Časopis pro pěstování matematiky

Ján Plesník

Diametrically critical tournaments

Časopis pro pěstování matematiky, Vol. 100 (1975), No. 4, 361--370

Persistent URL: `http://dml.cz/dmlcz/117889`

Časopis pro pěstování matematiky, roč. 100 (1975), Praha

DIAMETRICALLY CRITICAL TOURNAMENTS

JÁN PLESNÍK, Bratislava

(Received June 20, 1974)

In the whole paper, all notions not defined here will be used in the sense of [1]. Given a *tournament* T, $V(T)$ and $E(T)$ denote its *point set* and *line set*, respectively. For a set $M \subset V(T)$, $T(M)$ denotes the *induced subgraph* (*subtournament*) of T with point set M. If $u, v \in V(T)$, then the distance from u to v is denoted by $d_T(u, v)$. (Distinguish from the definition given in [2].) The *diameter* of T is denoted by $d(T)$.

The paper [3] deals with diametrically critical graphs and digraphs in general. In this paper we shall study e-critical (i.e. line-critical) and v-critical (i.e. point-critical) tournaments defined in the same way. A *tournament* T with a finite diameter d is called *e-critical* (*v-critical*) if $d(T - x) > d$ for any line $x \in E(T)$ ($d(T - u) > d$ for any point $u \in V(T)$, respectively). A line $x \in E(T)$ is said to be *superfluous* if $d(T - x) = d(T)$. Hence a tournament is e-critical if and only if no its line is superfluous. A common name for both the e-critical and the v-critical tournaments is that in the title of the paper. It is the purpose of this note to study the diametrically critical (mainly e-critical) finite tournaments.

In Figs. 1, 2 and 3 all e-critical tournaments with $p \leqq 6$ points are shown. This can be proved e.g. by exhaustion using a list of all tournaments with $p \leqq 6$ points (e.g. [2], pp. 91–95). In the sequel, however, we shall give also a reasonable proof of this assertion. As can be seen, these three tournaments are also v-critical tournaments with diameter 2. Thus according to the following theorem we can construct infinitely many e-critical as well as v-critical tournaments with diameter 2.

Theorem 1. *Let T be a tournament with $p > 1$ points, where $V(T) = \{v_1, v_2, \ldots, v_p\}$. Let $\bar{T}$ be a tournament with $p + 2$ points, where $V(\bar{T}) = V(T) \cup \{u', u''\}$ and $E(\bar{T}) = E(T) \cup \{u'u'', v_1u', v_2u', \ldots, v_pu', u''v_1, u''v_2, \ldots, u''v_p\}$* (cf. Fig. 4). *Then T is e-critical (v-critical) with diameter 2 if and only if $\bar{T}$ is e-critical (v-critical, respectively) with diameter 2.*

The proof is obvious.

Note that the tournament in Fig. 3 can be constructed from the tournament of Fig. 1 by Theorem 1. A tournament $\overline{T}$ is said to be *reducible* if there is a tournament T such that T and $\overline{T}$ both satisfy the assumptions of Theorem 1; in the opposite case $\overline{T}$ is said to be *irreducible.* So the tournaments of Figs. 1 and 2 are irreducible while the tournament of Fig. 3 is reducible.

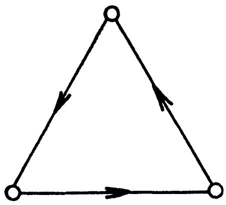

Fig. 1.

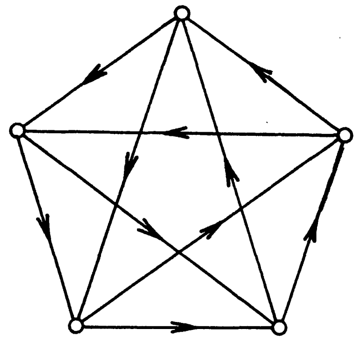

Fig. 2.

