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Applications of Mathematics, Vol. 50 (2005), No. 3, 203–217

Persistent URL: http://dml.cz/dmlcz/134603

50 (2005) APPLICATIONS OF MATHEMATICS No. 3, 203–217

ON A CONTACT PROBLEM FOR A VISCOELASTIC VON KÁRMÁN PLATE AND ITS SEMIDISCRETIZATION

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Abstract. We deal with the system describing moderately large deflections of thin viscoelastic plates with an inner obstacle. In the case of a long memory the system consists of an integro-differential 4th order variational inequality for the deflection and an equation with a biharmonic left-hand side and an integro-differential right-hand side for the Airy stress function. The existence of a solution in a special case of the Dirichlet-Prony series is verified by transforming the problem into a sequence of stationary variational inequalities of von Kármán type. We derive conditions for applying the Banach fixed point theorem enabling us to solve the biharmonic variational inequalities for each time step.

Keywords: von Kármán system, viscoelastic plate, integro-differential variational inequality, semidiscretization, Banach fixed point theorem

MSC 2000: 49J40, 65R20, 74D10, 74K20

1. Introduction

We study the quasistationary contact problem for the viscoelastic von Kármán system endowed with long memory of the exponential type. We have considered the general case in [2]. We will consider in this case the relaxation function in the form of a Dirichlet-Prony series $E(t) = \sum\limits_{m=0}^{k} E_i e^{-\beta_m t}$, $\beta_0 = 0$, $\beta_m > 0$, $i = 1, \ldots, k$, which will enable us to obtain less restrictive conditions for the existence of a solution. In the special case $k = 1$ the originally long memory integro-differential stress-strain relations can be transformed to a first-order differential system with respect to time [1]. In both cases a weak formulation of the problem can be expressed as one canonical nonstationary variational inequality with respect to admissible deflections. Using time discretization we obtain a finite sequence of stationary variational inequalities with nonlinear operators in Hilbert spaces. In addition to the papers mentioned above we will express their solutions as fixed points of the corresponding operators, which will enable us to solve variational inequalities and equations with biharmonic operators for every time step.

Stationary unilateral boundary value problems for the von Kármán equations were investigated for the first time by Naumann [9] and John [6] with unilateral conditions on the boundary and an inner obstacle, respectively.

Nonstationary von Kármán systems are nowadays studied mainly in the dynamic form. In the case of viscoelastic plates the short or long memory term appears mostly in the linear part of the equation for the deflection ([8]). We will consider here the model of the plate, in which the memory term appears in the variational inequality for the deflection as well as in the nonlinear right-hand side of the equation for the Airy stress function. Semidiscretization with respect to time can be considered the most suitable method to overcome this type of singularity.

2. Formulation of the problem

We assume a thin isotropic plate occupying the domain

$$Q = \{(x,z) \in \mathbb{R}^3\,;\; x = (x_1,x_2) \in \Omega,\; -\tfrac{1}{2}h < z < \tfrac{1}{2}h\},$$

where Ω is a bounded simply connected domain in $\mathbb{R}^2$ with a Lipschitz boundary Γ and a unit outer normal vector $\overline{\nu}$. The plate is clamped on its boundary and subjected to a perpendicular load $f(t,x)$, $t>0$, $x \in \Omega$.

Let us set

$$[v,w] = \partial_{11}v\partial_{22}w + \partial_{22}v\partial_{11}w - 2\partial_{12}v\partial_{12}w, \quad v,w \in H^2(\Omega). \tag{1}$$

We recall that in the elastic case the well known von Kármán system for the deflection w and the Airy stress function Φ has the form ([4])

$$\frac{h^3E}{12(1-\mu^2)}\Delta^2 w - [\Phi,w] = f(x) \quad \text{in } \Omega, \quad w = \frac{\partial w}{\partial \overline{\nu}} = 0 \quad \text{on } \Gamma,$$
$$\Delta^2\Phi = -\frac{Eh}{2}[w,w] \quad \text{in } \Omega, \quad \Phi = \frac{\partial \Phi}{\partial \overline{\nu}} = 0 \quad \text{on } \Gamma$$

with the Poisson ratio $\mu \in \left(0,\frac{1}{2}\right)$ and the Young modulus $E>0$. The stress-strain relations for the isotropic viscoelastic long memory material are of the form

$$\sigma_{ij} = \frac{E(0)}{1-\mu^2}[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}] + \frac{E'}{1-\mu^2} * [(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}](t), \tag{2}$$
$$i,j \in \{1,2\}, \quad \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22}$$

with a positive decreasing relaxation function $E \in C^1(\mathbb{R}^+)$ and a convolution product $f * g(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s$.

