Alexei Rodionov

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EXPLICIT SOLUTION FOR LAMÉ AND OTHER PDE SYSTEMS*

Alexei Rodionov, Medelin

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Abstract. We provide a general series form solution for second-order linear PDE system with constant coefficients and prove a convergence theorem. The equations of three dimensional elastic equilibrium are solved as an example. Another convergence theorem is proved for this particular system. We also consider a possibility to represent solutions in a finite form as partial sums of the series with terms depending on several complex variables.

Keywords: systems of linear partial differential equations, constant coefficient, explicit solutions, several complex variables, elasticity

MSC 2000: 35E20, 34A05, 32A99

1. Introduction

The series $W = \sum\limits_{|n|=0}^{\infty} \bar{z}^n W_n(z)$ known as holomorphic expansions (HE) were applied in [1] for solving PDEs. In the present paper we apply them to linear second order PDE systems with constant coefficients. The series form solution for a wide class of systems is constructed and the convergence theorem is proved. Some systems relevant to a number of physical and technical problems are not covered by this theorem and need special consideration. We treat threedimensional Lamé's elasticity system as an example and investigate the convergence of the HE solution for this particular case.

The possibility to present solutions in a finite form as partial sums of holomorphic expansions seems attractive. In this article we study polynomial solutions and compare our results with those published in [2], [3]. We also study finite solutions of another nature defined by arbitrary holomorphic functions. Kolosov-Muskhelishvili formula for the plane elastic equilibrium is shown to be a special case of finite solution for the above mentioned Lamé's system.

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We do not consider boundary value problems but just systems of PDEs. The type and dimension do not matter.

The following notation is used:

$x = (x_1, x_2, \dots, x_{m+1})$—vector in $\mathbb{R}^{m+1}$;

$l = (l_1, l_2, \dots, l_{m+1})$—differential indicator with respect to x;

$\partial^{|l|}/\partial x^l = \partial^{|l|}/\partial x_1^{l_1} \dots x_{m+1}^{l_{m+1}}$;

$z = (z_1, z_2, \dots, z_m)$—vector in $\mathbb{C}^m$;

$n = (n_1, n_2, \dots, n_m)$ where n_j are non negative integers;

$z^n = z_1^{n_1} z_2^{n_2} \dots z_m^{n_m}$;

$\zeta = (z_2, z_3, \dots, z_m)$;

$\nu = (n_2, n_3, \dots, n_m)$;

$|n| = n_1 + n_2 + \dots + n_m$.

We use Cauchy operators

$d_z^k = 2^{-k}(\partial/\partial x - \mathrm{i}\partial/\partial y)^k$; $\quad d_{\bar{z}}^k = 2^{-k}(\partial/\partial x + \mathrm{i}\partial/\partial y)^k$ if $z = x + \mathrm{i}y$ and $k \in \mathbb{N}$;

$d_z^n = d_{z_1}^{n_1} d_{z_2}^{n_2} \dots d_{z_m}^{n_m}$;

$d_\zeta^\nu = d_{z_2}^{n_2} d_{z_3}^{n_3} \dots d_{z_m}^{n_m}$;

G—simply connected domain in $\mathbb{C}^m$;

G_1—some compact in G;

$H(G) = \{(f_1(z), f_2(z), \dots, f_k(z))\colon\ f_j(z)$ a function holomorphic in $G\}$. The case $k = 1$ is also considered;

$\|z\| = \max\limits_j |z_j|$;

$\|f(z)\| = \max\limits_{j \leqslant k,\, z \in G_1} |f_j(z)|$ for $f(z) \in H(G)$;

$\|A\| = \max\limits_i \sum\limits_j |a_{i,j}|$.

Unfortunately some symbols used in this article have more than one meaning but the context will make things clear.

2. Formal solution for second-order linear PDE system with constant coefficients

Let us consider the system of k second order constant coefficients PDEs

$$Lu = (P + Q + R)u = f \tag{1}$$

with $u = (u_1(x), u_2(x), \dots, u_k(x))$, $x \in D \subset \mathbb{R}^{m+1}$ and $f = (f_1(x), \dots, f_k(x))$. The operators P, Q and R are homogeneous partial differential operators of order 0, 1 and 2, respectively. They have the forms $Q = \sum\limits_{|l|=1} Q_l \partial/\partial x^l$, $R = \sum\limits_{|l|=2} R_l \partial^2/\partial x^l$ where P, R_l, Q_l are $k \times k$ real matrices.