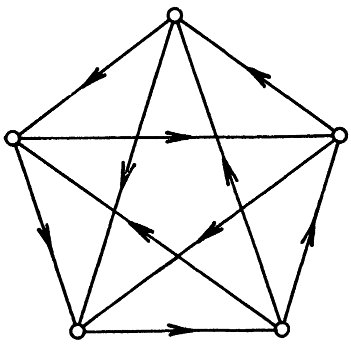

Fig. 3.

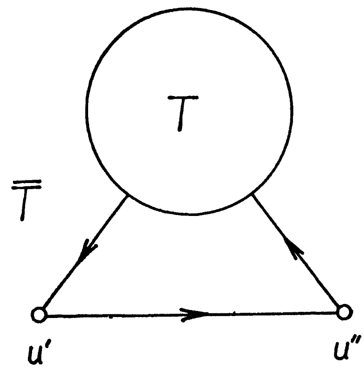

Fig. 4.

Theorem 2. *Except for the two tournaments of Figs.* 1 *and* 2 *every e-critical tournament with diameter* 2 *is reducible.*

Proof. Consider an e-critical tournament T with a diameter $d \geqq 2$. Let $w_1, w_2 \in \in V(T)$ and $x \in E(T)$ be such that

(1) $$d_{T-x}(w_1, w_2) > d$$

with

(2) $$d_T(w_1, w_2) = \min_{y \in E(T)} \min_{\substack{u,v \in V(T) \\ d_{T-y}(u,v) > d}} d_T(u, v) .$$

Put

(3) $$m = d_T(w_1, w_2).$$

The shortest $w_1 - w_2$ path is of the form $u_0 u_1 \dots u_m$, where $u_0 = w_1$, $u_m = w_2$ and $x = u_k u_{k+1}$ for some k with $0 \leqq k \leqq m - 1$. The set $V' = V(T) - \{u_0, u_1, \dots \dots, u_m\}$ can be decomposed into the following four classes:

$$A = \{v \in V' \mid w_1 v,\ w_2 v \in E(T)\}, \quad B = \{v \in V' \mid v w_1,\ v w_2 \in E(T)\},$$
$$C = \{v \in V' \mid v w_1,\ w_2 v \in E(T)\}, \quad D = \{v \in V' \mid w_1 v,\ v w_2 \in E(T)\}.$$

According to (1) we have

(4) $$D = \emptyset$$

and obviously

(5) $$1 \leqq m \leqq d.$$

After this general introduction we shall suppose that T is an e-critical tournament with diameter $d = 2$. So by (5) it is sufficient to consider the following three cases.

(I) $m = 1$.

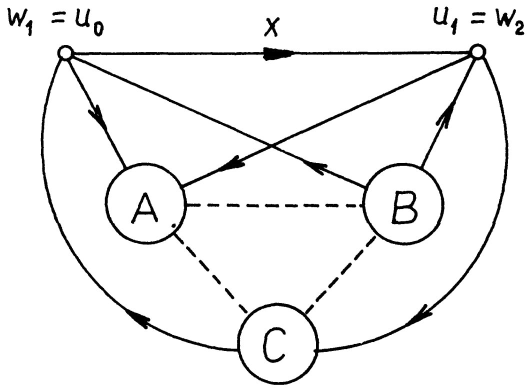

Fig. 5.

It can be easily seen (cf. Fig. 5) that if $A = \emptyset$ and $B \neq \emptyset$, then $d_T(u_0, b) > 2$ for any $b \in B$. Analogously if $A \neq \emptyset$ and $B = \emptyset$, then $d_T(a, u_1) > 2$ for any $a \in A$. If $A = = \emptyset = B$, then either $|C| = 1$ and we have the tournament from Fig. 1, or $|C| > 1$ and T is reducible. Therefore we can assume

(6) $$A \neq \emptyset \neq B.$$

If $C = \emptyset$, then $d_T(u_1, u_0) > 2$ and therefore

(7) $$C \neq \emptyset.$$

One sees that for any line $u_1 a$, where $a \in A$, at least one of the following two assertions is true:

(i) $d_{T-u_1a}(u_1, a) > 2$,

(ii) there is $b \in B$ with $d_{T-u_1a}(u_1, b) > 2$.