Let us define the material function

$$D(t) = \frac{h^3}{12(1-\mu^2)}\,E(t), \quad t \geqslant 0. \tag{3}$$

Applying an approach similar to that in the elastic case by Ciarlet and Rabier ([4]) the following integro-differential von Kármán system for the deflection w and the Airy stress function Φ can be derived:

$$D(0)\Delta^2 w + D' * \Delta^2 w - [\Phi, w] = f(t,x), \quad t > 0,\ x \in \Omega; \qquad w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma, \tag{4}$$

$$\Delta^2 \Phi = -\frac{h}{2}(E(0)[w,w] + E' * [w,w]), \quad t > 0,\ x \in \Omega; \qquad \Phi = \frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on } \Gamma. \tag{5}$$

We denote by $V = H_0^2(\Omega)$ the Hilbert space of all functions from the Sobolev space $H^2(\Omega)$ with zero traces and with zero traces of the derivatives on the boundary Γ. We set

$$(\!(u,v)\!) = \int_\Omega \Delta u \Delta v\,\mathrm{d}x, \quad \|u\| = (\!(u,u)\!)^{1/2},$$

the scalar product and the norm in V. The scalar product and the norm in the space $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$ and $\|\cdot\|_0$, $\langle\cdot,\cdot\rangle$ denotes the duality between V and V^*, $\|\cdot\|_*$ is a norm in the dual space V^*.

We will assume the plane $z = 0$ to be the inner obstacle for the deflection of the plate. The convex closed cone

$$K = \{w \in V \mid w(x) \geqslant 0 \quad \text{for all } x \in \Omega\}$$

is then an admissible set of deflections. Instead of the problem (4), (5) we shall then investigate a unilateral problem in the following variational form:

To find a couple $\{w, \Phi\}\colon\ [0,T) \to K \times V$ fulfilling

$$(\!(D(0)w(t) + D' * w(t), v - w(t))\!) \geqslant ([\Phi(t), w(t)], v - w(t)) + \langle f(t), v - w(t)\rangle \quad \text{for all } v \in K, \tag{6}$$

$$(\!(\Phi(t), \varphi)\!) = -\frac{h}{2}(E(0)[w(t),w(t)] + E' * [w(t),w(t)], \varphi) \quad \text{for all } \varphi \in V. \tag{7}$$

In order to convert the system (6), (7) into one Volterra integral variational inequality in the space V we introduce a bilinear operator $B\colon V\times V\to V$ defined by the uniquely solvable equation

$$((B(u,v),\varphi)) = \int_\Omega [u,v]\varphi\,\mathrm{d}x \quad \text{for all } \varphi\in V. \tag{8}$$

The operator $B\colon V\times V\to V$ is bilinear, compact and fulfils the relations [4]

$$((B(u,v),w)) = ((u,B(v,w))) \quad \text{for all } u,v,w\in V, \tag{9}$$

$$\begin{aligned} &|((B(B(v,v),v)-B(B(w,w),w),v-w))| \\ &\quad\leqslant \|B\|^2\max\{\|v\|^2,\|w\|^2\}\|v-w\|^2 \quad \text{for all } v,w\in V. \end{aligned} \tag{10}$$

Let us define a function $q\colon [0,T)\to V$ uniquely defined as a solution of the equation

$$((q(t),v)) = \frac{1}{D(0)}\langle f(t),v\rangle \quad \text{for all } v\in V. \tag{11}$$

Using the expression (8) and the relation (9) for determining the Airy stress function $\Phi(t)$ we obtain the following variational inequality equivalent with the system (6), (7):

To find a function $w\colon [0,T)\to K$ fulfilling

$$\begin{aligned} &((w(t)+g*w(t),v-w(t))) \\ &\qquad + a((B(w(t),w(t))+g*B(w,w)(t),B(w(t),v-w(t)))) \\ &\quad\geqslant ((q(t),v-w(t))) \quad \text{for all } v\in K \end{aligned} \tag{12}$$

with

$$g(t)=\frac{D'(t)}{D(0)}=\frac{E'(t)}{E(0)},\quad a=\frac{hE(0)}{2D(0)}=\frac{6(1-\mu^2)}{h^2}.$$

3. The case of Dirichlet-Prony series

3.1. Semidiscretization of the problem

We will find a solution to the variational inequality (12) by converting it into a sequence of stationary problems having the form of the canonical von Kármán variational inequalities. We shall use Rothe's method or the method of lines in a way similar to [7], where nonlinear integro-differential equations have been solved.

