Časopis pro pěstování matematiky

Pavel Krbec; Jaroslav Kurzweil

Kneser's theorem for multivalued differential delay equations

Časopis pro pěstování matematiky, Vol. 104 (1979), No. 1, 1--8

Persistent URL: `http://dml.cz/dmlcz/117996`

ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY

Vydává Matematický ústav ČSAV, Praha

SVAZEK 104 * PRAHA 12. 2. 1979 * ČÍSLO 1

KNESER'S THEOREM FOR MULTIVALUED DIFFERENTIAL DELAY EQUATIONS

PAVEL KRBEC and JAROSLAV KURZWEIL, Praha

(Received April 23, 1976)

I. Introduction and Definitions. Let P and Q be metric spaces. The set of all nonempty and compact subsets of P is denoted by $\Omega(P)$, the set of all nonempty, compact and convex subsets of the Euclidean n-dimensional space R^n is denoted by $\mathscr{K}(n)$. The closed convex hull of a set A, $A \subset R^n$ is denoted by $\overline{\text{co}}\, A$. The interior of B, $B \subset P$ is denoted by Int B. A mapping $F : Q \to \Omega(P)$ is *upper-semicontinous on Q* if for every $x \in Q$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $F(B_Q(x, \delta)) \subset$ $\subset B_P(F(x), \varepsilon)$, where $B_Q(x, \delta)$ and $B_P(F(x), \varepsilon)$ are respectively the δ-neighbourhood of x in Q and the ε-neighbourhood of the set $F(x)$ in P.

If P is compact then $F : Q \to \Omega(P)$ is upper-semicontinuous if and only if $\lim_{n\to\infty} x_n =$ $= x$, $\lim_{n\to\infty} y_n = y$, $y_n \in F(x_n)$ implies $y \in F(x)$ (see KURATOWSKI [6], vol. II, p. 69).

Let I be a compact interval in R and let C_I be the space of all continuous functions from I to R_n with the maximum norm, let us write simply C for $C_{\langle -1,0\rangle}$. Similarly, the ball $\{x \in R^n \mid \|x - y\| \leqq r\}$ is denoted by $B(y, r)$.

Let $J = \langle \bar{t}, \beta \rangle$, $\bar{t} < \beta$, and let $F : J \times C \to \mathscr{K}(n)$. We shall investigate certain solution sets of the multivalued differential delay equation

$$\dot{x}(t) \in F(t, x_t(\cdot)) \tag{1}$$

where $x_t(s) \overset{def}{=} x(t + s)$, $s \in \langle -1, 0\rangle$ and F fulfils the usual conditions for the existence of solutions. A continuous function $x(\cdot) : \langle \bar{t} - 1, \beta \rangle \to R^n$ is called *a solution of equation* (1) *on* $J = \langle \bar{t}, \beta \rangle$ if it is absolutely continuous on J and if $\dot{x}(t) \in F(t, x_t(\cdot))$ a. e. on J. Let $\tilde{x} \in C$. The set of all solutions $x(\cdot)$ on J with the property $x_{\bar{t}}(\cdot)|_{\langle -1,0\rangle} = \tilde{x}(\cdot)$ will be denoted by $\mathscr{S}(\bar{t}, \tilde{x}, J)$. We shall prove that the set $\mathscr{S}(\bar{t}, \tilde{x}, J)$ is a continuum, β being sufficiently close to $\bar{t}$. This assertion was proved by KNESER [5] for ordinary differential equations and is well-known as Kneser's theorem.

II. Some Preliminary Lemmas. To prove the generalization of Kneser's theorem the following three lemmas are needed:

Lemma 1. *Let P be a compact metric space, let $P_k \subset P$, $k = 1, 2, \ldots$ be continua. Let*

$$Q = \{x \in P \mid \text{there exists a sequence } \{p_{k_i}\}_{i=1}^{\infty},\ p_{k_i} \in P_{k_i},\ x = \lim p_{k_i}\}$$

and

$$Q \subset P_k\,, \quad k = 1, 2, \ldots$$

(*i.e.* $Q = \lim P_n$).

Then Q is a continuum.

For the (easy) proof see Kuratowski [6] vol. II, p. 179, th. 4.

