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COVARIANT CONSTRUCTIONS IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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Consider a linear differential homogeneous equation of the n-th order, $n \geqq 2$, of the form

(1) $$y^{(n)} + p_{n-2}(x)\, y^{(n-2)} + p_{n-3}(x)\, y^{(n-3)} + \ldots + p_0(x)\, y = 0 \quad \text{on} \quad I,$$

or simply $P_n(y, x; I) = 0$, where $I \subset \mathbb{R}$ is an open interval and $p_i \in C^0(I)$, $i = 1, \ldots$ $\ldots, n-2$, are real functions. Moreover, we shall suppose $p_{n-2} \in C^{n-2}(I)$. For a fixed integer n, $n \geqq 2$, let D_n denote the set of all n-th order linear differential equations of the type (1).

It is known [7] that the most general pointwise transformation that globally transforms all solutions of each equation $P_n(y, x; I) = 0$ from D_n into all solutions of an equation $Q_n(z, t; J) = 0$ from D_n, i.e., into

$$z^{(n)} + q_{n-2}(t)\, z^{(n-2)} + q_{n-3}(t)\, z^{(n-3)} + \ldots + q_0(t)\, z = 0 \quad \text{on} \quad J\,,$$

is

(2) $$z(t) = c\left|\mathrm{d}h(t)/\mathrm{d}t\right|^{(1-n)/2} y(h(t))\,,$$

where

(3) $$h \in C^{n+1}(J)\,, \quad \mathrm{d}h(t)/\mathrm{d}t \neq 0 \text{ on } J\,, \quad h(J) = I\,,$$

and $c \neq 0$ is a real constant.

To express that an equation $P_n(y, x; I) = 0$ is globally transformed into $Q_n(z, t; J) = 0$ in the sense of the relation (2), we shall simply write $h(P_n) = Q_n$.

Denote by A the set of all real functions $f\colon \mathbb{R} \to \mathbb{R}$ being expressible in a power series (centred at zero) that converges on the whole $\mathbb{R}$. Let AD_n denote all differential equations from D_n whose coefficients are in A.

We shall consider mappings F_n of AD_2 into AD_n constructed in the following way: for each integer n, $n \geqq 2$, there are n functions

F_{ni}: $\mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$, $i = 1, \ldots, n$, such that for each couple of linearly independent solutions u_1, u_2 of an equation from AD_2,

$$(p) \qquad u'' + p(x)\, u = 0\,, \quad p \in A\,,$$

the n functions $x \mapsto F_{ni}(u_1(x), u_2(x))$, $x \in \mathbb{R}$, form an n-tuple of linearly independent solutions of an equation from AD_n. Moreover, we require this n-th order equation from AD_n to depend only on the original equation (p) and not on the choice of its solutions u_1 and u_2. The n-th order equation from AD_n constructed in this way will be denoted by $F_n(p)$.

The aim of this paper is to study the mappings, or constructions F_n satisfying

$$(4) \qquad F_n\, h(p) = h\, F_n(p)$$

for all $p \in A$ and each h whenever $h(p)$ is defined.

We shall prove that the commutativity condition (4) characterizes the construction F_n, and $F_n(p)$ is the so-called iterative equation to (p). It occurs that this F_n is the only construction covariant with respect to transformations.

The category of linear differential equations as objects and their global transformations as morphisms was introduced in [6]. For covariant functors in the theory of categories, see e.g. [5].

Theorem. *Let n be a fixed integer, $n \geqq 2$, and let F_n: $AD_2 \to AD_n$ be a mapping satisfying (4). Then F_n is uniquely determined and is described by the following construction.*

If u_1 and u_2 denote two linearly independent solutions of (p), then

$$(5) \qquad y_i(x) = u_1^{n-i}(x) \,.\, u_2^{i-1}(x)\,, \quad i = 1, \ldots, n\,,$$

are n linearly independent solutions of the equation $F_n(p) \in AD_n$. The equation $F_n(p)$ is well-defined, i.e., it does not depend on the particular choice of solutions u_1 and u_2 of (p).