In the domain $G = \{(x,y)\colon x \in D,\, y = (y_1, y_2, \dots, y_{m-1}) \in D_1 \subset \mathbb{R}^{m-1}\}$ the system (1) is equivalent to

$$Lw(x,y) = \varphi(x,y), \quad \varphi(x,y) = f(x) \quad \forall\,(x,y) \in G,$$
$$\frac{\partial w}{\partial y_j} = 0, \quad j = 1, 2, \dots, m-1. \tag{2}$$

Let us define complex variables $z_1 = x_1 + \mathrm{i}x_2$, $z_2 = x_3 + \mathrm{i}y_1, \dots, z_m = x_{m+1} + \mathrm{i}y_{m-1}$ and consider G to be a simply connected domain in $\mathbb{C}^m$. The exchange of real partial differentiation in L for the Cauchy operators yields

$$\begin{aligned} L = {} & A d_{\bar z_1}^2 + B d_{\bar z_1} d_{z_1} + d_{\bar z_1} \sum_{|\nu|=1} C_\nu d_\zeta^\nu + D d_{\bar z_1} \\ & + \bar A d_{z_1}^2 + d_{z_1} \sum_{|\nu|=1} \overline{C}_\nu d_\zeta^\nu + \overline{D} d_{z_1} + \sum_{|\nu|=2} E_\nu d_\zeta^\nu + \sum_{|\nu|=1} F_\nu d_\zeta^\nu + P \end{aligned} \tag{3}$$

where

$$\begin{aligned} A &= R_{(2)} - R_{(0,2)} - \mathrm{i}R_{(1,1)}, & B &= 2(R_{(2)} + R_{(0,2)}), \\ C_\nu &= 2(R_{(1,0,\nu)} - \mathrm{i}R_{(0,1,\nu)}), & D &= Q_{(1,0)} - \mathrm{i}Q_{(0,1)}, \\ E_\nu &= 4R_{(0,0,\nu)}, & F_\nu &= 2Q_{(0,0,\nu)}. \end{aligned} \tag{4}$$

All the subindexes of the real matrices on the right-hand side of (4) are vectors of dimension $m+1$. Note that we do not write right zeros in subindexes. It means, for instance, that $R_{(0,2)}$ should be understood as $R_{(0,2,0,\dots,0)}$. The elements of $\bar A$, $\overline{C}_\nu$, $\overline{D}$ are conjugate to those of A, C_ν, D.

We can apply the operator L to a complex variable, complex valued vector function $W(z) = (W^1, W^2, \dots, W^k)$, $\Re(W^j) = w_j$, $\Im(W^j) = v_j$ and consider the system

$$LW(z) = g(z),$$
$$(d_{z_j} - d_{\bar z_j})W(z) = 0, \quad j = 2, \dots, m, \tag{5}$$

where $g(z)$ is a complex valued function with real (imaginary) part equal to $\varphi(x,y)$ in (2). We call (5) the complex analog for (2) because the real (imaginary) part of W satisfies (2). Throughout this paper we concentrate on the study of the complex analog.

R e m a r k 1. The complex analog for (2) is not unique. We can organize complex variables in a different way and obtain a system different from (5). We also have some freedom to choose $g(z)$ in (5) because we only need the real or the imaginary part of this function to be equal to $\varphi(x,y)$ in (2).

Let us search for coefficients of the formal HE solution of (5), that is, for the vector functions $W_n(z) = (W_n^1(z), W_n^2(z), \ldots, W_n^k(z)) \in H(G)$ which make the sum of the series

$$W = \sum_{|n|=0}^{\infty} \bar{z}^n W_n(z) \tag{6}$$

satisfy (5). All the necessary information on holomorphic expansions can be found in [1].

Substituting (6) in (5), we obtain the following equations for the HE coefficients:

$$A(n_1+2)(n_1+1)W_{n_1+2} - (n_1+1)M_1 W_{n_1+1} - M_0 W_n = g_n, \tag{7}$$

and

$$W_{n_j+1} = \frac{1}{n_j+1} d_{z_j} W_n, \quad j = 2, 3, \ldots, m. \tag{8}$$

Henceforth, we will write W_{n_j+1} for $W_{n_1,n_2,\ldots,n_{j-1},n_j+1,n_{j+1},\ldots,n_m}$ in order to shorten the expressions and we consider $g(z)$ to have a holomorphic expansion

$$g(z) = \sum_{|n|=0}^{\infty} \bar{z}^n g_n(z).$$

The detailed forms of the operators M_1 and M_0 are

$$-M_0 = \bar{A} d_{z_1}^2 + d_{z_1} \sum_{|\nu|=1} \overline{C}_\nu d_\zeta^\nu + \overline{D} d_{z_1} + \sum_{|\nu|=2} E_\nu d_\zeta^\nu + \sum_{|\nu|=1} F_\nu d_\zeta^\nu + P \tag{9}$$

and

$$-M_1 = B d_{z_1} + \sum_{|\nu|=1} C_\nu d_\zeta^\nu + D. \tag{10}$$

The equalities (7) and (8) are necessary and sufficient for W to be a formal solution of (5). See [1] for the proof.