Before formulating the semidiscrete scheme let us introduce some additional assumptions about the right-hand side q. We assume that the pressure f in the inequality (6) belongs to the Sobolev space $W^{2,1}(0,T;V^*)$, $T>0$ of time dependent functions. The right-hand side of the canonical inequality (12) then fulfils

$$q \in W^{2,1}(0,T;V), \quad T>0.$$

The relaxation function E and the kernel g have the form

$$\begin{aligned} (13) \quad E(t) &= E_0 + \sum_{m=1}^{k} E_m \mathrm{e}^{-\beta_m t}, \quad \beta_m > 0,\\ g(t) &= -\sum_{m=1}^{k} \alpha_m \mathrm{e}^{-\beta_m t}, \quad \alpha_m = \beta_m E_m \Bigl(\sum_{j=0}^{k} E_j\Bigr)^{-1} > 0, \quad i = 1,\dots,k. \end{aligned}$$

For fixed $n \in \mathbb{N}$ we set

$$\begin{aligned} \tau = \frac{T}{n}, \quad t_i = i\tau, \quad w_i = w(t_i), \quad i = 0,1,\dots,n;\\ \delta w_j = \frac{1}{\tau}(w_j - w_{j-1}), \quad j = 1,\dots,n \end{aligned}$$

and substitute the problem (12) by the finite sequence of stationary variational inequalities

$$\begin{aligned} (14) \qquad & (\!(w_0, v - w_0)\!) + a(\!(B(w_0,w_0), B(w_0, v-w_0))\!)\\ & \geqslant (\!(q_0, v - w_0)\!) \quad \text{for all } v \in K, \end{aligned}$$

$$\begin{aligned} (15) \qquad & (\!(w_i, v - w_i)\!) + a(\!(B(w_i,w_i), B(w_i, v-w_i))\!)\\ & -\tau \sum_{m=1}^{k} \alpha_m \mathrm{e}^{-\beta_m t_i} \sum_{j=0}^{i-1} \mathrm{e}^{\beta_m t_j} \bigl[(\!(w_j, v - w_i)\!) + a(\!(B(w_j,w_j), B(w_i, v-w_i))\!)\bigr]\\ & \geqslant (\!(q_i, v - w_i)\!) \quad \text{for all } v \in K, \ i = 1,\dots,n, \end{aligned}$$

where we set $q_i = q(t_i)$, $i = 0,1,\dots,n$.

These variational inequalities have solutions $w_0 \in K$ and $w_i \in K$, $i = 1, \ldots, n$, respectively. They are solutions of the minimizing problems

$$J_i(w_i) = \min_{v \in K} J_i(v), \quad i = 0, 1, \ldots, n, \tag{16}$$

where

$$J_0(v) = \frac{1}{2}\|v\|^2 + \frac{a}{4}\|B(v,v)\|^2 - (\!(q_0, v)\!), \tag{17}$$

$$\begin{aligned} J_i(v) = {} & \frac{1}{2}\|v\|^2 + \frac{a}{4}\|B(v,v)\|^2 \\ & - \tau \sum_{m=1}^{k} \alpha_m \mathrm{e}^{-\beta_m t_i} \sum_{j=0}^{i-1} \mathrm{e}^{\beta_m t_j} [(\!(w_j, v)\!) + (\!(B(w_j, w_j), B(v,v))\!)] - (\!(q_i, v)\!), \\ & \qquad v \in V,\ i = 1, \ldots, n. \end{aligned} \tag{18}$$

The functionals J_i are weakly lower semicontinuous and coercive. Hence for every $i = 0, 1, \ldots, n$ there exists an element $w_i \in V$ fulfilling the minimum condition

$$J_i(w_i) = \min_{v \in K} J_i(v)$$

and solving the discrete variational inequalities (14), (15).

3.2. Uniform a priori estimates

We proceed with uniform a priori estimates. After setting $v = 0$ we obtain directly from (14) the estimate

$$\|w_0\|^2 + a\|B(w_0, w_0)\|^2 \leqslant \|q_0\|^2. \tag{19}$$

Let us denote

$$\omega_i = \|w_i\|^2 + a\|B(w_i, w_i)\|^2, \quad i = 1, \ldots, n. \tag{20}$$

We obtain from (15) the inequality

$$\begin{aligned} \omega_i \leqslant {} & (\!(q_i, w_i)\!) \\ & - \tau \sum_{m=1}^{k} \alpha_m \sum_{j=0}^{i-1} \mathrm{e}^{-\beta_m (i-j)\tau} [(\!(w_j, w_i)\!) + a(\!(B(w_j, w_j), B(w_i, w_i))\!)]. \end{aligned} \tag{21}$$

Using the inequality

$$\begin{aligned} & \sum_{j=0}^{i-1} \mathrm{e}^{-\beta_m (i-j)\tau} [(\!(w_j, w_i)\!) + (\!(B(w_j, w_j), B(w_i, w_i))\!)] \\ & \qquad \leqslant \frac{1 - \mathrm{e}^{-i\beta_m \tau}}{\mathrm{e}^{\beta_m \tau} - 1} \sqrt{\omega_i} \max_{j=0,\ldots,i-1} \{\sqrt{\omega_j}\} \end{aligned}$$

we obtain from (21) the estimate

$$\sqrt{\omega_i} \leqslant \sum_{m=1}^{k} \frac{\alpha_m}{\beta_m} \max_{j=0,\dots,i-1} \{\sqrt{\omega_j}\} + \|q_i\|, \quad i = 1, \dots, n.$$