Moreover, the mapping F_n given by (5) can be extended to $\tilde{F}_n$ defined on the subset of the second order equations from D_2 with coefficients of class $C^{n-2}(I)$, $I \subset \mathbb{R}$, and this extension is an injection to D_n.

Proof. Consider an equation (p), $p \in A$, and its linearly independent solutions u_1 and u_2. Evidently $u_1 \in A$, $u_2 \in A$, and the Wroński determinant $\mathrm{W}(u_1, u_2) = k = \text{const.} \neq 0$. Choose arbitrary $h \in A$ satisfying (3) for $I = J = \mathbb{R}$. Denote by $(q) := h(p)$ the equation obtained from equation (p) by means of the transformation h, i.e.

$$(q) \qquad v'' + q(t)\, v = 0\,, \quad q \in A\,.$$

Due to (2), see also [2, Chap. 11] or [4, Chap. 7],

$$v_i(t) = |\mathrm{d}h(t)/\mathrm{d}t|^{-1/2}\, u_i(h(t))\,; \quad i = 1, 2\,; \quad t \in \mathbb{R}$$

are two linearly independent solutions of (q).

Let a mapping F_n satisfy the assumptions of the theorem. Consider the linear differential equations of the n-th order,

$$F_n(p) \quad \text{and} \quad F_n(q)\,.$$

Hence $y_i(x) = F_{ni}(u_1(x), u_2(x))$, $i = 1, \ldots, n$, are linearly independent solutions of $F_n(p)$. Due to (4), $h\,F_n(p)$ is defined because $h(p) = (q)$ exists. In fact, the same change $x \mapsto h(t)$ occurs in both of the transformations of the equation (p) to (q) and the equation $F_n(p)$ to $F_n(q)$, since the mapping F_n is a pointwise transformation and the independent variable, x, is not changed when mapping (p) to $F_n(p)$. The equation $h\,F_n(p)$ coincides with the equation $F_n(q)$, as follows from (4). Moreover, due to (2),

$$z_i(t) := \left|\mathrm{d}h(t)/\mathrm{d}t\right|^{(1-n)/2}\, y_i(h(t))\,, \quad i = 1, \ldots, n\,,$$

are linearly independent solutions of $F_n(q)$.

At the same time, $F_n(q)$ is $F_n(h(p))$, i.e.,

$$F_{ni}(v_1(t), v_2(t)) = F_{ni}(\left|h'(t)\right|^{-1/2}\, u_1(h(t)), \left|h'(t)\right|^{-1/2}\, u_2(h(t)))\,, \quad i = 1, \ldots, n\,,$$

are n linearly independent solutions of $F_n(q)$. Hence

$$\begin{pmatrix} z_1(t) \\ \ldots \\ z_n(t) \end{pmatrix} = \left|\mathrm{d}h(t)/\mathrm{d}t\right|^{(1-n)/2} . \begin{pmatrix} F_{n1}(u_1(h(t)),\, u_2(h(t))) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ F_{nn}(u_1(h(t)),\, u_2(h(t))) \end{pmatrix} =$$

$$= \mathbf{C} . \begin{pmatrix} F_{n1}(\left|\mathrm{d}h(t)/\mathrm{d}t\right|^{-1/2}\, u_1(h(t)),\ \left|\mathrm{d}h(t)/\mathrm{d}t\right|^{-1/2}\, u_2(h(t))) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_{nn}(\left|\mathrm{d}h(t)/\mathrm{d}t\right|^{-1/2}\, u_1(h(t)),\ \left|\mathrm{d}h(t)/\mathrm{d}t\right|^{-1/2}\, u_2(h(t))) \end{pmatrix}$$

for a unique nonsingular constant n by n matrix $\mathbf{C}$. Since (p), u_1, u_2, and h ($h' \neq 0$, $u_1^2 + u_2^2 > 0$) were arbitrarily chosen, the last relation for $\left|\mathrm{d}h(t)/\mathrm{d}t\right|^{-1/2} =: a$, $u_1(h(t)) =: r$, and $u_2(h(t)) =: s$ reads