3. Convergence theorem

This section deals with the homogeneous system (5) and we consider the matrix A in (3) to be nonsingular. From (7) we find

$$W_{n_1+2,0} = \frac{1}{(n_1+2)(n_1+1)} A^{-1}((n_1+1)M_1 W_{n_1+1,0} + M_0 W_{n_1,0}) \tag{11}$$

and (8) yields

$$W_{n_1,\nu} = \frac{1}{\nu!} d_\zeta^\nu W_{n_1,0}. \tag{12}$$

These equalities hold for all non negative integers n_1 and all ν as well. They establish one-to-one correspondence between the set of formal HE solutions for (5) and the Cartesian square of $H(G)$. In fact, provided $W_0 \in H(G)$ and $W_1 \in H(G)$ we put $W_{0,0,\ldots,0} = W_0$, $W_{1,0,\ldots,0} = W_1$ and use (11) to find $W_{n_1,0,\ldots,0}$ for $n_1 > 1$. The equality (12) defines $W_n = W_{n_1,\nu}$. The reverse correspondence is evident.

The convergence of the formal solution depends on the norms of W_n. We evaluate them and the norms of all other functions on some fixed compact $G_1 \subset G$.

Lemma 1. *Let $V \in H(G)$ satisfy the inequalities $\|d_z^n V\| < at^{|n|}$ for all n and some fixed constants a and $t \geqslant 1$. Let $P = \sum\limits_n A_n d_z^n$ be a differential polynomial of degree N_1, the total number of terms being N_2 and $\Lambda = \max\limits_n \|A_n\|$. Then, the inequality $\|PV\| \leqslant a\Lambda N_2 t^{N_1}$ holds.*

Proof. $\|PV\| \leqslant \sum\limits_n \|A_n\| \, \|d_z^n V\| \leqslant at^{N_1} \sum\limits_n \|A_n\| \leqslant a\Lambda N_2 t^{N_1}$. $\square$

Examples of functions satisfying the conditions of Lemma 1.

(1) $V(z) = (V_1(z), V_2(z), \ldots, V_k(z))$ where $V_j(z)$ are polynomials. This function has only a finite number of nonzero derivatives uniformly bounded on G_1 by some constant a. Take $t = 1$.

(2) Let $a_1, \ldots, a_k$ be fixed vectors in $\mathbb{C}^m$. We write za_j for the inner product in $\mathbb{C}^m$. The function $H(G) \ni V(z) = (\exp(za_1), \exp(za_2), \ldots, \exp(za_k))$ has derivatives $d_z^n V(z) = (\bar{a}_1^n \exp(za_1), \bar{a}_2^n \exp(za_2), \ldots, \bar{a}_k^n \exp(za_k))$. Take $t = \max\limits_j \{1, \|a_j\|\}$ and $a = \|V\|$.

The list of examples can be enlarged easily.

In the next theorem we state sufficient conditions for the convergence of the formal solution for the system (5).

Theorem 1. *Let $W_1 \in H(G)$ and $W_2 \in H(G)$ satisfy the conditions of Lemma* 1. *The formal* HE *solution for* (5) *defined by W_1 and W_2 converges uniformly on G_1.*

P r o o f. We write $W_{n_1,0}$ for $W_{n_1,0,\dots,0}$. The expression for $W_{n_1,0}$ provided by (11) is a sum of differential operators applied to W_0 or W_1. We have

$$W_{2,0} = \frac{1}{2}(A^{-1}M_1W_1 + A^{-1}M_0W_0),$$
$$W_{3,0} = \frac{1}{6}(A^{-1}A^{-1}M_1^2W_1 + A^{-1}A^{-1}M_1M_0W_0 + A^{-1}M_0W_1),$$

and so on. There are no similar terms and hence $\mathrm{Num}(W_{n_1,0}) = \mathrm{Num}(W_{n_1-1,0}) + \mathrm{Num}(W_{n_1-2,0})$, where $\mathrm{Num}(W_{n_1,0})$ denotes the number of summands in the expression for $W_{n_1,0}$. Consequently,

$$W_{n_1,0} = \frac{1}{n_1!}\sum_{k=1}^{F_{n_1+1}} T_kW_j.$$

Here F_k is the kth Fibonacci number, W_j is either W_0 or W_1 and T_k are differential operators depending on k and also on n_1 but the last dependence is not important for us.