Applying the expressions for the coefficients α_m in (13) we arrive at the uniform a priori estimate

$$[\|w_i\|^2 + a\|B(w_i, w_i)\|^2]^{1/2} \leqslant M \max_{t\in[0,T]} \|q(t)\|, \tag{22}$$
$$i = 1, \dots, n, \quad M = \Big(\sum_{m=0}^{k} E_m\Big) E_0^{-1}.$$

3.3. A priori estimates of the differences

In order to achieve convergence of the scheme we need estimates of the differences δw_i. We set

$$g_j = -\sum_{m=1}^{k} \alpha_m \mathrm{e}^{-\beta_m j\tau}, \tag{23}$$
$$\delta g_j = \frac{g_j - g_{j-1}}{\tau} = \frac{1 - \mathrm{e}^{\beta_m \tau}}{\tau} g_j, \quad j = 1, \dots, n.$$

After setting $v = w_1$ in (14), $j = k$ and subsequently $i = j$, $v = w_{j-1}$ $i = j-1$, $v = w_j$ in (15) we obtain after subtracting the inequalities

$$\|\delta w_1\|^2 + a(\!(\delta B(B(w_1, w_1), w_1), \delta w_1)\!) + g_1(\!(w_0 + aB(B(w_0, w_0), \delta w_1)\!)$$
$$\leqslant (\!(\delta q_1, \delta w_1)\!),$$

$$\begin{aligned}
&\|\delta w_j\|^2 + a(\!(\delta B(B(w_j, w_j), w_j), \delta w_j)\!) \\
&\quad + \Big(\!\!\Big(g_j w_0 + \tau \sum_{k=1}^{j-1} g_{j-k} \delta w_k, \delta w_j\Big)\!\!\Big) \\
&\quad + a\Big(\!\!\Big(B\Big(\sum_{k=0}^{j-2} \tau \delta g_{j-k} B(w_k, w_k) + g_1 B(w_{j-1}, w_{j-1}), w_j\Big), \delta w_j\Big)\!\!\Big) \\
&\quad + \tau a\Big(\!\!\Big(B\Big(\sum_{k=0}^{j-2} g_{j-1-k} B(w_k, w_k), \delta w_j\Big), \delta w_j\Big)\!\!\Big) \\
&\leqslant (\!(\delta q_j, \delta w_j)\!), \quad j = 2, \dots, i,
\end{aligned}$$

and

$$
\begin{aligned}
\tau\sum_{j=1}^{i}\|\delta w_j\|^2 &+ a(\!(\delta B(B(w_j,w_j),w_j),\delta w_j)\!)\\
&+\tau\sum_{j=1}^{i}(\!(g_j w_0 + aB(g_1 B(w_{j-1},w_{j-1}),w_j),\delta w_j)\!)\\
&+\tau^2\sum_{j=2}^{i}\left(\!\!\left(\sum_{k=1}^{j-1} g_{j-k}\delta w_k + aB\left(\sum_{k=0}^{j-2}\delta g_{j-k}B(w_k,w_k),w_j\right),\delta w_j\right)\!\!\right)\\
&+\tau^2 a\sum_{j=2}^{i}\left(\!\!\left(\sum_{k=0}^{j-2} g_{j-1-k}B(w_k,w_k),B(\delta w_j,\delta w_j)\right)\!\!\right)\\
&\leqslant \tau\sum_{j=1}^{i}(\!(\delta q_j,\delta w_j)\!).
\end{aligned}
$$

Using the properties (9), (10) of the operator B we obtain the inequality

$$
\begin{aligned}
\tau\sum_{j=1}^{i}\|\delta w_j\|^2 &\leqslant 2a\|B\|^2\max_{j\in\{0,\dots,i\}}\|w_j\|^2\tau\sum_{j=1}^{i}\|\delta w_j\|^2\\
&+2a\|B\|\tau\max_{j\in\{2,\dots,i\}}\left\|\sum_{k=0}^{j-2} g_{j-1-k}B(w_k,w_k)\right\|\tau\sum_{j=1}^{i}\|\delta w_j\|^2\\
&+2\tau\sum_{j=1}^{i}\left\|g_j w_0 + aB\left(g_1 B(w_{j-1},w_{j-1})\right.\right.\\
&\qquad\left.\left.+\tau\sum_{k=0}^{j-2}\delta g_{j-k}B(w_k,w_k),w_j\right)\right\|^2\\
&+2\tau^3\sum_{j=2}^{i}\left\|\sum_{k=1}^{j-1} g_{j-k}\delta w_k\right\|^2 + 2\tau\sum_{j=1}^{i}\|\delta q_j\|^2.
\end{aligned}
\tag{24}
$$