Lemma 2. *Let I be a compact interval, $\tau \in I$, let functions $p_k : I \to R^n$, $k = 1, 2, \ldots$ be integrable and let there exist an integrable function $\xi : I \to R$ such that for every $k = 1, 2, \ldots$ the inequality $\|p_k(t)\| < \xi(t)$ holds for all $t \in I$.*

Let $P_i(t) = \overline{\text{co}}\,\{p_i(t), p_{i+1}(t), \ldots\}$ $i = 1, 2, \ldots$ and let $P(t) = \bigcap_{i=1}^{\infty} P_i(t)$. Suppose that

$$q_k(t) = \int_{\tau}^{t} p_k(\sigma)\,\mathrm{d}\sigma \to q(t) \quad \textit{for} \quad k \to \infty,\ t \in J\,.$$

Then

$$\|q(t) - q(s)\| \leqq \left|\int_{s}^{t} \xi(\sigma)\,\mathrm{d}\sigma\right|$$

for each s, t from I and $\dot{q}(t) \in P(t)$ a. e. on I.

S k e t c h o f t h e p r o o f: We have $q_k(\cdot) \to q(\cdot)$ in C_I for $k \to \infty$. Let $t_1, t_2 \in I$. Then

$$\|q(t_2) - q(t_1)\| \leqq \overline{\lim_{k \to \infty}} \int_{t_1}^{t_2} \|p_k(\sigma)\|\,\mathrm{d}\sigma \leqq \int_{t_1}^{t_2} \xi(\sigma)\,\mathrm{d}\sigma\,.$$

Hence the function $q(\cdot)$ is absolutely continuous which implies the existence of the derivative $\dot{q}(t)$ a. e. in I. The sequence $\{\dot{q}_k\}_{k=1}^{\infty}$ has the following properties

1) $\sup_{n=1,2\ldots} \int_I \|\dot{q}_k(t)\|\,\mathrm{d}t < \infty\,,$

2) $\lim_{k \to \infty} \int_{\tau_1}^{\tau_2} \dot{q}_k(t)\,\mathrm{d}t = \int_{\tau_1}^{\tau_2} \dot{q}(t)\,\mathrm{d}t$ for every $\tau_1, \tau_2 \in I\,.$

Hence $\lim \int_M \dot{q}_k = \int_M \dot{q}$ for every measurable M, $M \subset I$ which implies $\dot{q}_k \to \dot{q}$ weakly in L_1 (see Dunford-Schwartz [7] p. 316). The set $P_k = \{u \in L_1(I) \mid u(t) \in P_k(t)$ a. e. in $I\}$ is convex. Let $u_n \in P_k$ for $n = 1, 2, \ldots$ and $u_n \to u$ in $L_1(I)$

for $n \to \infty$. Then there exists a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ with the property $u_{n_i}(t) \to u(t)$ a. e. in I for $i \to \infty$. The set $P_k(t)$ is closed. Hence $u(t) \in P_k(t)$ a. e. in I. It means $u \in P_k$, which proves that the set P_k is strongly closed in $L_1(I)$. The inclusion $P_k \supset \supset P_{k+1}$ for $k = 1, 2, \ldots$ implies that the set $P = \bigcap_{k=1}^{\infty} P_k$ is also convex and strongly closed in $L_1(I)$. Hence the set P is also weakly closed in $L_1(I)$. But $\dot{q}_k \to \dot{q}$ weakly in $L_1(I)$ and $\dot{q}_k \in P_k$ for $k = 1, 2, \ldots$ which implies $\dot{q} \in P$.

Lemma 3. *Let P and Q be metric spaces, Q connected and let a mapping $\Phi : Q \to \Omega(P)$ be upper-semicontinuous with the property that $\Phi(a)$ is a continuum for every $a \in Q$. Then the set $\bigcup_{a \in Q} \Phi(a)$ is connected.*

Proof: If the assertion were false there would exist two nonempty sets P_1, P_2 such that $\bar{P}_1 \cap P_2 = \emptyset = P_1 \cap \bar{P}_2$ and $\bigcup_{a \in Q} \Phi(a) = P_1 \cup P_2$. If $a \in Q$, $\Phi(a) \cap P_j \neq \emptyset$ then $\Phi(a) \subset P_j$ for $j = 1, 2$. Hence the sets