$$\mathbf{C} . \begin{pmatrix} F_{n1}(ar, as) \\ \ldots\ldots \\ F_{nn}(ar, as) \end{pmatrix} = a^{n-1} \begin{pmatrix} F_{n1}(r, s) \\ \ldots\ldots \\ F_{nn}(r, s) \end{pmatrix}$$

for $F_{ni}\colon \mathbb{R}^2 - \{(0, 0)\} \to \mathbb{R}$ and all $a > 0$. By specifying $a := 1$ we get

$$\mathbf{C} . \begin{pmatrix} F_{n1}(r, s) \\ \ldots\ldots \\ F_{nn}(r, s) \end{pmatrix} = \begin{pmatrix} F_{n1}(r, s) \\ \ldots\ldots \\ F_{nn}(r, s) \end{pmatrix} .$$

The last relation is satisfied for the unit matrix $\mathbf{I}$ instead of $\mathbf{C}$. Due to the uniqueness of $\mathbf{C}$, $\mathbf{C} = \mathbf{I}$. We have

$$(6) \qquad F_{ni}(ar, as) = a^{n-1}\, F_{ni}(r, s)\,, \quad i = 1, \ldots, n\,,$$

i.e., each F_{ni} is a homogeneous function in two variables of the order $n-1$. Following the method of "specification of variables", see J. Aczél [1], put $a := 1/r$ for $r > 0$. Then

$$F_{ni}(1, s/r) = r^{1-n} F_{ni}(r, s) \quad \text{or} \tag{7}$$

$$F_{ni}(r, s) = r^{n-1} F_{ni}(1, s/r) = r^{n-1} G_{ni}(s/r), \quad r > 0,$$

where G_{ni}: $\mathbb{R} \to \mathbb{R}$ are defined on the whole $\mathbb{R}$.

For a moment, let $p \equiv 0$ on $\mathbb{R}$ in (p). Chose $u_1(x) = 1$ and $u_2(x) = x$ for $x \in \mathbb{R}$. Evidently $p, u_1, u_2 \in A$. Since $F_n(p) \in AD_n$, we have

$$F_{ni}(1, x) \in A \quad \text{for} \quad i = 1, \ldots, n.$$

Thus

$$G_{ni}(x) = F_{ni}(1, x) \in A,$$

or

$$G_{ni}(x) = a_{ni0} + a_{ni1}x + a_{ni2}x^2 + \ldots = \sum_{j=0}^{\infty} a_{nij}x^j, \tag{8}$$

where

$$\limsup_{j \to \infty} |a_{nij}|^{1/j} = 0.$$

For the same equation (p), i.e. with $p \equiv 0$ on $\mathbb{R}$, change the order of its solutions u_1, u_2. Again $F_{ni}(x, 1)$, $i = 1, \ldots, n$, are linearly independent solutions of the equation $F_n(p)$ from AD_n, and hence

$$F_{ni}(x, 1) \in A, \quad i = 1, \ldots, n.$$

We have

$$F_{ni}(x, 1) = \sum_{j=0}^{\infty} b_{nij}x^j \tag{9}$$

with $\limsup_{j \to \infty} |b_{nij}|^{1/j} = 0$.

Due to (7), we can write

$$F_{ni}(x, 1) = x^{n-1} G_{ni}(1/x) \quad \text{for} \quad x > 0$$

or, by comparing (8) and (9),

$$\sum_{j=0}^{\infty} b_{nij}x^j = x^{n-1} \sum_{j=0}^{\infty} a_{nij}x^{-j} \quad \text{on} \quad \mathbb{R}_+. \tag{10}$$

Define

$$H(x) := \sum_{\nu=1}^{\infty} a_{ni,\nu+n-1}x^{-\nu} - \sum_{\nu=n}^{\infty} b_{ni\nu}x^{\nu} + (a_{ni,n-1} - b_{ni0}) +$$

$$+ (a_{ni,n-2} - b_{ni1})\, x + \ldots + (a_{ni1} - b_{ni,n-2})\, x^{n-2} + (a_{ni0} - b_{ni,n-1})\, x^{n-1}.$$

From (10) we have $H(x) = 0$ for all $x \in \mathbb{R}_+$. Due to the conditions on a_{nij} and b_{nij} in the relations (8) and (9), the complex function $H(z)$ of the complex variable z vanishes on C. Hence all coefficients in the expansion of $H(z)$ must be zeros.