Every T_k is the product of matrices A^{-1} and operators M_1 and M_0. The number of factors A^{-1} is less than n_1 and the total number of M_1 and M_0 is less than n_1 as well.

The operator M_0 defined by (9) is a differential polynomial of degree 2 and the number of terms is equal to $\frac{1}{2}(m+1)(m+2)$. The operator M_1 has degree 1 and $m+1$ terms. Consequently, the degree of T_k does not surpass $2n_1$ and the number of terms is less than $(\frac{1}{2}(m+1)(m+2))^{n_1}$. The number of matrix factors in the coefficients of T_k is less than $2n_1$. We denote by α_1 the maximum norm of the matrices encountered in (9), (10) and put $\alpha = \max\{\alpha_1, \|A^{-1}\|, 1\}$. Then the inequality $\|AB\| \leqslant \|A\|\,\|B\|$ and Lemma 1 imply

$$\|T_kW_j\| \leqslant a\Big[\alpha^2t^2\frac{(m+1)(m+2)}{2}\Big]^{n_1}.$$

Binet's formula yields

$$F_{n_1} = \frac{((1+\sqrt{5})^{n_1} - (1-\sqrt{5})^{n_1})}{\sqrt{5}\,2^{n_1}} \leqslant 2\cdot 4^{n_1}\cdot 2^{-1}\cdot 2^{-n_1} = 2^{n_1}$$

and the inequality

$$\|W_{n_1,0}\| \leqslant \frac{1}{n_1!}2a[\alpha^2t^2(m+1)(m+2)]^{n_1}$$

holds. We put $\beta = \alpha^2 t^2 (m+1)(m+2)$ and conclude the proof with the estimation

$$\|W_n\| = \|W_{n_1,\nu}\| \leqslant \frac{1}{n_1!} 2a[\alpha^2 t^2 (m+1)(m+2)]^{n_1} t^{|\nu|} \leqslant \frac{1}{n_1!} 2a\beta^{|n|}$$

which is sufficient for the uniform convergence of (6). $\square$

The convergence theorem for a non-homogeneous system is not difficult to prove. It can be done using the estimates from Theorem 1 and the theory developed in [1]. The condition $\|g_n(z)\| \leqslant at^{|n|}$ will do perfectly well.

Remark 2. The results of [1] ensure the existence and uniform convergence of the derivatives of the convergent HE solution.

4. Threedimensional Lamé's system for the linear elasticity problem

In the previous section we restricted the class of systems assuming the matrix A to be nonsingular. This does not hold, however, in many cases which are particularly important for applications. One of these is the threedimensional Lamé's system

$$(\lambda + \mu)\frac{\partial \Theta}{\partial x_j} + \mu \Delta u_j = 0, \quad j = 1, 2, 3, \tag{13}$$

where λ and μ are Lamé's constants and $\Theta = \sum\limits_{j=1}^{3} \partial u_j / \partial x_j$. This system describes linear displacements in elastic medium.

We put $a = \lambda + \mu$, $b = \mu$ and transform (13) to the form (1). The nonzero matrices involved in the operators P, Q and R are

$$R_{(2,0,0)} = \begin{pmatrix} a+b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \qquad R_{(0,2,0)} = \begin{pmatrix} b & 0 & 0 \\ 0 & a+b & 0 \\ 0 & 0 & b \end{pmatrix},$$

$$R_{(0,0,2)} = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a+b \end{pmatrix}, \qquad R_{(1,0,1)} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix},$$

$$R_{(0,1,1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix}, \qquad R_{(1,1,0)} = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the system (13) we have $k = 3$, $m = 2$, $x = (x_1, x_2, x_3)$, $z = (z_1, z_2)$, $n = (n_1, n_2)$ and we enjoy the possibility to write sub-indices and degrees in their complete form without making expressions too awkward. It can be noticed that the matrix A defined by (4) is singular and we cannot use (11). We proceed by taking into account that the complex analog (5) is not just an algebraic system and there is no need to consider $d_{z_1}W$ and $d^2_{z_1}W$ as independent values. We construct the complex analog for (13) following the steps described in Section 1 and change unknown functions $V^1 = W^1 + \mathrm{i}W^2$, $V^2 = W^1 - \mathrm{i}W^2$ and $V^3 = 2W^3$. The first system in the complex analog transforms to