As T is a tournament, there is at most one $a \in A$ with (i) but without (ii). If such a point a exists, then it is denoted by a_0 and we have immediately

(8) *for any* $a \in A - \{a_0\}$ *it is* $a_0 a \in E(T)$,

(9) *for any* $c \in C$ *it is* $a_0 c \in E(T)$,

(10) *there is at least one* $b \in B$ *with* $a_0 b \in E(T)$ (for otherwise $d_T(a_0, u_1) > 2$).

Thus for any $a \in A - \{a_0\}$ the assertion (ii) holds. Choose arbitrarily one of such points b and denote it by $f(a)$. In this way we have defined a mapping $f : A - - \{a_0\} \to B$. Obviously

(11) *for any* $a \in A - \{a_0\}$ *we have* $af(a) \in E(T)$, *but* $a' f(a) \notin E(T)$ *whenever* $a' \in A - \{a\}$. *Especially* $f(a) \neq f(a')$ *whenever* $a \neq a'$.

Further we see that

(12) *for any* $f(a)$ *and any* $c \in C$ *it is* $f(a)\, c \in E(T)$.

Now we shall consider the lines bu_0 for $b \in B$. One can find out that there is at most one $b \in B$ with $d_{T-bu_0}(b, u_0) > 2$ and with $d_{T-bu_0}(a, u_0) \leqq 2$ for any $a \in A$. Denoting this point b (if any) by b_0, we see that

(13) *for any* $b \in B - \{b_0\}$ *it is* $bb_0 \in E(T)$,

(14) *for any* $c \in C$ *it is* $cb_0 \in E(T)$,

(15) *there is at least one* $a \in A$ *with* $ab_0 \in E(T)$ (for otherwise $d_T(u_0, b_0) > 2$).

Thus for any $b \in B - \{b_0\}$ there is $a \in A$ with $d_{T-bu_0}(a, u_0) > 2$. One of such points a can be chosen arbitrarily and denoted by $g(b)$. For this mapping $g : B - \{b_0\} \to A$, we have

(16) *for any* $b \in B - \{b_0\}$ *it is* $g(b)\, b \in E(T)$, *but* $g(b)\, b' \notin E(T)$ *whenever* $b' \in \in B - \{b\}$. *Especially,* $g(b) \neq g(b')$ *whenever* $b \neq b'$.

Further we have

(17) *for any* $g(b)$ *and any* $c \in C$ *it is* $cg(b) \in E(T)$.

Now we assert that

(18) *there is no point* $b \in B - \{b_0\}$ *with* $a_0 = g(b)$.

In the opposite case the assertion (17) contradicts (9) (see also (7)). Analogously we have

(19) *there is no point* $a \in A - \{a_0\}$ *with* $f(a) = b_0$.

Put $A - \{a_0\} = \{a_1, a_2, \ldots, a_r\}$ and $B - \{b_0\} = \{b_1, b_2, \ldots, b_s\}$. There are the following five possibilities (a) to (e):

(a) *Neither* a_0 *nor* b_0 *exist.* Then $r \geqq 1$ and $s \geqq 1$ (cf. (6)). Without loss of generality we can put $f(a_i) = b_i$ $(i = 1, 2, \ldots, r)$ (see (11)). Then by (16) we have $g(b_j) = a_j$ $(j = 1, 2, \ldots, r)$, $r = s$ and there is no line $a_i b_j$ except for the case $i = j$ $(i, j = 1, 2, \ldots, s)$.

(b) *The point* a_0 *exists but the point* b_0 *does not exist.* Then we can assume $a_0 b_1 \in E(T)$ (see (10)) and $f(a_i) = b_{i+1}$ $(i = 1, 2, \ldots, r)$ (cf. (11)). However, $g(b_1) \neq a_0$ by (18) and $g(b_1) \neq a_i$ $(i = 1, 2, \ldots, r)$ by (16), i.e. $g(b_1)$ is not defined and therefore the case (b) is not possible.