$$Q_j = \{a \in Q \mid \Phi(a) \subset P_j\}, \quad j = 1, 2$$

are disjoint, nonempty and $Q = Q_1 \cup Q_2$. The set Q is connected, hence $Q_1 \cap \bar{Q}_2 \neq \emptyset$ or $\bar{Q}_1 \cap Q_2 \neq \emptyset$. Let us suppose $Q_1 \cap \bar{Q}_2 \neq \emptyset$ and let $a \in Q_1 \cap \bar{Q}_2$. Then there exists a sequence $\{a_j\}_{j=1}^{\infty}$ of elements from Q_2 such that $a_j \to a$ as $j \to \infty$. The mapping Φ is upper-semicontinuous. Hence for every $\varepsilon > 0$ there exists a positive integer n such that for every positive integer $j, j > n$, the relation $\Phi(a_j) \subset B(\Phi(a), \varepsilon)$ holds which yields $\overline{\bigcup_{j=1}^{\infty} \Phi(a_j)} \cap \Phi(a) \neq \emptyset$. Since $\bigcup_{j=1}^{\infty} \Phi(a_j) \subset P_2$ and $\Phi(a) \subset P_1$ we obtain $\bar{P}_2 \cap P_1 \neq \emptyset$ and this contradiction proves the assertion of the lemma.

It is well-known that Kneser's theorem is of a local character and we may formulate it without loss of generality as follows:

Theorem 1. *Let $\bar{t}$ and β be real numbers, $\bar{t} < \beta$, and let $J = \langle \bar{t}, \beta \rangle$. Let $\eta : : J \to \langle 0, \infty)$ be a real function such that $\int_{\bar{t}}^{\beta} \eta(t)\, dt < 1$ and let $F : J \times B_C(o, 2) \to \to \mathscr{K}(n)$ be a mapping with the following properties:*

(i) *$F(t, \cdot)$ is upper-semicontinuous on $B_C(o, 2)$ for almost every $t \in J$;*

(ii) *if $\psi : \langle \bar{t} - 1, \beta \rangle \to B(o, 2)$ is continuous then there exists a measurable function $\xi : J \to R^n$ such that*

$$\xi(t) \in F(t, \psi_t(\cdot)) \quad a.\ e.\ on\ J\,;$$

(iii) $$F(t, x_t(\cdot)) \subset B(o, \eta(t)) \quad on \quad J \times B_C(o, 2)\,.$$

Let $M \subset B_C(o, 1)$ be a continuum in C. Then the set $\mathscr{S}(\bar{t}, M, J] = \bigcup_{\tilde{x} \in M} \mathscr{S}(\bar{t}, \tilde{x}, J)$ is a continuum in C_J.

Remark 1. The image of a connected set by a continuous function is a connected set. Hence the section $\{y = y(t) \mid y(\cdot) \in \mathscr{S}(\bar{t}, M, J)\}$ is connected in R^n for every $t \in J$.

Remark 2. The assumptions of Theorem 1 imply that

$$\mathscr{S}(\bar{t}, o, J) \subset \operatorname{Int} B_{C_J}(o, 1) .$$

Remark 3. The supposition (ii) is valid if (i) is valid and if the set $\{t \in J \mid F(t, u) \cap K \neq \emptyset\}$ is Lebesgue measurable for every $u \in C_{\langle -1,0\rangle}$ and for every compact set K, $K \subset R^n$. For the proof see HUKUHARA [4] and CASTAING [1], [2]. Moreover, the assumption (ii) may be replaced without loss of generality by a stronger assumption

(iv) to every $\varepsilon > 0$ there exists a measurable set $A_\varepsilon \subset J$ such that $\mu(J - A_\varepsilon) < \varepsilon$ and the function $F|_{A_\varepsilon \times B_C(o,2)}$ is upper-semicontinuous. See JARNÍK, KURZWEIL [7].

Proof of the theorem. Let us suppose that $\mathscr{S}(\bar{t}, \tilde{x}, J)$ is a continuum. It follows from (iii) that the functions from $\mathscr{S}(\bar{t}, B_C(o, 1), J)$ are equibounded and equicontinuous. It is an easy consequence of Lemma 2 that $\mathscr{S}(\bar{t}, \overline{B_C(o, 1)}, J)$ is closed in C_J. Hence $\mathscr{S}(\bar{t}, \overline{B_C(o, 1)}, J)$ is compact. This and Lemma 2 yield upper-semicontinuity of the mapping $\mathscr{S}(\bar{t}, \cdot, J)$. The assertion of the theorem is then a consequence of Lemma 3.