Let us assume that there exist $\tau_1 \leqslant \tau_0$ and $\theta\in(0,1)$ such that

$$
2a\|B\|\left(\|B\|\,\|w_j^\tau\|^2 + \tau\left\|\sum_{k=0}^{j-2} g_{j-1-k}B(w_k^\tau,w_k^\tau)\right\|\right)\leqslant 1-\theta
\tag{25}
$$

$$
\text{for every } \tau\leqslant\tau_1 \quad\text{and } j=1,\dots,n,\quad \text{where } n=\frac{T}{\tau}.
$$

We set here $w_j \equiv w_j^\tau$ in order to point out the dependence on τ. Using the expression (23) for the kernel function g we obtain that the bound

$$2a\|B\|^2\left(1+\sum_{m=1}^{k}\frac{\alpha_m}{\beta_m}\right)\max_{j=0,\dots,i}\|w_j^\tau\|^2 \leqslant 1-\theta \tag{26}$$

implies the bound (25). Comparing with the a priori estimate (22) and the expression (13) for the coefficients α_m we can see that the bound

$$\max_{t\in[0,T]}\|q(t)\| \leqslant \frac{E_0}{2\sqrt{a}\|B\|\sum\limits_{m=0}^{k}E_m} \tag{27}$$

is sufficient for fulfilling the estimates (26) and (25).

Combining the inequality (24), the a priori estimate (22) and the differentiability assumption $q \in W^{2,1}(0,T;V)$ we obtain the inequality

$$\tau\sum_{j=1}^{i}\|\delta w_j^\tau\|^2 \leqslant C_1 + C_2\tau^2\sum_{j=2}^{i}\sum_{k=0}^{j-2}\|\delta w_k^\tau\|^2 \quad \text{for every } \tau\in(0,\tau_1). \tag{28}$$

The discrete Gronwall lemma then implies the a priori estimate

$$\tau\sum_{j=1}^{n}\|\delta w_j^\tau\|^2 \leqslant C_3 \quad i=1,\dots,n; \quad 0<\tau\leqslant\tau_1. \tag{29}$$

3.4. The convergence of the semidiscrete scheme

Let us define the following segment line and step functions determined by the values w_i, δw_i:

$$\begin{aligned} &w_n\colon [0,T]\to V, \quad w_n(t) = w_{i-1} + (t-t_i)\delta w_i, \\ &\qquad t_{i-1}\leqslant t\leqslant t_i, \quad i=1,\dots,n, \\ &\overline{w}_n\colon [0,T]\to V, \quad \overline{w}_n(0)=w_0, \quad \overline{w}_n(t)=w_i, \\ &\qquad t_{i-1}< t\leqslant t_i, \quad i=1,\dots,n, \\ &\tilde{w}_n\colon [0,T]\to V, \quad \tilde{w}_n(t)=0, \quad 0\leqslant t\leqslant t_1, \\ &\tilde{w}_n(t)=w_{i-1}, \quad t_i<t\leqslant t_{i+1}, \quad i=1,\dots,n-1. \end{aligned} \tag{30}$$

The a priori estimates (22), (29) imply that the sequence of functions $\{w_n\}$ is bounded in the space $W^{2,1}(0,T;V)$:

$$\|w_n\|_{W^{2,1}(0,T;V)} \leqslant C_4, \quad n\in\mathbb{N}. \tag{31}$$

Hence there exists its subsequence (again denoted by $\{w_n\}$) and a function $w \in W^{2,1}(0,T;V)$ such that

$$w_n \rightharpoonup w \quad \text{in } W^{2,1}(0,T;V), \tag{32}$$

$$w_n(t) \rightharpoonup w(t), \quad \overline{w}_n(t) \rightharpoonup w(t) \quad \text{in } V \quad \text{for every } t \in [0,T], \tag{33}$$

$$w_n \rightharpoonup^* w, \quad \overline{w}_n \rightharpoonup^* w, \quad \tilde{w}_n \rightharpoonup^* w \quad \text{in } L^\infty(0,T;V), \tag{34}$$

$$w_n \to w, \quad \overline{w}_n \to w, \quad \tilde{w}_n \to w \quad \text{in } L^p(0,T;W^{r,1}(\Omega)), \quad p > 1,\ r > 1. \tag{35}$$

Let us set

$$B(w,w)(t) = U(t), \quad B(w_n,w_n)(t) = U_n(t), \quad t \in [0,T], \quad n = 1,2,\ldots \tag{36}$$