From (8) we get

$$G_{ni}(x) = a_{ni0} + a_{ni1}x + \ldots + a_{ni,n-1}x^{n-1} \quad \text{on} \quad \mathbb{R}\,,$$

and due to (7),

$$F_{ni}(r, s) = \sum_{j=1}^{n} a_{ni,j-1} r^{n-j} s^{j-1} \quad \text{on} \quad \mathbb{R}^2 - \{0,0\}, \quad i = 1, \ldots, n\,.$$

These $F_{ni}(u_1(x), u_2(x))$, $i = 1, \ldots, n$, should be linearly independent. On the other hand, another linear combination with constant coefficients giving n linearly independent functions determines the same linear differential equation. Hence we may choose

$$F_{ni}(r, s) = r^{n-i} s^{i-1}\,, \quad i = 1, \ldots, n\,,$$

if we show that

(i) $u_1^{n-i}(x)\, u_2^{i-1}(x)$, $i = 1, \ldots, n$, are of class A with nonvanishing Wroński determinant on $\mathbb{R}$,

(ii) the n-th order linear differential equation having these n functions as solutions is of class AD_n,

(iii) the differential equation is the same if another pair of solutions of the equation (p) is taken.

Since $u_1, u_2 \in A$, we have $u_1^{n-i} u_2^{i-1} \in A$ for $i = 1, \ldots, n$. In view of $u_1 u_2' - u_1' u_2 = k = \text{const.} \neq 0$, we have the Wroński determinant

$$\begin{aligned}
& W(u_1^{n-1}, u_1^{n-2} u_2, \ldots, u_2^{n-1}) = \\
& = W(u_1^{n-1}\, 1, u_1^{n-1}(u_2/u_1), \ldots, u_1^{n-1}(u_2/u_1)^{n-1}) = \\
& = (u_1^{n-1})^n \left(\frac{\mathrm{d}}{\mathrm{d}t}(u_2/u_1)\right)^{n(n-1)/2} W(1, x, \ldots, x^{n-1}) = \\
& = u_1^{n(n-1)}((u_2' u_1 - u_2 u_1')\, u_1^{-2})^{n(n-1)/2}\, 0!\, 1! \ldots (n-1)! = \\
& = K = \text{constant} \neq 0, \text{ except at isolated zeros of the solution } u_1.
\end{aligned}$$

However, this Wroński determinant is at least of class $C^1(\mathbb{R})$, hence it is a nonzero constant on $\mathbb{R}$, and the condition (i) is verified.

Due to the fact that the Wroński determinant is a constant, the property (i) implies (ii).

For each $i \in 1, \ldots, n$ and constants $c_{11}, c_{12}, c_{21}, c_{22}$ such that $c_{11}c_{22} - c_{12}c_{21} \neq 0$, we have

$$(c_{11}u_1 + c_{12}u_2)^{n-i}(c_{21}u_1 + c_{22}u_2)^{i-1} = \sum_{j=1}^{n} d_j u_1^{n-j} u_2^{j-1}\,,$$

where d_j are suitable constants. Hence the property (iii) is also established.

It remains to show that the construction F_n can be extended from AD_2 onto the second order equations from D_2,

$$(\tilde{p}) \qquad u'' + \tilde{p}(x)\, u = 0\,,$$

where $\tilde{p} \in C^{n-2}(I), I \subset \mathbb{R}$. This subset of equations $(\tilde{p})$ from D_2 will be denoted by $\tilde{D}_2$.