It remains to prove that for an arbitrary $\tilde{x} \in C$, $\mathscr{S}(\bar{t}, \tilde{x}, J)$ is a continuum. It will be convenient to introduce some notation. Let k be a positive integer and let numbers $\sigma_1, \sigma_2, \ldots, \sigma_k$ be such that $\bar{t} = \sigma_0 < \sigma_1 < \ldots < \sigma_k = \beta$ and $\sigma_{i+1} - \sigma_i = (1/k)(\beta - \bar{t})$, $i = 0, 1, \ldots, k - 1$. Let $\mathscr{X}_{k,j}$, $j = 1, 2, \ldots, k$ denote the set of all pairs (v, u) of mappings from $\langle \sigma_0 - 1, \sigma_j \rangle$ into $B(o, 2)$ with the properties

1) $v(t) = u(t) = \tilde{x}(t)$ for every $t \in \langle \sigma_0 - 1, \sigma_0 \rangle$;

2) $\|v(t_2) - v(t_1)\| \leqq \int_{t_1}^{t_2} \eta(t)\, dt$ for every t_1, t_2 such that $\sigma_0 \leqq t_1 \leqq t_2 \leqq \sigma_j$;

3) $\|u(t_2) - u(t_1)\| \leqq \int_{t_1}^{t_2} \eta(t)\, dt$ for every t_1, t_2 such that there exists $i \in \{0, 1, \ldots, j - 1\}$ such that $\sigma_i \leqq t_1 \leqq t_2 < \sigma_{i+1}$;

4) $\dot{v}(t) \in F(t, v_t^{i+1}(\cdot))$ a. e. on (σ_i, σ_{i+1}), $i = 0, 1, 2, \ldots, j - 1$ where $v^{i+1}(\sigma) = u(\sigma)$ on $\langle \sigma_i, \sigma_{i+1})$ and $v^{i+1}(\sigma) = v(\sigma)$ on $\langle \sigma_0 - 1, \sigma_i)$;

5) $u(\sigma_i) = v(\sigma_i)$ for every $i \in \{0, 1, \ldots, j - 1\}$.

First we need to show that the set $\mathscr{X}_{k,j}$ is nonempty for all positive integers k and all $j = 1, 2, \ldots, k$. Let a positive integer k be chosen and let $u(\cdot) = v(\cdot) = \tilde{x}(\cdot)$ on $\langle \sigma_0 - 1, \sigma_0 \rangle$, $u(t) = v(\sigma_0)$ for all $t \in (\sigma_0, \sigma_1)$. It follows from (ii) that there exists a measurable selection $\xi^1(\cdot)$ such that $\xi^1(t) \in F(t, v_t^1(\cdot))$ for a. e. $t \in (\sigma_0, \sigma_1)$

(where $v^1(\sigma) = \text{const} = v(\sigma_0)$ for all $\sigma \in \langle\sigma_0, \sigma_1)$) and $v^1(\sigma) = v(\sigma) = \tilde{x}(\sigma)$ for every $\sigma \in \langle\sigma_0 - 1, \sigma_0\rangle$. We define $v(t) = v(\sigma_0) + \int_{\sigma_0}^{t} \xi^1(\tau)\,d\tau$ for every t in $\langle\sigma_0, \sigma_1\rangle$ and $u(\sigma_1) = v(\sigma_1)$. Thus $\mathscr{X}_{k,1}$ is nonempty. From (iii) we obtain $v(t) \in B(o, 1 + \int_{\sigma_0}^{\sigma_1} \eta(t)\,dt) \subset B(o, 2)$ for every $t \in \langle\sigma_0, \sigma_1\rangle$.

Assume that $\mathscr{X}_{kj}$ is nonempty for some $j < k$. Then there exist functions u, v defined on $\langle\sigma_0 - 1, \sigma_j\rangle$, with properties 1), ..., 5). We define $u(t) = v(\sigma_j)$, $v^{j+1}(t) = u(t)$ for all $t \in \langle\sigma_j, \sigma_{j+1})$ and $v^{j+1}(t) = v(t)$ for all t in $\langle\sigma_0 - 1, \sigma_j)$. Then it is clear that the mapping $v^{j+1}(\cdot) : \langle\sigma_0 - 1, \sigma_{j+1}) \to B(o, 2)$ is continuous and as a consequence of assumptions (ii) and (iii) we obtain a measurable function $\xi^{j+1}(\cdot) : \langle\sigma_j, \sigma_{j+1}) \to R^n$ such that $\xi^{j+1}(t) \in F(t, u_t(\cdot)) \subset B(o, \eta(t))$ for a. e. $t \in \langle\sigma_j, \sigma_{j+1}\rangle$. Hence it is possible to define