Using the properties of the operator B we obtain the convergences

$$U_n \rightharpoonup U \quad \text{in} \quad W^{2,1}(0,T;V), \tag{37}$$

$$U_n(t) \rightharpoonup U(t), \quad \overline{U}_n(t) \rightharpoonup U(t) \quad \text{in } V \quad \text{for every } t \in [0,T], \tag{38}$$

$$U_n \rightharpoonup^* U, \quad \overline{U}_n \rightharpoonup^* U, \quad \tilde{U}_n \rightharpoonup^* U \quad \text{in } L^\infty(0,T;V). \tag{39}$$

The operator $G\colon\ L^p(0,T;V) \to L^p(0,T;V)$, $p > 1$ defined by

$$(Gu)(t) = \int_0^t g(t-s)u(s)\,\mathrm{d}s, \quad u \in L^p(0,T;V)$$

is linear and continuous and the convergences

$$G\tilde{w}_n \rightharpoonup Gw, \quad G\tilde{U}_n \rightharpoonup GU, \quad \overline{G\tilde{w}_n} \rightharpoonup Gw, \quad \overline{G\tilde{U}_n} \rightharpoonup GU \quad \text{in } L^p(0,T;V) \tag{40}$$

follow.

The form (13) of the kernel function implies the coercivity property

$$\int_0^T (((I+G)(v)(t), v(t)))\,\mathrm{d}t \geqslant M^{-1} \int_0^T \|v(t)\|^2\,\mathrm{d}t \quad \text{for all } v \in L^2(0,T;V),$$
$$M = \Big(\sum_{m=0}^k E_m\Big) E_0^{-1}.$$

The variational inequalities (14), (15) can be expressed in the integral form

$$\int_0^T ((\overline{w}_n + \overline{G\tilde{w}_n}, v - \overline{w}_n))\,\mathrm{d}t + a\int_0^T ((\overline{U}_n + \overline{G\tilde{U}_n}, B(\overline{w}_n, v) - \overline{U}_n))\,\mathrm{d}t$$
$$\leqslant \int_0^T ((\bar{q}_n(t), v - \overline{w}_n))\,\mathrm{d}t \quad \text{for all } v \in \mathcal{K},$$
$$\mathcal{K} = \{v \in L^2(0,T;V) \mid v(t) \in K \quad \text{for a.e. } t \in [0,T]\}.$$

Using the limiting process and the weak lower semicontinuity of the functional $v \in \int_0^T (((I+G)(v)(t), v(t)))\,\mathrm{d}t$ over $L^2(0,T;V)$ we arrive at the inequality

$$\begin{aligned}\int_0^T &((w(t)+g*w(t), v(t)-w(t)))\,\mathrm{d}t \\ &+ a\int_0^T ((B(w,w)(t)+g*B(w,w)(t), B(w(t), v(t)-w(t))))\,\mathrm{d}t \\ &\leqslant \int_0^T ((q(t), v(t)-w(t)))\,\mathrm{d}t \quad \text{for all } v\in\mathcal{K}.\end{aligned}$$

Applying the Lebesgue mean value limit theorem we obtain in a way similar to that in [5, Chapt. 1.5.6] that the variational inequality (11) is fulfilled.

3.5. Existence and uniqueness theorems

Theorem 3.1. *Let the kernel function g have the form* (13) *and let $q \in W^{2,1}([0,T],V)$ fulfil the condition* (27).

Then there exists a unique solution $w \in W^{2,1}(0,T;V)$ of the variational inequality (12).

A sequence $\{w_n\}$ of segment line functions defined by discrete values w_i fulfilling the inequalities (14), (15) *such that a weak and strong convergence* (32)–(35) *holds.*

Proof. We have derived above a solution as the limit of a subsequence of segment line functions $\{w_n\}$. It remains to verify uniqueness, which will imply the convergence of the whole sequence $\{w_n\}$.

Let w_1 and w_2 be two solutions of the variational inequality (12). Their difference $u = w_2 - w_1$ then fulfils the inequality

$$\begin{aligned}\|u(t)\|^2 &+ a((B(B(w_2,w_2)(t), w_2(t)) - B(B(w_1,w_1)(t), w_1(t)), u(t))) \\ &\leqslant -\int_0^t g(t-s)((u(s), u(t)))\,\mathrm{d}s \\ &\quad - \int_0^t g(t-s)a((B(B(w_2,w_2)(s), w_2(t)) - B(B(w_1,w_1)(t), w_1(t)), u(t)))\,\mathrm{d}s.\end{aligned}$$