In fact, since $\tilde{p} \in C^{n-2}(I)$, we have $u_1 \in C^n(I)$ and $u_2 \in C^n(I)$ for each pair of linearly independent solutions u_1 and u_2 of $(\tilde{p})$. All steps of introducing the mapping F_n have required derivatives at most of the order n. Hence we may define in the same manner the equation

$$\tilde{F}_n(\tilde{p})$$

as the unique n-th order linear differential equation having the n functions $u_1^{n-i} . u_2^{i-1}$ as its solutions. The coefficients of the equation $\tilde{F}_n(\tilde{p})$ are continuous, the coefficient of the $(n-1)$-st derivative is zero because the Wroński determinant of the solutions is a nonzero constant, and the coefficient of the $(n-2)$-nd derivative is of class $C^{n-2}(I)$ if the leading coefficient is 1, because solutions are of class $C^n(I)$. Hence $\tilde{F}_n\colon \tilde{D}_2 \to D_n$. Denote $\tilde{D}_n := \tilde{F}_n(\tilde{D}_2)$.

In fact, the equation $\tilde{F}_n(\tilde{p})$ coincides with the so-called iterative equation generated by $\tilde{p}$, and it can be written in the form

$$\tilde{F}_n(\tilde{p}) = y^{(n)} + \binom{n+1}{3} \tilde{p}(x)\, y^{(n-2)} + 2\binom{n+1}{4} \tilde{p}'(x)\, y^{(n-3)} + \ldots = 0\,,$$

see, e.g. [3]. We can see that for different $(\tilde{p})$ we get different $\tilde{F}_n(\tilde{p})$, hence $\tilde{F}_n$ is an injection. Q.E.D.

CONCLUSION

We have proved that a special construction of iterative equations is unique and in this sense natural, if the commutativity of constructions with transformations and a regularity condition are required.

Due to the injectivity of the mapping $\tilde{F}_n$, we can complete the relation (4) to

$$\tilde{F}_n\, h(\tilde{p}) = h\, \tilde{F}_n(\tilde{p}) \quad \text{and} \quad h\, \tilde{F}_n^{-1}(\tilde{P}_n) = \tilde{F}_n^{-1}\, h(\tilde{P}_n)$$

for all $\tilde{p} \in \tilde{D}_2$ and $\tilde{P}_n \in \tilde{D}_n$ whenever h is defined.

Remark. In the proof of the theorem the regularity assumption on F_n was used only to establish that F_n maps the equation $u'' = 0$ on $\mathbb{R}$ to an equation from AD_n. Hence the assertion of the theorem remains true if its assumption is weakened in this sense.

Corollary. *The mapping F_n is a covariant functor from the category AD_2 to the category AD_n.*

Indeed, one can check that

$$h(p) = (p) \quad \text{implies} \quad h\, F_n(p) = F_n(p)\,,$$

and

$$h\, k(p) = (q) \quad \text{implies} \quad hk\, F_n(p) = F_n(q)\,,$$

because

$$\left|(hk)'\right|^{(1-n)/2} y(hk) = \left|k'\right|^{(1-n)/2} \left(\left|h'\right|^{(1-n)/2} y(h)\right)(k)\,.$$

References

[1] *J. Aczél:* Lectures on Functional Equations and their Applications. Academic Press, New York and London, 1966.

[2] *O. Borůvka:* Linear Differential Transformations of the Second Order. The English Univ. Press, London, 1971.

[3] *Z. Hustý:* Die Iteration homogener linearer Differentialgleichungen. Publ. Fac. Sci. Univ. J. E. Purkyně (Brno) *449* (1964), 23—56.

[4] *J. Kurzweil:* Ordinary Differential Equations in the Real Domain. SNTL Praha & Elsevier, Amsterdam, 1986.

[5] *Ch Maxwell:* Coincidences of mappings. Global Analysis-Analysis on Manifolds, edited by T. M. Rassias, Teubner-Texte zur Mathematik, Leipzig 1983.

[6] *F. Neuman:* Categorial approach to global transformations of the *n*-th order linear differential equations. Čas. pěst. mat. *102* (1977), 350—355.

[7] *F. Neuman:* Criterion of global equivalence of linear differential equations. Proc. Roy. Soc. Edinburgh 97A (1984), 217—221.

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