$$v(t) = v(\sigma_j) + \int_{\sigma_j}^{t} \xi^{j+1}(\sigma)\,d\sigma \quad \text{for all} \quad t \in \langle\sigma_j, \sigma_{j+1}\rangle\,,$$

$$u(\sigma_{j+1}) = v(\sigma_{j+1})$$

and it is clear that the relations

$$u(t) \in B\left(o, 1 + \int_{\sigma_0}^{\sigma_j} \eta(\tau)\,d\tau\right) \subset B(o, 2)\,,$$

$$v(t) \in B\left(o, 1 + \int_{\sigma_0}^{\sigma_{j+1}} \eta(\tau)\,d\tau\right) \subset B(o, 2)$$

hold for every $t \in \langle\sigma_0, \sigma_{j+1}\rangle$.

Therefore, the functions $u(\cdot)$ and $v(\cdot)$ with properties 1), ..., 5) can be defined on the interval $\langle\sigma_0, \sigma_{j+1}\rangle$. Hence every set $\mathscr{X}_{k,j}$, $k = 1, 2, \ldots, j = 1, 2, \ldots, k$ is nonempty.

Let $\mathscr{X} = \bigcup_{k=1}^{\infty} \mathscr{X}_{k,k}$ and let us define for $j = 1, 2, \ldots, k$ the sets $\mathscr{C}_{k,j} = \{(v, u) |\ v \in C_{\langle\sigma_0 - 1, \sigma_j\rangle},\ u\colon \langle\sigma_0 - 1, \sigma_j\rangle \to R^n,\ u(\sigma_i) = v(\sigma_i)$ for every $i = 0, 1, \ldots, j$, $u(t) = v(t) = \tilde{x}(t)$ for every $t \in \langle\sigma_0 - 1, \sigma_0\rangle$, for every $i = 1, 2, \ldots, j$ there exists $\hat{u}^i \in C_{\langle\sigma_{i-1}, \sigma_i\rangle}$ such that $u|_{\langle\sigma_{i-1}, \sigma_i)} = \hat{u}^i\}$,

$$\mathscr{C} = \bigcup_{k=1}^{\infty} \mathscr{C}_{k,k}$$

and

$$\varrho_j((v_1, u_1)\,,\ (v_2, u_2)) = \sup_{t \in \langle\sigma_0, \sigma_j\rangle} \|v_1(t) - v_2(t)\| + \sup_{t \in \langle\sigma_0, \sigma_j)} \|u_1(t) - u_2(t)\|$$

for $(v_1, u_1), (v_2, u_2) \in \mathscr{C}_{k,j}$ where $\|\cdot\|$ denotes the Euclidean norm in R^n.

Then the pairs $(\mathscr{C}_{k,j}, \varrho_{k,j})$ and $(\mathscr{C}, \varrho_{k,k})$ are metric spaces; let us denote them $\mathscr{C}_{k,j}$ and $\mathscr{C}$, respectively. It is easy to see that a set A closed in $\mathscr{C}_{k,j}$ is compact if and only if the first components of elements from A are equicontinuous and equibounded on $\langle\sigma_0, \sigma_j\rangle$ and the second components are equibounded on $\langle\sigma_0, \sigma_j\rangle$ and equicontinuous on $\langle \sigma_{i-1}, \sigma_i)$ for every $i = 1, 2, \ldots, j$. We prove that the set $\mathscr{X}_{k,j}$ is compact in $\mathscr{C}_{k,j}$. In view of the conditions 1), 2), 3) it is sufficient to prove that $\mathscr{X}_{k,j}$ is closed in $\mathscr{C}_{k,j}$.