(c) *The point* b_0 *exists but the point* a_0 *does not exist.* This case is not possible. The proof is similar to that of (b) and can be left to the reader.

(d) *The points* a_0 *and* b_0 *both exist and* $a_0 b_0 \in E(T)$. Then we can assume $f(a_i) = b_i$ $(i = 1, 2, \ldots, r)$ (see (11)). The case $s > r$ is not possible ($g(b_{r+1})$ would not exist by (16)). Hence $r = s$. Further, according to (11) and (16) we have $f(a_i) = b_i$, $g(b_i) = a_i$ $(i = 1, 2, \ldots, s)$ and there is no line $a_i b_j$ except for the case $i = j$ $(i, j = 0, 1, \ldots, s)$.

(e) *The points* a_0 *and* b_0 *both exist and* $a_0 b_0 \notin E(T)$. We can assume $a_0 b_1 \in E(T)$ (see (10)) and $a_1 b_0 \in E(T)$ (see (15)). Then by (11) we can assume $f(a_i) = b_i$ $(i = 2, 3, \ldots, r)$. According to (16) the case $s > r$ is impossible ($g(b_{r+1})$ would not exist). Thus $r = s$ and $g(b_i) = a_i$ $(i = 2, 3, \ldots, s)$. Then $f(a_1) = b_0$ (cf. (11)) which contradicts (19). Hence the case (e) is not possible.

Thus there are only two possible cases, namely (a) and (d). According to (9), (12), (14) and (17) we obtain the direction of every line connecting C with A or B. So the illustration in Fig. 6 can be used for both cases (a) and (d). (Note that there is no line $a_i b_j$ with $i \neq j$ $(i, j = 0, 1, \ldots, s)$.) Now we assert

(20) *if for some* i *and* j *both the lines* $a_i a_j$ *and* $b_i b_j$ *exist, then the line* $b_i a_j$ *is superfluous.*

This fact can be easily verified with the aid of Fig. 6. Indeed, for any $a \in A - \{a_0\}$ and $b \in B - \{b_0\}$ there are paths $b_i u_0 a$, $b u_0 a_j$, $a_i a_j$ and $b_i b_j$ not containing the line $b_i a_j$. Further we assert that

(21) *$T(A)$ and $T(B)$ are acyclic tournaments and moreover, if we put a_i for b_i ($i = 0, 1. \ldots, s$), then $T(A)$ and $T(B)$ appear to be mutually converse tournaments.*

Indeed, if $T(A)$ contains a cycle, then it contains also a cycle of length 3, say, $a_ia_ja_ka_i$. Then by (20) we have $b_jb_i \in E(T)$. However, then the line b_ja_i appears to be superfluous (cf. Fig. 6). (Note that the paths b_jb_i and $a_ja_ka_i$ exist.)

Now we are going to consider the cases (a) and (d).

The case (a). Consider the sink (see (21)), say, a_n in $T(A)$. Then b_n is the source in $T(B)$ and we see that T is reducible (put $a_n = u'$ and $b_n = u''$ in Theorem 1) which contradicts our assumption.

