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A NOTE IN INVERSE AND DUAL SEMIGROUPS

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In studying the structure of inverse semigroups, L. Márkl [2] has proved the following conditions on semigroups with 0 are equivalent:

(i) S is an inverse semigroup and the union of a finite number of its 0-minimal left (right) ideals.

(ii) S is the union of a finite number of its quasi-ideals, and all these quasi-ideals form a special complete system.

(iii) S is an inverse semigroup and the union of a finite number of its 0-minimal quasi-ideals.

(iv) S is an inverse semigroup with finitely many idempotents and every non-zero idempotent is primitive.

However, it has been noticed by him that the following condition:

(v) S is a 0-direct union of finitely many two-sided ideals which are completely 0-simple inverse subsemigroups of S.

is weaker than any one of the conditions (i) to (iv).

In this note, we observe that if each summand in (v) satisfies min-r (that is, Minimum condition on right ideals), then the above conditions are in fact all equaivalent. We shall prove that any one of these conditions is a necessary and sufficiency condition for a semigroup to be semisimple and dual. Thus a characterization theorem for semisimple dual semigroup is obtained. Throughout the paper, every semigroup has 0 and contains more than one element. The reader is referred to O. Steinfeld [7] for all terminology and definitions not given here.

Definition 1. Let A be a subset of a semigroup S. Denote

$$r(A) = \{x \in S \mid Ax = 0\}$$

and

$$\ell(A) = \{x \in S \mid xA = 0\}.$$

A semigroup S is called *dual semigroup* if $\ell r(L) = L$ for every left ideal L in S and $r\ell(R) = R$ for every right ideal R in S.

Definition 2. A semigroup S with min-r is said to be *semisimple* if the radical of S is zero. In other words, a semigroup S is semisimple if S satisfies the descending chain condition on its right ideals and contains no non-trivial nilpotent ideals.

Unlike in ring theory, a semisimple semigroup need not be a dual semigroup. The following is an example.

Example 3. Let $S = \{0, a, b, c, d\}$ with Cayley table

·	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
d	0	a	b	c	d

Then S is a semisimple semigroup. Since $\{0, a, c\}$ is a right ideal of S such that $r\ell\{0, a, c\} = S$, S is not a dual semigroup.

Lemma 4. *Let S be a dual semigroup. Then S satisfies* min-r *(*min-ℓ; max-r; max-ℓ*) if and only if S satisfies* max-ℓ *(*max-r; min-ℓ; min-r*).*

P r o o f ($\Rightarrow$). Consider the ascending chain of left ideals of S: $L_1 \subseteq L_2 \subseteq \ldots$. Then $r(L_1) \supseteq r(L_2) \subseteq \ldots$ is a descending chain of right ideals of S. By min-r, this chain will be terminated, say, at the nth step. Thus, there exists a positive integer n such that $r(L_n) = r(L_m)$ for all $m \geqslant n$. Apply the duality of S, we have $L_n = \ell r(L_n) = \ell r(L_m) = L_m$ for all $m \geqslant n$. This implies that S satisfies max-ℓ.
($\Leftarrow$). Follows dually as the "only if" part. $\qquad\square$

Lemma 5. *If a semigroup is a finite 0-direct union of 0-minimal right (left) ideals of S, then S satisfies both* min-r *and* max-r *(*min-ℓ *and* max-ℓ*).*

P r o o f. By assumption, $S = \bigcup_{i=1}^{n} R_i$, where each R_i is a 0-minimal right ideal of S. Let R be an arbitrary non-zero right ideal of S. Then $R \subseteq \bigcup_{i=1}^{k} R_{n_i}$, $\{n_i \mid i = 1, 2, \ldots, k\} \subseteq \{1, 2, \ldots, n\}$. Clearly $R \cap R_{n_i} \neq \{0\}$ for all $i = 1, 2, \ldots, k$. Since each R_{n_i} is 0-minimal, so $R_{n_i} = R \cap R_{n_i} \subseteq R$ for all $i = 1, 2, \ldots, k$. Therefore $\bigcup_{i=1}^{k} R_{n_i} \subseteq R$. Consequently $R = \bigcup_{i=1}^{k} R_{n_i}$. This shows that every right ideal of S is also a finite 0-direct union of 0-minimal right ideals of S. This implies that S satisfies both min-r and max-r. $\qquad\square$

Lemma 6. *Let S be a dual semigroup with zero radical. Then the following conditions are equaivalent:*

(i) $\overline{\overline{E}} < \infty$.