Lemma 4. *Let k and j be positive integers, $j \leqq k$. Then the set $\mathscr{X}_{k,j}$ is closed in $\mathscr{C}_{k,j}$.*

Proof of the lemma. Let $(v_n, u_n) \in \mathscr{X}_{k,j}$ and $(v_n, u_n) \to (v, u)$ in $\mathscr{C}_{k,j}$. Then the conditions 1), 2), 3) and 5) hold for (v, u) and it is sufficient to prove 4). Applying Lemma 2 to $\{v_n\}_{n=1}^{\infty}$ we obtain $\dot{v}(t) \in P(t)$ a. e. on $\langle \sigma_0, \sigma_j)$ where $P(t) = \bigcap_{n=1}^{\infty} P_n(t)$ and $P_n(t) = \overline{\text{co}}\, \{\dot{v}_n(t), \dot{v}_{n+1}(t), \ldots\}$. As $(v_n, u_n) \to (v, u)$ in $\mathscr{C}_{k,j}$, it follows that $v_n^i(t) \to v^i(t)$ for $i = 1, 2, \ldots, j$, $t \in \langle \sigma_0, \sigma_j \rangle$, $n \to \infty$. Choose $\eta > 0$. (i) implies that for almost every $t \in \langle \sigma_{i-1}, \sigma_i \rangle$, $i = 1, 2, \ldots, j$ there exists such an $n_0(t)$ that $F(t, v_{nt}^i(\cdot)) \in B(F(t, v_t^i(\cdot)), \eta)$ for $n > n_0(t)$. We have $\dot{v}_n(t) \in F(t, v_{nt}^i(\cdot))$ a. e. on $\langle \sigma_{i-1}, \sigma_i \rangle$, therefore $\dot{v}_n(t) \in B(F(t, v_t^i(\cdot)), \eta)$ for $n > n_0(t)$ a. e. on $\langle \sigma_{i-1}, \sigma_i \rangle$, $i = 1, 2, \ldots, j$. The set $\overline{B(F(t, v_t^i(\cdot)), \eta)}$ is compact and convex, hence

$$P_n(t) = \overline{\text{co}}\, \{\dot{v}_n(t), \dot{v}_{n+1}(t), \ldots\} \subset \overline{B(F(t, v_t^i(\cdot)), \eta)}$$

for $n > n_0(t)$ a. e. on $\langle \sigma_{i-1}, \sigma_i \rangle$. Since $\dot{v}(t) \in P(t) = \bigcap_{n=1}^{\infty} P_n(t)$ a. e. on $\langle \sigma_0, \sigma_j \rangle$ we obtain $\dot{v}(t) \in \overline{B(F(t, v_t^i(\cdot)), \eta)}$ a. e. on $\langle \sigma_{i-1}, \sigma_i \rangle$ for every η and for every $i = 1, 2, \ldots, j$. The set $F(t, v_t^i(\cdot))$ is compact. Hence $\dot{v}(t) \in \in F(t, v_t^i(\cdot))$ a. e. on $\langle \sigma_{i-1}, \sigma_i \rangle$ for every $i = 1, 2, \ldots, j$. Therefore $(v, u) \in \mathscr{X}_{k,j}$ and Lemma 4 is proved.

Now we can go back to the proof of the theorem.

Let us denote

$$\mathscr{Y} = \{(v, u) \in \mathscr{X} \mid v = u\}\,.$$

It is clear that v is a solution of (1) if and only if $(v, v) \in \mathscr{Y}$ so that $\mathscr{Y} = \mathscr{S}(\bar{t}, \tilde{x}, J)$ and $\mathscr{X}_{k,k} \supset \mathscr{Y}$ for every $k = 1, 2, \ldots$. We want to prove $\mathscr{Y} = \text{Lim}\, \mathscr{X}_{k,k}$ in $\mathscr{C}$ i. e. $\mathscr{Y} = \{(v, u) \in \mathscr{X} \mid$ there exists $(v_{k_j}, u_{k_j}) \in \mathscr{X}_{k_j,k_j}$, $j = 1, 2, \ldots$ such that $(v_{k_j}, u_{k_j}) \to \to (v, u)$ in $\mathscr{C}\}$.

It is sufficient to prove that if

$$(v_{k_j}, u_{k_j}) \in \mathscr{X}_{k_j,k_j} \quad \text{and} \quad (v_{k_j}, u_{k_j}) \to (v, u) \quad \text{in } \mathscr{C} \text{ for } \; j \to \infty$$

then $v = u$ and the function v is a solution of (1). Since $v_{k_j}(t_i) = u_{k_j}(t_i)$ for $t_i = = \bar{t} + i\,.\,(\beta - \bar{t})/k_j$, $i = 0, 1, \ldots, k_j$ it follows from 2), 3) and 5) that $u_{k_j} \to v$ for $j \to \infty$ uniformly on $\langle \bar{t}, \beta \rangle$. Hence $u = v$ on $\langle \bar{t} - 1, \beta \rangle$. It remains to prove that v is a solution of (1) — the proof parallels that of Lemma 4 and is omitted.