$$\begin{aligned}(14)\qquad & ad^2_{\bar z_1}V^2 + (a+2b)d_{\bar z_1}d_{z_1}(V^1+V^2) + ad_{\bar z_1}d_{z_2}V^3 \\ & \quad + ad^2_{z_1}V^1 + 2bd^2_{z_2}(V^1+V^2) + ad_{z_1}d_{z_2}V^3 = 0,\\ (15)\qquad & ad^2_{\bar z_1}V^2 + (a+2b)d_{\bar z_1}d_{z_1}(V^1-V^2) + ad_{\bar z_1}d_{z_2}V^3 \\ & \quad - ad^2_{z_1}V^1 + 2bd^2_{z_2}(V^1-V^2) - ad_{z_1}d_{z_2}V^3 = 0,\\ (16)\qquad & ad_{\bar z_1}d_{z_2}V^2 + bd_{\bar z_1}d_{z_1}V^3 + ad_{z_1}d_{z_2}V^1 + (a+b)d^2_{z_2}V^3 = 0,\end{aligned}$$

and the second system stays unchanged,

$$(17)\qquad d_{z_2}V^j = d_{\bar z_2}V^j.$$

These equations enable us to express $d_{\bar z_1}V^j$ in terms of derivatives with respect to z_j but not to $\bar z_j$. We subtract (15) from (14) to find $d_{\bar z_1}V^2$. The equation (16) provides $d_{\bar z_1}V^3$ and finally, from (14), we have $d_{\bar z_1}V^1$. The explicit expressions are

$$\begin{aligned}(18)\qquad d_{\bar z_1}V^1 &= -\frac{1}{a+2b}(2bJd^2_{z_2}V^1 + aJ^3d^4_{z_2}V^2 - aJ^2d^3_{z_2}V^3),\\ (19)\qquad d_{\bar z_1}V^2 &= -\frac{1}{a+2b}(ad_{z_1}V^1 + 2bJd^2_{z_2}V^2 + ad_{z_2}V^3),\\ (20)\qquad d_{\bar z_1}V^3 &= -\frac{1}{a+2b}(2ad_{z_2}V^1 - 2aJ^2d^3_{z_2}V^2 + (3a+2b)Jd^2_{z_2}V^3),\end{aligned}$$

where $J\varphi(z) = \int_0^{z_1}\varphi(\xi,\zeta)\,\mathrm{d}\xi$. To find the coefficients of the HE solution for (14)–(17) we substitute the holomorphic expansion of $V^j = \sum\limits_{n_1,n_2} \bar z_1^{n_1}\bar z_2^{n_2}V^j_{n_1,n_2}$ into (17)–(20), group similar terms, equate to zero the coefficients at $\bar z_1^{n_1}\bar z_2^{n_2}$ and solve the equations obtained with respect to $V^j_{n_1+1,n_2}$ and $V^j_{n_1,n_2+1}$. The solutions are

$$\begin{aligned}V^j_{n_1,n_2+1} &= \frac{d_{z_2}V^j_{n_1,n_2}}{n_2+1},\\ V^1_{n_1+1,n_2} &= -\frac{2bJd^2_{z_2}V^1_{n_1,n_2} + aJ^3d^4_{z_2}V^2_{n_1,n_2} - aJ^2d^3_{z_2}V^3_{n_1,n_2}}{(a+2b)(n_1+1)},\\ V^2_{n_1+1,n_2} &= -\frac{ad_{z_1}V^1_{n_1,n_2} + 2bJd^2_{z_2}V^2_{n_1,n_2} + ad_{z_2}V^3_{n_1,n_2}}{(a+2b)(n_1+1)},\\ V^3_{n_1+1,n_2} &= -\frac{2ad_{z_2}V^1_{n_1,n_2} - 2aJ^2d^3_{z_2}V^2_{n_1,n_2} + (3a+2b)Jd^2_{z_2}V^3_{n_1,n_2}}{(a+2b)(n_1+1)}.\end{aligned}$$