Let $w_\xi = w_1 + \xi(w_2 - w_1)$, $\xi \in R$. The relations

$$\begin{aligned}&((B(B(w_2,w_2), w_2)(t) - B(B(w_1,w_1), w_1)(t), u(t))) \\ &\qquad = \int_0^1 [2\|B(u(t), w_\xi(t))\|^2 + ((B(w_\xi, w_\xi)(t), B(u(t), u(t))))]\,\mathrm{d}\xi, \\ &((B(B(w_2,w_2))(s), w_2(t)) - B(B(w_1,w_1))(s), w_1)(t)), u(t))) \\ &\qquad = \int_0^1 [2((B(u,w_\xi)(s)), B(u,w_\xi)(t))) + ((B(w_\xi,w_\xi))(t), B(u,u)(t)))]\,\mathrm{d}s\end{aligned}$$

hold, from which we obtain the inequality

$$
\begin{aligned}
(41) \qquad \|u\|^2 &+ a\int_0^1 [2\|B(u(t),w_\xi(t))\|^2 \,\mathrm{d}\xi \\
&+ a\int_0^1 ((B(w_\xi,w_\xi)(t),B(u(t),u(t))))\,\mathrm{d}\xi \\
\leqslant &- \int_0^t g(t-s)[((u(s),u(t))) \\
&+ 2a\int_0^1 ((B(u,w_\xi)(s),B(u,w_\xi)(t)))\,\mathrm{d}\xi]\,\mathrm{d}s \\
&- \int_0^t g(t-s)\int_0^1 ((B(w_\xi,w_\xi)(s),B(u,u)(t)))\,\mathrm{d}\xi\,\mathrm{d}s.
\end{aligned}
$$

Using the same approach as in the discrete case the following estimates analogous to the a priori estimates (22) can be derived:

$$\max_{t\in[0,T]}\|w_i(t)\| \leqslant M\max_{t\in[0,T]}\|q(t)\|,\quad i=1,2.$$

The condition (27) then implies

$$a\left|\int_0^1 ((B(w_\xi,w_\xi)(t),B(u(t),u(t))))\,\mathrm{d}\xi\right| \leqslant (1-\theta)\|u\|^2,\quad 0<\theta<1.$$

We obtain then a constant C_5 and the inequality

$$
\begin{aligned}
&[\|u(t)\|^2 + 2\alpha\int_0^1 \|B(u(t),w_\xi(t))\|^2\,\mathrm{d}\xi] \\
\leqslant C_5 &\int_0^t \left[\|u(s)\|^2 + 2\alpha\int_0^1 \|B(u(s),w_\xi(s))\|^2\,\mathrm{d}\xi\right]\mathrm{d}s \quad \text{for all } t\in[0,T].
\end{aligned}
$$

The uniqueness of the solution follows by applying the Gronwall lemma. □

The previous theorem directly implies

Theorem 3.2. *Let the relaxation function have the form*

$$E(t) = E_0 + \sum_{m=1}^{k} E_m \mathrm{e}^{-\beta_m t},\quad \beta_m > 0.$$

Let $f\in W^{2,1}([0,T],V^*)$ *possess the bound*

$$\max_{t\in[0,T]}\|f(t)\|_* \leqslant \frac{h^4E_0}{24\sqrt{6}(1-\mu^2)^{3/2}\|B\|}.$$

Then there exists a unique solution $\{w,\Phi\}\in W^{2,1}(0,T;V)^2$ *of the system* (6), (7).

R e m a r k 3.3. More general homogeneous boundary conditions for the deflection w can be considered. It is possible to consider also a convex set K with a nonzero inner obstacle. Deriving a priori estimates is more complicated in that case.

4. Solving the semidiscretized problem

The Banach fixed point theorem can be applied to solving the semidiscretized variational inequalities (14), (15). We can express them as nonlinear operator equations

$$w_i = A_i(w_i), \quad i = 0, 1, \ldots, N, \tag{42}$$

$$A_0(w) = P_K(q_0 - aB(B(w,w),w)), \tag{43}$$

$$\begin{aligned} A_i(w) = P_K\Biggl(q_i + \tau \sum_{m=1}^{k} \alpha_m \mathrm{e}^{-\beta_m t_i} \\ \times \sum_{j=0}^{i-1} \mathrm{e}^{\beta_m t_j} [w_j + aB(B(w_j, w_j), w)] - aB(B(w,w),w)\Biggr), \end{aligned} \tag{44}$$

where $P_K\colon V \to K$ is the projection operator onto the closed convex set $K \subset V$. Using the property $\|P_K(u) - P_K(v)\| \leqslant \|u - v\|$ for all $u, v \in V$ and the uniform a priori estimates (22) we can derive bounds of the right-hand sides in order to ensure the contractivity of the operators A_i. We impose a more restrictive bound on the right-hand side q than in (27).

