ON DE BLASI'S DIFFERENTIATION THEORY FOR MULTIFUNCTIONS

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1. DEFINITION OF DIFFERENTIABILITY

Let K be an abstract convex cone, i.e. K is a nonempty set in which an addition $x + y$ and positive scalar multiplication $0.x = 0$, $t.x = 0$, $1.x = x$, $t.(x + y) = t.x + t.y$, $t.(s.x) = (t.s).x$ for all $x, y \in K$ and $t, s \geqq 0$. We say that K is a topological abstract convex cone if K is a topological space such that the addition and scalar multiplication are continuous.

Throughout this and the next sections we will assume:

1) K is a topological abstract convex cone, w its topology, and another topology w' on K is given such that $x_n \to 0$ in w' implies $x_n \to 0$ in w.

2) $K_0 \subset K$ is a subcone of K, i.e. $t.K_0 \subset K_0$ for $t \geqq 0$.

3) The topology w is semimetrizable by a semimetric d which is a metric on K_0.

4) If $x \in K$ and $y, z \in K_0$ then $d(x + y, x + z) = d(y, z)$.

5) The semimetric d is positively homogeneous, i.e. $d(t.x, t.y) = t.d(x, y)$ for all $x, y \in K$ and $t \geqq 0$.

Remark that condition 4) implies the law of cancellation: $x + y = x + z$ implies $y = z$, but the converse is not true in general (see [4]).

Now, let f be a map from an open subset U of a normed space X into K. We say that f is differentiable at $x \in U$ if there exists a w'-continuous and positively homogeneous map $T(x)$ from X into K_0 such that

$$d(f(x + h), f(x) + T(x)(h)) = o(h) \tag{1}$$

where $o(h)/\|h\|$ tends to 0 if h tends to 0.

Of course, if $K = K_0 = Y$ is a normed space, $w = w'$ is its norm topology and $d(x, y) = \|x - y\|$, then we get the classical Fréchet differentiation theory.

Using the methods of M. Boudourides and J. Schinas ([2], [3]) we can develop our theory.

Proposition 1. *There exists at most one map $T(x)$ satisfying the condition* (1).

Proof. Suppose that there exist two maps $T(x)$ and $S(x)$ satisfying (1). Thus we get $d(T(x)(h), S(x)(h)) = d(f(x) + T(x)(h), f(x) + S(x)(h)) \leqq o_1(h) + o_2(h)$. By the homogeneity we obtain $d(T(x)(h/\|h\|), S(x)(h/\|h\|)) \leqq o_1(h)/\|h\| + o_2(h)/\|h\|$. Take $h_n = n^{-1}v$, where $v \in X$ is arbitrary such that $\|v\| = 1$. We get $d(T(x)(v), S(x)(v)) \leqq n\, o_1(n^{-1}v) + n\, o_2(n^{-1}v)$ tends to 0 if n tends to infinity. Therefore $T(x)(v) = S(x)(v)$. Hence $T(x) = S(x,)$ which completes the proof.

The unique map $T(x)$ defined by (1) will be called the derivative of f at the point x, and will be denoted by $f'(x)$.

Proposition 2. *If $f: U \to K$ is differentiable at $x \in U$, then $\|f'(x)\|_1 = \sup \{\|h\|^{-1} d(f'(x)(h), 0) : h \neq 0\}$ is finite.*

Proof. We have $\|f'(x)\|_1 = \sup \{d(f'(x)(v), 0) : \|v\| = 1\}$. Suppose that $\|f'(x)\|_1 = \infty$. For each n there exists v_n, $\|v_n\| = 1$ and $d(f'(x)(v_n), 0) > n$. Then $d(f'(x)(n^{-1}v_n), 0) > 1$. Since $n^{-1}v_n$ tends to 0 and $f'(x)$ is w'-continuous at 0 then $d(f'(x)(n^{-1}v_n), 0)$ tends to 0 and we get a contradiction.

2. MEAN VALUE THEOREM

First we state some helpful results on differentiable maps from X to K.