The explicit expressions for V_{n_1,n_2} are given by

$$(21)\quad V_{n_1,n_2}=\frac{(-1)^{n_1}}{n_1!n_2!}d_{z_2}^{n_2}(J^3d_{z_2}^4A_1+J^2d_{z_2}^3A_2+Jd_{z_2}^2A_3+d_{z_1}A_4+d_{z_2}A_5)^{n_1}V_{0,0},$$

where

$$A_1=\frac{a}{a+2b}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\qquad A_2=\frac{-a}{a+2b}\begin{pmatrix}0&0&1\\0&0&0\\0&2&0\end{pmatrix},$$

$$A_3=\frac{2b}{a+2b}\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{3a+2b}{2b}\end{pmatrix},\qquad A_4=\frac{a}{a+2b}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},$$

$$A_5=\frac{a}{a+2b}\begin{pmatrix}0&0&0\\0&0&1\\2&0&0\end{pmatrix}.$$

We introduce the operator $T_r=d_{z_1}^{i(1)}J^{j(1)}\dots d_{z_1}^{i(m)}J^{j(m)}=\prod\limits_{k=1}^{m}d_{z_1}^{i(k)}J^{j(k)}$. Here $i(k)$ and $j(k)$ take non negative integer values and $r=\sum\limits_{k=1}^{m}(i(k)+j(k))$. The operator T_r is not wholly defined by r but it is not important in what follows.

For the convergence theorem we consider G_1 to be the closure of a simply connected bounded domain which guarantees the estimate $\|JV\|\leqslant R\|V\|\ \forall\, V\in H(G)$ and some fixed constant R.

Lemma 2 will be proved for a function holomorphic in the domain G of an arbitrary dimension though in this paper we need it for the functions of two variables only.

Lemma 2. *For a function V satisfying the conditions of Lemma 1, the operator T_r defined above and the constants R, $\beta=a\mathrm{e}^{Rt}$ and $\tau=\max\{t,R\}$ the inequality $\|T_rd_z^nV\|\leqslant\beta\tau^{|n|+r}$ holds.*

Proof. For each factor in the operator T_r we have $d_{z_1}^{i(k)}J^{j(k)}=d_{z_1}^{i(k)-j(k)}$ if $i(k)\geqslant j(k)$ or $d_{z_1}^{i(k)}J^{j(k)}=J^{j(k)-i(k)}$ if the reverse inequality holds. Therefore $T_r=\prod\limits_{k=1}^{l}d_{z_1}^{i(k,l)}J^{j(k,l)}$ and $l\leqslant[\frac12m]+1$. Here $i(k,l)$ and $j(k,l)$ take non negative integer values.

To conserve the form of the operator we add $d_{z_1}^0$ to the first term or J^0 to the last one if necessary. The inequality $\sum\limits_{k=1}^{l}(i(k,l)+j(k,l))\leqslant r$ takes place. We follow with the process and arrive at the equality $T_r=d_{z_1}^iJ^j$ or $T_r=J^jd_{z_1}^i$ with $i+j\leqslant r$.

If $T_r = d_{z_1}^i J^j$ we have

$$\|T_r d_z^n V\| = \begin{cases} \|d_{z_1}^{i-j} d_z^n V\| \leqslant a t^{|n|+i-j} \leqslant a t^{|n|+r}, & \text{if } i \geqslant j, \\ \|J^{j-i} d_z^n V\| \leqslant R^{j-i} \|d_z^n V\| \leqslant a R^r t^{|n|}, & \text{if } j \geqslant i. \end{cases}$$

Considering the second possibility, namely $T_r = J^j d_{z_1}^i$ with $j \geqslant i$, and keeping in mind that $|z_1| \leqslant R$ we get

$$\begin{aligned} \|T_r d_z^n V\| = \|J^j d_{z_1}^i d_z^n V\| &= \left\| J^{j-i} \left(d_z^n V(z) - \sum_{k=0}^{i-1} \frac{1}{k!} z_1^k d_{z_1}^k d_z^n V(0,\zeta) \right) \right\| \\ &\leqslant R^{j-i} \left\| \sum_{k=i}^{\infty} \frac{1}{k!} z_1^k d_{z_1}^k d_z^n V(0,\zeta) \right\| \leqslant a R^{j-i} R^i t^{i+|n|} \sum_{k=0}^{\infty} \frac{1}{k!} R^k t^k \\ &\leqslant a \mathrm{e}^{Rt} R^j t^{i+|n|}. \end{aligned}$$

If $i \geqslant j$ we also have

$$\|T_r d_z^n V\| \leqslant a \mathrm{e}^{Rt} R^j t^{i+|n|},$$

and with the evident inequality $|i+j| \leqslant r$ we complete the proof. $\square$

Theorem 2. *The* HE *solution for the system* (14)–(17) *converges uniformly on the compact* G_1 *if* $V_{0,0}$ *satisfies the conditions of* Lemma 1.