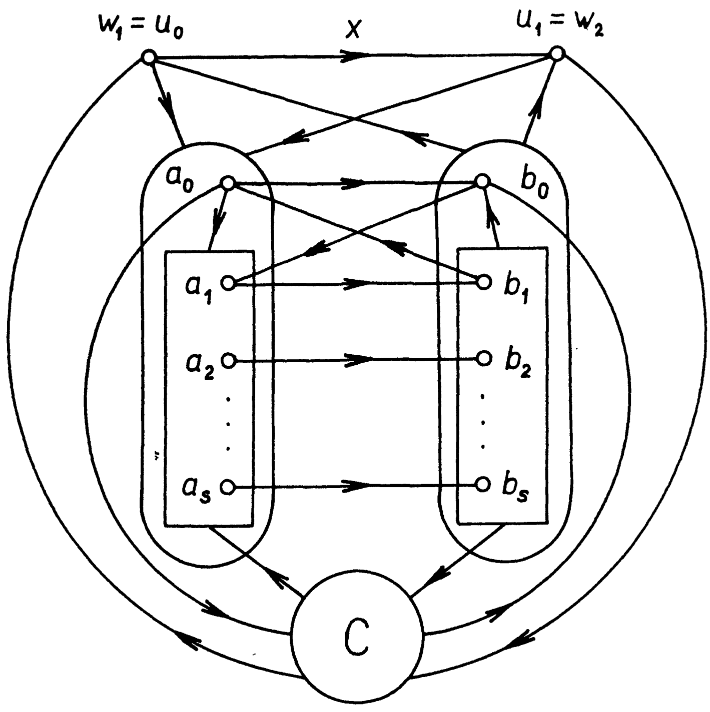

Fig. 6.

The case (d). At first let us consider the case $s = 0$. If $|C| = 1$, then we obtain the tournament in Fig. 2. If $|C| > 1$, then there are $c, c' \in C$ with $c'c \in E(T)$ and the line a_0c appears to be superfluous as can be easily seen. If $s \geqq 1$, then using the same proof as in the case (a) we can show that T is reducible.

This completes the proof of the case (I).

(II) $m = 2$ and $k = 0$.

Then T contains the line $y = u_2u_0$ and we have $d_{T-y}(u_1, u_0) > 2$ or $d_{T-y}(u_2, u_1) > 2$ (cf. Fig. 7). In the first case we have $bu_1 \in E(T)$ for any $b \in B$. If we put $x' = u_1u_2$, then $d_{T-x'}(u_1, u_2) > 2$ which contradicts our assumption on m (see (2) and (3)). The other possibility gives $u_1a \in E(T)$ for any $a \in A$. Then, however, $d_{T-x}(u_0, u_1) > 2$ which contradicts (2) and (3) again. So the case (II) is impossible.

(III) $m = 2$ and $k = 1$.

Like (II) this case appears to be impossible, too.

This completes the proof of Theorem 2.

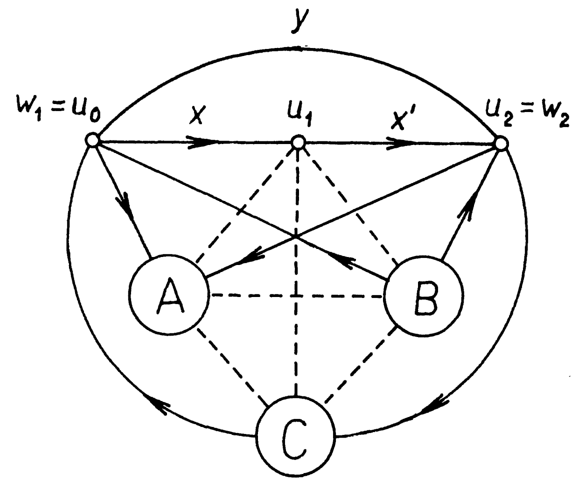

Fig. 7.

Corollary 2.1. *There is no e-critical tournament with diameter 2 and with even number of points.*

Corollary 2.2. *There is exactly one e-critical tournament with diameter 2 and with three points. For every odd $p \geqq 5$, there are exactly two e-critical tournaments with diameter 2 and with p points.*

Proof. It is sufficient to show that for any tournament $\overline{T}$ constructed by Theorem 1, there is exactly one tournament T from which $\overline{T}$ can be constructed. This follows, however, from the properties of the points u' and u''.

Corollary 2.3. *Every e-critical tournament with diameter 2 is v-critical.*

The converse of Corollary 2.3 is not true, however, This can be seen with the aid of the tournament T from Fig. 8. T has diameter 2. Since $d_{T-v_1}(v_6, v_3) > 2$,

$d_{T-v_2}(v_4, v_3) > 2$ and analogously (owing to the symmetry) for all other points, T is v-critical. However, $d(T - v_1v_2) = 2$, i.e., T is not e-critical.