Let

$$\max_{t\in[0,T]} \|q(t)\| \leqslant \frac{3E_0}{4(\sqrt{3}+2)\sqrt{a}\|B\| \sum\limits_{m=0}^{k} E_k}. \tag{45}$$

Then we can conclude:

1. There exists $\delta_0 \in (0,1)$ such that

$$\|A_0(u) - A_0(v)\| \leqslant \delta_0 \|u - v\|, \quad \forall u, v \in M_0,$$
$$M_0 = \left\{v \in V\colon \|v\| \leqslant \frac{\sqrt{\delta_0}}{\sqrt{3a}\|B\|}\right\}.$$

2. There exists $\delta_1 \in (0, \frac{3}{4}]$ such that

$$\|A_i(u) - A_i(v)\| \leqslant \delta \|u - v\|, \quad \forall u, v \in M, \quad \delta = \delta_1 \left(1 + \frac{\sum\limits_{m=1}^{k} E_k}{3\sum\limits_{m=0}^{k} E_k}\right),$$
$$M = \left\{v \in V\colon \|v\| \leqslant \frac{\sqrt{\delta_1}}{\sqrt{3a}\|B\|}\right\}.$$

If the right-hand side q corresponding to the right-hand side f of the original problem has the bound (45) then we can solve the equations (42) and hence the variational inequalities (14), (15) using the iterations $w^n = A_i(w_{n-1})$, $i = 0, 1, \ldots, N$ with starting points $w^0 = 0$. We are solving the variational inequalities with the same operator Δ^2 and the Dirichlet boundary conditions. Using the definition (8) of the operator B we obtain systems consisting of an elliptic variational inequality for the deflection and an elliptic equation with the biharmonic operator for the Airy stress function:

$$w_0^1 \in K, \quad ((w_0^1, v - w_0^1)) \geqslant \frac{1}{D(0)} \langle f(0), v - w_0^1 \rangle \quad \forall v \in K,$$

$$\Delta^2 \Phi_0^1 = -\frac{h}{2E(0)} [w_0^1, w_0^1], \quad \Phi_0^1 = \partial_\nu \Phi_0^1 = 0 \quad \text{on } \Gamma,$$

$$w_0^{n+1} \in K,$$

$$((w_0^{n+1}, v - w_0^{n+1})) \geqslant \frac{1}{D(0)} (([\Phi_0^n, w_0^n], v - w_0^n) + \langle f(0), v - w_0^{n+1} \rangle) \quad \forall v \in K,$$

$$\Delta^2 \Phi_0^{n+1} = -\frac{h}{2} (E_0 + E_1) [w_0^{n+1}, w_0^{n+1}], \quad \Phi_0^{n+1} = \partial_\nu \Phi_0^{n+1} = 0 \quad \text{on } \Gamma,$$

$$w_i^1 \in K,$$

$$((w_i^1, v - w_i^1)) \geqslant \frac{1}{D(0)} \langle f_i, v - w_{i-1} \rangle + \tau \sum_{m=1}^{k} \alpha_m \mathrm{e}^{-\beta_m t_i} \sum_{j=0}^{i-1} e^{\beta_m t_j} ((w_j, v_i^1)) \quad \forall v \in K,$$

$$\Delta^2 \Phi_i^1 = -\frac{h}{2E(0)} \Big([w_i^1, w_i^1] + \tau \sum_{m=1}^{k} \alpha_m \mathrm{e}^{-\beta_m t_i} \sum_{j=0}^{i-1} \mathrm{e}^{\beta_m t_j} [w_{i-1}, w_{i-1}] \Big),$$

$$\Phi_i^1 = \partial_\nu \Phi_i^1 = 0 \quad \text{on } \Gamma,$$

$$w_i^{n+1} \in K,$$

$$((w_i^{n+1}, v - w_i^{n+1})) \geqslant \frac{1}{D(0)} (([\Phi_i^n, w_i^n], v - w_i^{n+1}) + \langle f_i, v - w_i^{n+1} \rangle) \quad \forall\, v \in K,$$

$$\Delta^2 \Phi_i^{n+1} = -\frac{h}{2E(0)} \Big([w_i^{n+1}, w_i^{n+1}] + \tau \sum_{m=1}^{k} \alpha_m \mathrm{e}^{-\beta_m t_i} \sum_{j=0}^{i-1} \mathrm{e}^{\beta_m t_j} [w_{i-1}, w_{i-1}] \Big),$$

$$\Phi_i^{n+1} = \partial_\nu \Phi_i^{n+1} = 0 \quad \text{on } \Gamma.$$

Remark 4.1. In the special case $k = 1$ the integral stress-strain relation (2) in the Dirichlet-Prony series can be transformed into differential relations and we obtain pseudoparabolic von Kármán variational inequalities. We have considered this case together with the application of the Banach fixed point theorem in [3].

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