P r o o f. We expand the expression (21) considering $n_2 = 0$. It consists of 5^{n_1} summands, each one of form

$$T = (J^3 d_{z_2}^4 A_1)^{i_1} (J^2 d_{z_2}^3 A_2)^{i_2} (J d_{z_2}^2 A_3)^{i_3} (d_{z_1} A_4)^{i_4} (d_{z_2} A_5)^{i_5}.$$

The factors can change their places but the sum of the degrees is constant and equal to n_1. The operators d_{z_1}, d_{z_2} and J commute with matrices A_j and d_{z_2} commutes with d_{z_1} and J so that we can transform T in

$$T = T_r d_{z_2}^{4i_1+3i_2+2i_3+i_5} \prod_{j=1}^{5} A_j^{i_j},$$

where $r = 3i_1 + 2i_2 + i_3 + i_4$. Applying Lemmas 1–2 we estimate

$$\|T V_{0,0}\| \leqslant \beta \tau^i \prod_{j=1}^{5} \|A_j\|^{i_j}$$

with $i = 7i_1 + 5i_2 + 3i_3 + i_4 + i_5$ and further

$$\|TV_{0,0}\| \leqslant \beta\tau^{7n_1}\alpha^{n_1},$$

where $\alpha = \max\limits_{j\leqslant 5}\{\|A_j\|, 1\}$.

The evident inequalities

$$\|V_{n_1,0}\| \leqslant \frac{1}{n_1!}\beta\tau^{7n_1}\alpha^{n_1}5^{n_1}$$

and

$$\|V_{n_1,n_2}\| \leqslant \frac{1}{n_1!n_2!}\beta(5\tau^7\alpha)^{n_1}t^{n_2}$$

complete the proof. □

5. Finite solutions

The possibility to construct solutions expressed in terms of elementary functions without any limit process (series summation, definite integration and so on) is attractive for scientists and those engaged in symbolic computation [4]. The results in this area cover mostly polynomial solutions. The HE techniques also permit construction of polynomial solutions.

Let us go back to the homogeneous system (5) and consider the case $P = D = 0$. We observe that each summand on the right-hand sides of (11) and (12) contains derivatives with respect to some z_j. To construct the HE solution for (5) we start with two arbitrary initial functions W_0 and W_1 and consequently apply (11) and (12) so that every additive term in the expanded expression for W_n contains differentiation and the order of derivatives grows uniformly and indefinitely when $|n|$ increases. For polynomials W_0 and W_1 this means $W_n \equiv 0$ for $|n|$ big enough. The solution achieved is a polynomial. Vice versa, each polynomial solution for (5) has a holomorphic expansion and the coefficients can be found by (11), (12) starting with appropriate polynomials W_0 and W_1. This does not add anything new to the theory developed in [2], [3]. The results presented in the above mentioned papers are more general and we just propose an alternative technique.

Imposing some more restrictions on (5) we construct non-polynomial finite solutions. If all the matrices in (9) and (10) are null except E_ν and F_ν and the matrix B is nonsingular then we transform (11) and (12) to

$$W_{n_1,\nu} = \frac{1}{\nu!}d_\zeta^\nu W_{n_1,0}$$

and

$$W_{n_1,0} = \frac{(-1)^{n_1}}{n_1!} B^{-n_1} J^{n_1} \Bigl(\sum_{|\nu|=2} E_\nu d_\zeta^\nu + \sum_{|\nu|=1} F_\nu d_\zeta^\nu \Bigr)^{n_1} W_0.$$

The characteristic of this case is that all the summands contain derivatives with respect to z_ν and starting with $W_0 = \varphi(z_1)\zeta^\nu$ we come to the finite solution which is not necessarily a polynomial with respect to z_1.

Considering Lamé's system we easily come to a polynomial solution if $V_{0,0}$ in (21) is a polynomial. The argumentation is the same as in the general case. The possibility to construct a non polynomial solution is not so evident and is due to the fact that the operator d_{z_1} in (21) has a nilpotent matrix A_4 as a coefficient. Hence each additive term in the expanded expression for V_{n_1,n_2} has a derivative with respect to z_2 for $n_1 + n_2$ big enough and the order of derivatives grows uniformly and indefinitely when $n_1 + n_2$ grows. So for every $V_{0,0} = \varphi(z_1) z_2^k$ a finite solution appears and it is not necessarily a polynomial with respect to z_1. An example of such a solution for Lamé's system is

$$\begin{aligned}
u_1 &= \Re\Bigl(-\frac{2(\lambda+\mu)}{\lambda+3\mu} x_3 \bar{z}_1 \tan^2 z_1 \Bigr),\\
u_2 &= \Im\Bigl(\frac{2(\lambda+\mu)}{\lambda+3\mu} x_3 \bar{z}_1 \tan^2 z_1 \Bigr),\\
u_3 &= \Re\Bigl(2x_3^2 \tan^2 z_1 - \frac{3\lambda+5\mu}{\lambda+3\mu} (\tan z_1 - z_1) \bar{z}_1 \Bigr).
\end{aligned}$$