Our goal now is to prove that the set $\mathscr{Y} = \mathscr{S}(\bar{t}, \tilde{x}, J)$ is a continuum; we shall apply Lemma 1.

Observe that $\mathscr{X} = \bigcup_{k=1}^{\infty} \mathscr{X}_{k,k}$ is a compact in $\mathscr{C}$. Let $P = \{(v_n, u_n)\}_{n=1}^{\infty}$ be a sequence in $\mathscr{X}$. If there exists such an i that $(v_n, u_n) \in \mathscr{X}_{i,i}$ for $n = 1, 2, \ldots$, then there exists a convergent subsequence ($\mathscr{X}_{i,i}$ being compact). Otherwise for every positive integer i there exists $k(i)$ such that $k(i) \geqq i$ and $(v_{k(i)}, u_{k(i)}) \in \mathscr{X}_{k(i),k(i)}$ then the subsequence $P' = \{v_{k(i)}\}_{i=1}^{\infty}$ is compact in $C_{\langle \bar{t}, \beta \rangle}$ (i. e. relatively compact) and there exists a subsequence of P' which is uniformly convergent on $\langle \bar{t}, \beta \rangle$.

Let us denote it again $\{v_{k(i)}\}_{i=1}^{\infty}$ and let $v_{k(i)} \to v$ in $C_{\langle \bar{t}, \beta \rangle}$ as $i \to \infty$. Then $u_{k(i)} \to v$ uniformly on $\langle \bar{t}, \beta \rangle$ as $i \to \infty$ i. e. $(v_{k(i)}, u_{k(i)}) \to (v, v)$ in $\mathscr{C}$ as $i \to \infty$. Hence $(v, v) \in \mathscr{Y} \subset \mathscr{X}$ and therefore the set $\mathscr{X}$ is compact.

To apply Lemma 1 we must prove that the set $\mathscr{X}_{k,k}$ is a continuum for every $k = 1, 2, \ldots$. In Lemma 4 we have proved that $\mathscr{X}_{k,k}$ is a compact. Let us prove by mathematical induction that $\mathscr{X}_{k,j}$ is connected for $j = 1, 2, \ldots, k$. Let

$$Q_0 = \{u : \langle \sigma_0 - 1, \sigma_1) \to B(o, 2) \mid \text{both conditions 1) and 3) with } j = 1 \text{ are valid}\} .$$

The set Q_0 is convex. Hence it is connected. For $u \in Q_0$ let

$$\Phi_0(u) = \{(v, u) \mid (v, u) \in \mathscr{X}_{k,1}\} .$$

The set $\Phi_0(u)$ is convex (cf. the definition of $\mathscr{X}_{k,j}$) and compact (as $\mathscr{X}_{k,1}$ is compact). For $(\hat{v}, \hat{u}) \in \mathscr{X}_{k,j}$, $1 \leqq j \leqq k$ let us denote

$$\Psi_j(\hat{v}, \hat{u}) = \{(v, u) \in \mathscr{X}_{k,j+1} \mid v|_{\langle \sigma_0, \sigma_j \rangle} = \hat{v},\ u|_{\langle \sigma_0, \sigma_j \rangle} = \hat{u}\} .$$

The set $\Psi_j(\hat{v}, \hat{u})$ is compact as $\mathscr{X}_{k,j+1}$ is compact.

Let us prove that $\Psi_j(\hat{v}, \hat{u})$ is connected. Let us denote

$$Q_j(\hat{u}) = \{u : \langle \sigma_0, \sigma_{j+1} \rangle \to B(o, 2) \mid u_{\langle \sigma_0, \sigma_j \rangle} = \hat{u} \text{ and condition 3) is valid}\}$$

and for $u \in Q_j(\hat{u})$ let

$$\Phi_j(u) = \{(v, u) \mid (v, u) \in \Psi_j(\hat{v}, \hat{u})\}$$

i. e.

$$\Phi_j(u) = \{(v, u) \mid (v, u) \in \mathscr{X}_{k,j+1},\ v|_{\langle \sigma_0, \sigma_j \rangle} = \hat{v},\ u|_{\langle \sigma_0, \sigma_j \rangle} = \hat{u}\} .$$