Note that there is no v-critical tournament with four points. Using the tournament from Fig. 8, Theorem 1 and Corollaries 2.2 and 2.3, we have

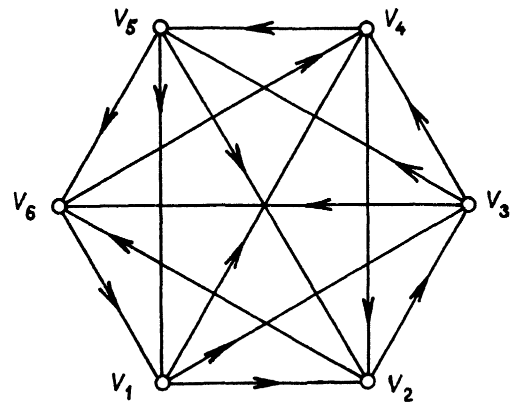

Fig. 8.

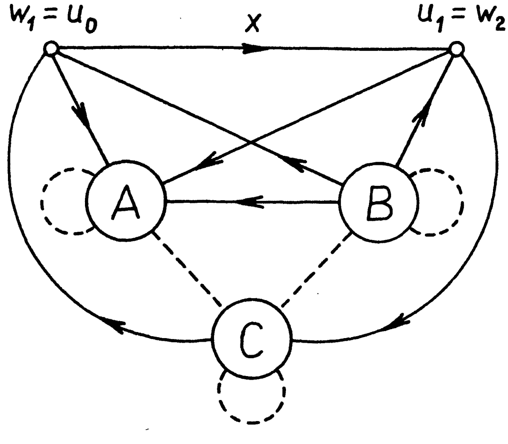

Fig. 9.

Theorem 3. *For $p = 3$ and any integer $p \geqq 5$, there exists a v-critical tournament with diameter 2 and with p points.*

All examples of e-critical tournaments that we have given up to here are with diameter 2. The following theorem justifies this fact.

Theorem 4. *There is no e-critical tournament with diameter $d \geqq 3$.*

Proof. Assume that there is an e-critical tournament T with a diameter $d \geqq 3$. Repeating the reasoning in the beginning of the proof of Theorem 2, we can obtain

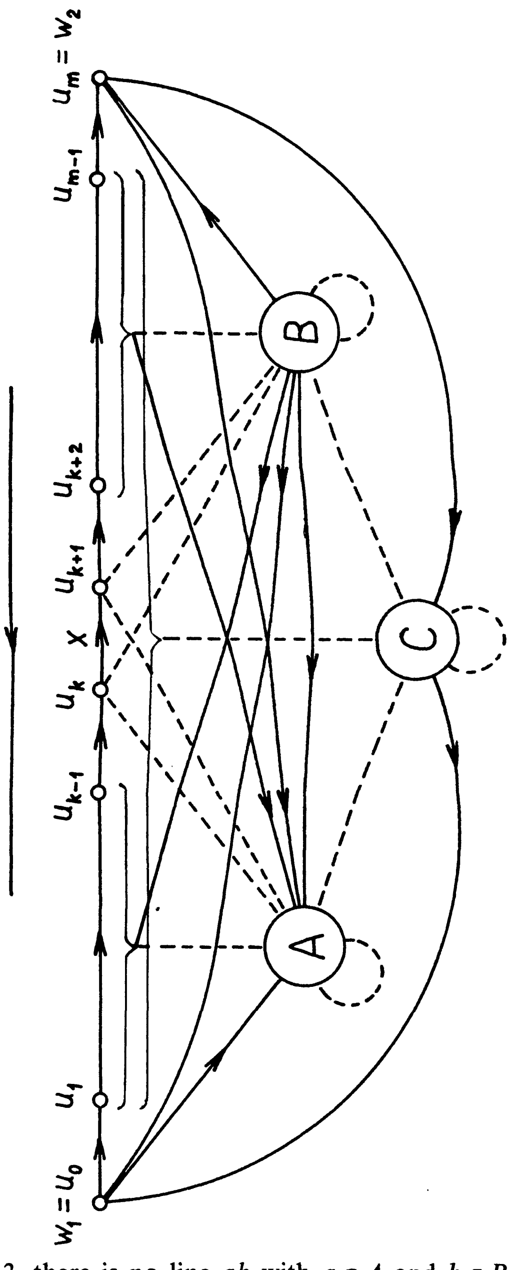

Fig. 10.