We can separate the parts and obtain

$$\begin{aligned}
u_1 &= -\frac{(\lambda+\mu) x_3 (2x_1 \sin(2x_1) \sinh(2x_2) - x_2(\sinh^2(2x_2) - \sin^2(2x_1)))}{2(\lambda+3\mu)(\cosh^2(2x_2) - \sin^2(2x_1))^2},\\
u_2 &= -\frac{(\lambda+\mu) x_3 (2x_2 \sin(2x_1) \sinh(2x_2) + x_1(\sinh^2(2x_2) - \sin^2(2x_1)))}{2(\lambda+3\mu)(\cosh^2(2x_2) - \sin^2(2x_1))^2},\\
u_3 &= \frac{x_3^2 \sin(2x_1) \sinh(2x_2)}{(\cosh^2(2x_2) - \sin^2(2x_1))^2} + \frac{(3\lambda+5\mu)(x_2 \sin(2x_1) - x_1 \sinh(2x_2))}{2(\lambda+3\mu)(\cosh^2(2x_2) - \sin^2(2x_1))}.
\end{aligned}$$

The functions given above solve the system (13). We use (21) and the initial function $V_{0,0} = \{0, 0, z_2^2 \tan^2(z_1)\}$ to solve the equations (14)–(17). The transformation of V to u is trivial and therefore is skipped. The solution obtained has singularities at $(\frac{1}{4}\pi + \frac{1}{2}\pi k, 0, x_3)$.

Let us consider the expression (21) for the initial functions $V_{0,0}$ which are independent of z_2. They will bring solutions for the twodimensional Lamé's problem. We write z for z_1 and n for n_1.

The expression (21) transforms to

$$V_n(z) = \frac{(-1)^n}{n!} d_z^n A_4^n V_0(z).$$

The square of the matrix A_4 is equal to zero and therefore $V_n \equiv 0$ for $n > 1$. For $V_1(z)$ we have $V_1^1(z) = 0$, $V_1^2(z) = -(a/(a+2b))d_z V_0^1$ and $V_1^3(z) = 0$ so that

$$\begin{aligned} V^1(z) &= V_0^1(z), \\ V^2(z) &= V_0^2(z) - \frac{a}{a+2b}\bar{z} d_z V_0^1(z), \\ V^3(z) &= V_0^3(z), \end{aligned}$$

where $V_0^1(z)$, $V_0^2(z)$ and $V_0^3(z)$ are arbitrary holomorphic functions. We can express $u_j(x_1, x_2)$ in terms of V^j but it seems more interesting to consider

$$\begin{aligned} \mu(u_1 + \mathrm{i}u_2) &= \frac{\mu}{2}(V^1 + \overline{V}^2) \\ &= \frac{\mu}{2}V_0^1 - \frac{\mu(\lambda+\mu)}{2(\lambda+3\mu)} z d_{\bar{z}} \overline{V}_0^1 + \frac{\mu}{2}\overline{V}_0^2 \\ &= \frac{\lambda+3\mu}{\lambda+\mu}\Big(\frac{\mu(\lambda+\mu)}{2(\lambda+3\mu)}V_0^1\Big) - \frac{\mu(\lambda+\mu)}{2(\lambda+3\mu)} z d_{\bar{z}} \overline{V}_0^1 + \frac{\mu}{2}\overline{V}_0^2. \end{aligned}$$

By taking $\kappa = \lambda + 3\mu/(\lambda+\mu)$, $\varphi(z) = \frac{1}{2}(\mu(\lambda+\mu)/(\lambda+3\mu))V_0^1(z)$ and $\psi(z) = -\frac{1}{2}\mu V_0^2(z)$ we arrive at

$$\mu(u_1 + \mathrm{i}u_2) = \kappa\varphi - z d_{\bar{z}}\bar{\varphi} - \overline{\psi},$$

which is the famous Kolosov-Muskhelishvili formula for the general solution of two-dimensional Lamé's problem.

R e m a r k 3. In [5] Lamé's system was carefully studied for the first time with the help of HE type series. However, they were used just as a technical tool and no mathematical background was developed.

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Author's address: *A. Rodionov*, Departamento de matemáticas, Universidad de Antioquia, AA 1226, Medellin, Colombia, e-mail: `alerod@matematicas.udea.edu.co`.