrak{g}_{12}$.

In this case the conditions on the components of the matrices of $\mathfrak{g}_{12}$ are:

$$\begin{array}{lll} a^k_{ij} = 0 & \text{if } j = 4,5,\ k = 1,2,3, & i = 1,2,3,4,5, \\ a^k_{ij} = 0 & \text{if } j = 3,\quad k = 1,2, & i = 1,2,3,4,5, \\ a^3_{i3} = a^4_{i4} + a^5_{i5} & \text{if} & i = 1,2,3,4,5, \\ a^1_{i2} = a^4_{i5} & \text{if} & i = 1,2,3,4,5, \\ a^2_{i1} = a^5_{i4} & \text{if} & i = 1,2,3,4,5, \\ a^1_{i1} = 2a^4_{i4} + a^5_{i5} & \text{if} & i = 1,2,3,4,5, \\ a^2_{i2} = a^4_{i4} + 2a^5_{i5} & \text{if} & i = 1,2,3,4,5, \end{array}$$

So, in terms of the coordinates of θ we get:

$$(4) \qquad \begin{cases} \theta^k_{ji} = -\theta^k_{ji} & \text{if } i = 1,2,3,4,5,\ j = 1,2,3,4,5,\ k = 1,2,3,4,5, \\ \theta^k_{45} = 0 & k = 1,2,3, \\ \theta^k_{34} = \theta^k_{35} = 0 & k = 1,2, \end{cases}$$

$$
(5) \qquad \begin{cases}
\theta^1_{12} = a^4_{15} - 2a^4_{24} - a^5_{25}, \\
\theta^2_{12} = a^4_{14} + 2a^5_{15} - a^5_{24}, \\
\theta^1_{13} = -2a^4_{34} - a^5_{35}, \\
\theta^2_{13} = -a^5_{34}, \\
\theta^1_{14} = -2a^4_{44} - a^5_{45}, \\
\theta^2_{14} = -a^5_{44}, \\
\theta^1_{15} = -2a^4_{54} - a^5_{55}, \\
\theta^2_{15} = -a^5_{54}, \\
\theta^1_{23} = -a^4_{35}, \\
\theta^2_{23} = -a^4_{34} - 2a^5_{35}, \\
\theta^1_{24} = -a^4_{45}, \\
\theta^2_{24} = -a^4_{44} - 2a^5_{45}, \\
\theta^1_{25} = -a^4_{55}, \\
\theta^2_{25} = -a^4_{54} - 2a^5_{55}.
\end{cases}
$$

Conditions (4) are the same that those ones we have when we consider $\mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}_{16}))$. From (5) the unique relations that we obtain among the components of θ are:

$$
(6) \qquad \begin{cases}
\theta^1_{14} + \theta^2_{24} = -3a^4_{44} - 3a^5_{45} - 3a^3_{43}, \\
\theta^1_{15} + \theta^2_{25} = -3a^4_{54} - 3a^5_{55} - 3a^3_{53}.
\end{cases}
$$

Since a^3_{i3} is the trace of the submatrix $\begin{pmatrix} a^4_{i4} & a^4_{i5} \\ a^5_{i4} & a^5_{i5} \end{pmatrix}$ which is arbitrary in the elements of $\mathfrak{g}_{12}$, the relations (6) do not impose any new condition on θ, and hence its components only must verify (4).

When we consider the structure function

$$
c^{12}\colon B_{G_{12}} \to \mathrm{Hom}(V \wedge V, V)/\mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}_{12}))
$$

for trying to reduce the G_{12}-structure, $B_{G_{12}}$, we must search the elements $a \in G_{12}$ such that there exists $\theta \in \mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}_{12}))$ verifying for every $u, v \in V$

$$
(7) \qquad [au, av] = a[u, v] + \theta(u, v).
$$

Since $B_{G_{12}}$ is the reduction of $B_{G_{16}}$, every element of G_{12} verifies the relation (3), similar to (7), but with a rest in $\mathscr{A}(\mathrm{Hom}(V, \mathfrak{g}_{16}))$, so they verify (7) because we are not imposing any new condition on θ. Hence the isotropy of any value $z = c(p)$, with $\pi(p) = x$, where x is a generic point of M, is all the group G_{12}, and therefore using this procedure no reduction is obtained on structure $B_{G_{12}}$.

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