Proposition 3. *If $f: U \to K$ is differentiable at $x \in U$, then f is lipschitzian at x.*

Proof. Let $\varepsilon > 0$ be arbitrary There exists $\delta > 0$ such that for $\|h\| < \delta$, $d(f(x + h), f(x) + f'(x)(h)) < \varepsilon\|h\|$. By Proposition 2 we have $d(f'(x)(h), 0) \leqq \|f'(x)\|_1 \|h\|$. Therefore, for $0 < \|h\| < \delta$ we get $d(f(x + h), f(x)) \leqq d(f(x + h), f(x) + f'(x)(h)) + d(f(x) + f'(x)(h), f(x)) \leqq \varepsilon\|h\| + d(f'(x)(h), 0) \leqq \varepsilon\|h\| + \|f'(x)\|_1 \|h\| = (\varepsilon + \|f'(x)\|_1)\|h\|$, which completes the proof.

Proposition 4. *If $f: U \to K$ is differentiable at x then f is w-continuous at x.* This follows by the Lipschitz condition from Proposition 3.

Now, let $g: [a, b] \to K$. We say that g is right differentiable at $t \in [a, b)$ if there exists a w' – continuous and positively homogeneous map $P(t): [0, \infty) \to K_0$ such that $d(g(t + h), g(t) + P(t)(h)) = o(h)$ where $o(h)/h \to 0$ if $h \to 0+$. The map $P(t)$ defined above is unique and will be denoted by $g'_+(t)$.

Proposition 5. *Let $g: [a, b] \to K$ be w-continuous, right differentiable on $[a, b)$ and let $\|g'_+(t)(1)\|_0 = d(g'_+(t)(1), 0) \leqq M$ for all $t \in [a, b)$. Then*

$$d(g(b), g(a)) \leqq M(b - a).$$

Proof. Let $J = \{t \in [a, b]$: for some $\varepsilon > 0$, $d(g(t), g(a)) > M(t - a) + \varepsilon(t - a) + \varepsilon\}$. It is sufficient to prove that the set J is empty. Since the semimetric d is w-continuous the set J is open. Suppose that J is nonempty and let c be the infimum of J. We have $c \in (a, b)$ for $a \notin J$ and $\|g'_+(c)(1)\|_0 \leqq M$. Let $\varepsilon > 0$. By the right differentiability there exists $\delta > 0$ such that for $t \in [c, c + \delta)$, $d(g(t), g(c) + g'_+(c)(t - c)) \leqq \varepsilon(t - c)$. Therefore, for $t \in [c, c + \delta)$ we get $d(g(t), g(c)) \leqq d(g(t), g(c) + g'_+(c)(t - c)) + d(g(c) + g'_+(c)(t - c), g(c)) \leqq \varepsilon(t - c) + d(g'_+(c)(t - c), 0) = \varepsilon(t - c) + (t - c)\|g'_+(c)(t - c)\|_0 \leqq \varepsilon(t - c) + M(t - c)$. But $c \notin J$ and consequent by $d(g(c), g(a)) \leqq M(c - a) + \varepsilon(c - a) + \varepsilon$. Hence $d(g(t), g(a)) \leqq d(g(t), g(c)) + d(g(c), g(a) \leqq \varepsilon(t - c) + M(t - c) + M(c - a) + \varepsilon(c - a) + \varepsilon = M(t - a) + \varepsilon(t - a) + \varepsilon$ for $t \in [c, c + \delta)$, which means that $t \notin J$, a contradiction. The proof is complete.

Proposition 6. *Let $f\colon U \to K$ be differentiable at $x \in U$ and continuous in a neighbourhood of x. Then the map $f'(x)$ is w-continuous.*

Proof. It is sufficient to show that $f'(x)$ is w-continuous on every ball $B(0, r)$ in X. Take $\varepsilon > 0$. The differentiability implies that there exists $r > 0$ such that for $\|h\| < r$, $d(f(x + h), f(x) + f'(x)(h)) \leqq \varepsilon\|h\|$. Take any $k \in B(0, r)$. By the w-continuity of f in a neighbourhood of x there exists $\delta > 0$, $\delta < r - \|k\|$ such that $d(f(x + k), f(x + h)) < \varepsilon$ for all $h \in B(k, \delta) \subset B(0, r)$. Therefore, for all $k \in B(0, r)$ and all $h \in B(k, \delta)$ we get $d(f'(x)(h), f'(x)(k)) = d(f(x) + f'(x)(h), f(x) + f'(x)(k)) \leqq d(f(x) + f'(x)(k), f(x + h)) + d(f(x + h), f(x + k)) + d(f(x + k), f(x) + f'(x)(k)) < \varepsilon\|h\| + \varepsilon + \varepsilon\|k\|$, which completes the proof.