(1) to (5). As $d \geqq 3$, there is no line ab with $a \in A$ and $b \in B$. Now it is sufficient to consider the following two cases.

(I) $m = 1$ (cf. Fig. 9).

(The dashed lines in Fig. 9 represent lines with unknown direction.) It is clear that $C \neq \emptyset$ (for otherwise $d_T(u_1, u_0) = \infty$). If $A \cup B = \emptyset$, then either $|C| > 1$ and any line c_1c_2 $(c_1, c_2 \in C)$ is superfluous or $|C| = 1$ and $d(T) = 2$ which is impossible, too. Therefore $A \cup B \neq \emptyset$. If $A = \emptyset$, then any line bu_0 where $b \in B$ is superfluous as can be verified (cf. Fig. 9). If $B = \emptyset$, then any line u_1a where $a \in A$ is superfluous. Thus $A \neq \emptyset$, $B \neq \emptyset$, and $C \neq \emptyset$. Then, however, any line ba, where $a \in A$ and $b \in B$, appears to be superfluous. Hence the case $m = 1$ is impossible.

(II) $m \geqq 2$.

As the path $u_0u_1 \dots u_m$ is a shortest $w_1 - w_2$ path, there is no line u_iu_j whenever $j - i \geqq 2$. According to (1), there is no line u_ib with $i \leqq k - 1$, $b \in B$, and no line au_j with $a \in A$, $j \geqq k + 2$. These facts are illustrated by Fig. 10. (In this figure, any full line has priority over a dashed line, e.g. if $k = 0$, then all lines u_ka with $a \in A$ exist.) If $A = \emptyset$, then $d_{T-u_0u_1}(u_0, u_1) = \infty$, i.e., by (2) and (3) we have $m = 1$ which contradicts our assumption. Therefore $A \neq \emptyset$. Analogously $B \neq \emptyset$ (for otherwise it would be $d_{T-u_{m-1}u_m}(u_{m-1}, u_m) = \infty$).

Now we assert that any line $z = b_0a_0$, where $a_0 \in A$ and $b_0 \in B$, is superfluous. As $d_{T-z}(b, a) \leqq 2$ for any $a \in A$ and any $b \in B$, it is sufficient to verify that

(i) for any path of the form b_0a_0v, where $v \notin A$, there is a $b_0 - v$ path of length not exceeding 2 and not containing the line z. This is clear (cf. Fig. 10) except the case $v = u_{k+1}$ with $k + 1 = m - 1$. In this case, however, there is no line u_kw with $w \in B$ (existence of such line would contradict (1)). Thus $b_0u_ku_{k+1}$ is the required path.

(ii) for any path of the form vb_0a_0, where $v \notin B$ there is a $v - a_0$ path of length not exceeding 2 and not containing the line z. This can be easily verified (cf. Fig. 10) except the case $v = u_1 = u_k$. In this case, however, there is no line wu_2 with $w \in A$ (see (1)). So we can take the path $u_1u_2a_0$.

Hence neither the case $m \geqq 2$ is possible and the theorem is proved.

Thus we have the full characterization of all e-critical tournaments. Nevertheless, we have not succeeded in proving or disproving the existence of a v-critical tournament with diameter $d \geqq 3$. We conjecture that there exists an integer d_0 such that there is no v-critical tournament with diameter $d \geqq d_0$.

References

[1] *Harary, F.:* Graph theory. Addison-Wesley, Reading 1969.
[2] *Moon, J. W.:* Topics on tournaments. Holt, Rinehart and Winston, New York 1968.
[3] *Plesník, J.:* Critical graphs of given diameter. Acta Fac. R. N. Univ. Math. *30* (1975), 71—93.

Author's address: 816 31 Bratislava, Mlynská dolina, (Prírodovedecká fakulta UK).