(ii) S *satisfies* max-r

(iii) S *satisfies* min-r

(iv) S *satisfies* max-ℓ

(v) S *satisfies* min-ℓ.

P r o o f . It was proved by St. Schwarz [3] that S is a 0-direct union of eS for all $e^2 = e \in S$, that is, $S = \bigcup\limits_{e \in E} eS$. It can by seen that each eS is a 0-minimal right ideal of S. In virtue of Schwarz's theorem and Lemma 5, we obtain (i) implies (ii) to (v). In view of Lemma 4, it remains to prove (ii) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i). For (ii) $\Rightarrow$ (i), we suppose that the set E is not finite. Then there exists a countable set of idempotents $\{e_i \mid i \in N\} \subseteq S$. Write $S_n = \bigcup\limits_{i=1}^{n} e_i S$. If $S_n = S_{n+1}$, then $\bigcup\limits_{i=1}^{n} e_i S = \bigcup\limits_{i=1}^{n+1} e_i S \supseteq e_{n+1} S$. This implies that $e_{n+1} S \cap e_k S \neq \{0\}$ for some k, which contradicts that the union of $e_i S$'s are 0-directed. Thus $S_n \subsetneqq S_{n+1}$. However, this result contradicts max-r. Consequently E must be a finite set. Thus (ii) $\Rightarrow$ (i). Similarly, (iv) $\Rightarrow$ (i). $\qquad\square$

Theorem 7. *The following conditions on a semigroups with 0 are equivalent:*

(i) S *is a semisimple dual semigroup.*

(ii) S *is a 0-direct union of finitely many two-sided ideals which are 0-simple dual subsemigroups of* S *with* min-r.

(iii) S *is a 0-direct union of finitely many two-sided ideals which are completely 0-simple inverse subsemigroups of* S *with* min-r.

(iv) S *is a regular semigroup with finitely many orthogonal idempotents.*

(v) S *is an inverse semigroup with finitely many idempotents and every non-zero idempotent of* S *is primitive.*

(vi) S *is an inverse semigroup and can be expressed as the union of a finite number of its quasi-ideals, and these quasi-ideals form a special complete system.*

(vii) S *is an inverse semigroup and can be expressed as the union of a finite number of its 0-minimal left (right) ideals.*

(viii) S *is an inverse semigroup and can be expressed as the union of a finite nunber of its 0-minimal quasi-ideals.*

P r o o f . In virtue of the characterization theorem for dual semigroups with zero radicals due to O. Steinfeld in [6] and Lemma 6, we have (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). By Corollary 10.13 in [7], conditions (v) to (viii) are all equivalent. It remains to show that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). By the Corollary stated in [5], we obtain that (ii) $\Leftrightarrow$ (iii). For (i) $\Leftrightarrow$ (ii), we let S be a semisimple dual semigroup. By Schwarz's decomposition theorem in [4], S is a 0-direct union of two-sided ideals $\{I_i \mid i \in \Gamma\}$ which are 0-simple dual subsemigroups of S. Since every 0-simple dual semigroup contains at

least one non-zero idempotent [5], so $E = \bigcup_{i \in \Gamma} (E \cap I_i)$. As $\overline{\overline{E}} < \infty$, by Lemma 6, Γ and $E \cap I_i$ are finite sets for all i. Hence all I_i's satisfy min-r by Lemma 6. Thus (i) $\Rightarrow$ (ii). Conversely, let S be a 0-direct union of finitely many two-sided ideals $\{I_1, I_2, \ldots, I_n\}$ which are 0-simple dual subsemigroups of S with min-r. Then by the converse of Schwarz's decomposition theorem proved in [1], S must be a dual semigroup with zero radical. Moreover, for $i = 1, 2, \ldots, n$, $E \cap I_i$ is a finite set since each I_i is a semisimple dual subsemigroup of S. Thus $E = \bigcup_{i=1}^{n} (E \cap I_i)$ is a finite set. By Lemma 6, S is a semisimple dual semigroup. Thus (ii) $\Rightarrow$ (i). Our proof is completed. $\qquad\qquad\square$

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