Proposition 7. *Let X and Y be normed spaces, U an open subset of X, $g\colon U \to Y$ differentiable at $x \in U$, let V be an open subset of Y, $f\colon V \to K$ differentiable at $g(x)$ which belongs to V. If $f'(g(x))$ is w-continuous then the composition $f \circ g$ is differentiable at x and $(f \circ g)'(x) = f'(g(x)) \circ g'(x)$.*

Proof. By the hypotheses we obtain $d(f(g(x + h)), f(g(x)) + f'(g(x))(g'(x))(h)) \leqq \leqq d(f(g(x) + g'(x)(h) + o(h)), f(g(x)) + d(f(g(x)), f(g(x)) + f'(g(x))(g'(x))(h)) = = o_1(g'(x)(h) + o(h)) + d(0, f'(g(x))(g'(x))(h)) = o_1(h) + \|h\|\, o(h)$, which completes the proof.

Finally we are ready to prove the following mean value theorem.

Theorem 1. *Let $f\colon U \to K$ be differentiable and let $[x_1, x_2] \subset U$ be a segment. Then*

$$d(f(x_2).f(x_1)) \leqq \|x_2 - x_1\| \sup\{\|f'(x)\|_1 : x \in [x_1, x_2]\}.$$

Proof. Let $g(t) = f((1 - t)x_1 + tx_2)$, $t \in [0, 1]$. By Proposition 7 the map g is right differentiable on $[0, 1)$, $g'_+(t)(h) = f'((1 - t)x_1 + tx_2)(hx_2 - hx_1)$, and $\|g'_+(t)(1)\|_0 = \|f'((1 - t)x_1 + tx_2)(x_2 - x_1)\|_0 \leqq \sup d(f'((1 - t)x_1 + tx_2) . (x_2 - x_1), 0) : t \in [0, 1]\} = \|x_2 - x_1\| \sup\{\|f'(x)(x_2 - x_1)/\|x_2 - x_1\|\| : x \in$

$\in [x_1, x_2]\} \leqq \|x_2 - x_1\| \sup \{\sup \{\|f'(x)(v)\|_0 : x \in [x_1, x_2]\} : \|v\| = 1\} =$
$= \|x_2 - x_1\| \sup \{\sup \{\|f'(x)(v)\|_0 : \|v\| = 1\} : x \in [x_1, x_2]\} =$
$= \|x_2 - x_1\| \sup \{\|f'(x)\|_1 : x \in [x_1, x_2]\} \equiv M$. If $M = \infty$ the theorem holds trivially. If $M < \infty$, then applying Proposition 5 we get the result.

Remark. We may also consider higher order derivatives for $f: U \to K$. For example, let f be differentiable on U. We say that f is twice differentiable at $x \in U$ if there exists a w'-continuous and positively two-homogeneous map $P(x)$ from $X \times X$ into K_0 such that $d_1(f'(x + h), f'(x) + P(x)(\cdot, h)) = o(h)$, where $d_1(P, Q) = \sup \{\|h\|^{-1} d(P(h), Q(h)) : h \neq 0\}$, see [3].

3. DIFFERENTIABILITY OF MULTIFUNCTIONS

Let Y be a Banach space, K the family of all nonempty and bounded subsets of Y and K_0 the family of all nonempty bounded convex and closed subsets of Y. K is an abstract convex cone with the following operations: $A + B = \{a + b : a \in A, b \in B\}$, $tA = \{ta : a \in A\}$ for $A, B \in K$ and $t \geqq 0$. Let d be the Hausdorff distance on K and let w be its topology, w' the upper Hausdorff topology on K, i.e. for $A_0 \in K$ and V a neighbourhood of 0 in Y, the set $V(A_0) = \{A \in K : A \subset A_0 + V\}$ is a neighbourhood of A_0. Then the conditions 1)–5) are satisfied and we get De Blasi's differentiation theory (see [1], [2], [3]).

References

[1] *F. S. De Blasi:* On the differentiability of multifunctions. Pacif. J. Math. *66* (1976), 67–81.

[2] *M. Boudourides, J. Schinas:* The mean value theorem for multifunctions. Bull. Math. Roumanie *25* (1981), no 2, 129–141.

[3] *M. Boudorides, J. Schinas:* Higher order differentiability of multifunctions. Nonlin. Anal. *5* (1981), no 5, 509–516.

[4] *P. Kranz:* A metrizable cancellative semigroup without translation invariant metric. Proc. Amer. Math. Soc. *66* (1977), no 1, 17–19.

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