Then $\Psi_j(\hat{v}, \hat{u}) = \bigcup_{u \in Q_j(\hat{u})} \Phi_j(u)$, the set $Q_j(\hat{u})$ is convex and for any $u \in Q_j(\hat{u})$ the set $\Phi_j(u)$ is convex (cf. the Definition of $\mathscr{X}_{k,j}$) and compact (as $\Psi_j(\hat{v}, \hat{u})$ is compact). To apply Lemma 3 it remains to prove that the mapping Φ_j is upper-semicontinuous. Let $u_n \to u$ in $Q_j(\hat{u})$, $(v_n, w_n) \to (v, w)$ in $\mathscr{X}_{k,j+1}$ for $n \to \infty$, $(v_n, w_n) \in \Phi_j(u_n)$. Then $w_n = u_n$, $w = u$. Hence $(v, u) \in \mathscr{X}_{k,j+1}$ i. e. $(v, u) \in \Phi_j(u)$ and Φ_j is upper-semicontinuous. Applying Lemma 3 we observe that $\Psi_j(\hat{v}, \hat{u})$ is connected and $\mathscr{X}_{k,1} = \bigcup_{u \in Q_0} \Phi_0(u)$

is connected. The mapping Ψ_j, $j = 1, 2, \ldots$ is an upper-semicontinuous mapping from $\mathscr{X}_{k,j}$ into the space of compact subsets of compact space $\mathscr{X}_{k,j+1}$. To prove this let $(\hat{v}_n, \hat{u}_n) \to (\hat{v}, \hat{u})$ in $\mathscr{X}_{k,j}$, $(v_n, u_n) \to (v, u)$ in $\mathscr{X}_{k,j+1}$ for $n \to \infty$ and $(v_n, u_n) \in \Psi_j(\hat{v}_n, \hat{u}_n)$ for $n = 1, 2, \ldots$. Then $v|_{\langle\sigma_0,\sigma_j\rangle} = \hat{v}$, $u|_{\langle\sigma_0,\sigma_j\rangle} = \hat{u}$ and $(v, u) \in \mathscr{X}_{k,j+1}$ i. e. $(v, u) \in \Psi_j(\hat{v}, \hat{u})$ which proves upper-semicontinuity of Ψ_j. Applying Lemma 3 we observe that the set $\mathscr{X}_{k,j+1} = \bigcup_{(\hat{v},\hat{u})\in\mathscr{X}_{k,j}} \Psi_j(\hat{v}, \hat{u})$ is connected, provided the set $\mathscr{X}_{k,j}$ is connected. The set $\mathscr{X}_{k,1}$ is connected and the principle of mathematical induction implies the connectedness of $\mathscr{X}_{k,k}$.

Since we have already proved that the set $\mathscr{X} = \bigcup_{k=1}^{\infty} \mathscr{X}_{k,k}$ is compact it follows from Lemma 1 that $\mathscr{Y} = \operatorname{Lim} \mathscr{X}_{k,k}$ is a continuum and the proof is complete.

Remark 5. Together with Kneser's theorem we have also proved the existence theorem.

References

[1] *Castaing Ch.:* Quelques problémes de mesurabilité lies à la theorie de la commande, C. R. Acad. Sc. Paris t. *262* (1966) Ser. A pp. 409—411.

[2] *Castaing Ch.:* Sur les équations differentiales multivoques, C. R. Acad. Sc. Paris t. *263* (1966) Sér. A pp. 63—66.

[3] *Dunford N., Schwartz J. T.:* Linear Operators, Part 1 (Russian), Izd. inostrannoj literatury, Moskva 1962.

[4] *Hukuhara M.:* Intégration des applications mesurables dont la valeur est un compact convexe, Funkcialaj Ekvacioj vol. 10, numero 3, decembro 1967.

[5] *Kneser H.:* Über die Lösungen eines Systems gewönlicher Differentialgleichungen das der Lipschitzen Bedingung nicht genügt, S. B. Preuss. Akad. Wiss. Phys.-Math. Kl., (1923) pp. 171—174.

[6] *Kuratowski K.:* Topology, vol. I, II (Russian), Izd. MIR, Moskva 1966.

[7] *Jarník J., Kurzweil J.:* On conditions on right hand sides of differential relations, Časopis pro pěstování matematiky *102* (1977) pp. 334—349.

Author's addresses: Pavel Krbec, 199 05 Praha 9, Beranových 130 (Výzkumný a zkušební letecký ústav); Jaroslav Kurzweil, 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).