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# NUMERICAL MODEL OF A PINE IN A WIND

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*Abstract.* Steady-state nonlinear differential equations govering the stem curve of a wind-loaded pine are derived and solved numerically. Comparison is made between the results computed and the data from photographs of a pine stem during strong wind. The pine breaking is solved at the end.

*Keywords*: elasticity, nonlinear differential equations, cubic spline

*MSC 2000*: 65C20, 65D07, 65L05

## 1. Introduction

In this paper the theory of elasticity is used for studying properties of the steady wind loaded pine and for answering the question: How strong must the wind be before the pine is broken?

The pine [*pinus silvestra*] grows frequently in Czech woods. Mature pine is about 20–25 meter tall topped with a spherical canopy of radius 1–2 meter. The stem shape is approximately a truncated cone. The stem is covered with thin bark and there are no branches around the stem. All these properties are applied for the construction of an elastic model and for the derivation of the governing equations in Sections 2 and 3. Using dimensional reduction a system of non-linear ordinary differential equations is obtained. This equations are solved numerically in Section 4. The results calculated are compared with a photo of a curved pine taken during a strong wind in the wood. In the last section breaking is studied. It is shown that wind drag can't break a healthy mature pine.

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## 2. Notation

We study the stem curve of the pine in the fixed rectangular system $(x, y, z)$ with its origin at the base of the stem. The pine is gravity-loaded in the direction $-z$ and top-loaded with the wind which blows in the direction of the $x$-axis. We suppose that external forces bend the pine in the $(x, z)$-plane without torsion.

We select an arbitrary point $P$ on the central line of the stem. The independent variable $s$ stands for the length of arc along the stem curve measured from the base to the point $P$. The dependent variable $\Theta$ stands for the size of the angle formed by the tangent to the stem at the point $P$ and the axis $z$ (see Fig. 1). Our aim is to find the angle $\Theta$ depending on the length of $s$.

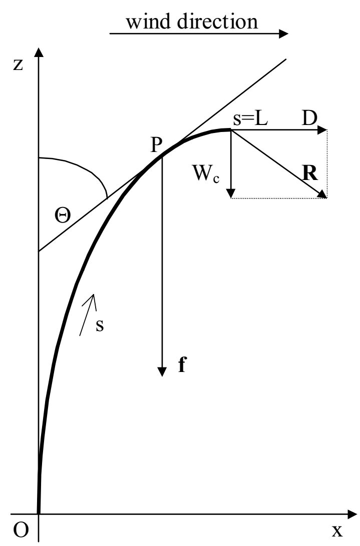


Figure 1. Notation.

We idealize the stem of the pine as a thin homogeneous isotropic column of length $L$. The stem shape is a truncated cone of radius $A$ at the base and of radius $a$ at the canopy. The stem radius at the point $P$ is

$$r(s) = \left(1 - \frac{s}{L}\right) A + \left(\frac{s}{L}\right) a.$$

We denote the total stem weight by $W_s = \frac{1}{3}\pi L(A^2 + Aa + a^2)g\varrho_s$, where $\varrho_s$ is the density of the stem material and $g$ is the gravity acceleration. Young's modulus of the stem material is $E$. The moment of inertia $I$ of a thin cylindrical stem of radius $r(s)$ is given by $\frac{1}{4}\pi r^4(s)$.

We idealize the canopy of the pine as a sphere of radius $a_c$. The total canopy weight $W_c$ can be estimated by $\frac{4}{3}\pi a_c^3 g\varrho_c$, where $\varrho_c$ is the estimated canopy density. From the experiments it is known that a good estimate for $\varrho_c$ is $10\,\mathrm{kg}\cdot\mathrm{m}^{-3}$. The wind drag on the canopy $D$ is determined from a standard formula $D = \frac{1}{2}\varrho_a A_c U^2 C_d$, where $\varrho_a$ is the density of air, $A_c = \pi a_c^2$ is the projected cross section of the canopy viewed from the direction of the wind, $U$ is the wind speed and $C_d$ is the drag coefficient. We set $C_d = 1$ by [3].

## 3. Derivation of equations

We cut the stem into small segments of length $\Delta s$. The relation between neighbour segments is given by the tension and shear forces together with the bending moment. We derive the equation for the segment so that the segment is in static equilibrium. Finally, we take the limit of $\Delta s$ to 0.

We construct a local rectangular system $(\mathbf{e_t}, \mathbf{e_n}, \mathbf{e_b})$ with the origin at the point $P$ (defined in the Section 2). A unit tangent vector $\mathbf{e_t} = (\sin\Theta, 0, \cos\Theta)$ is oriented from the base of the stem to the canopy. A unit principal normal $\mathbf{e_n} = (\cos\Theta, 0, -\sin\Theta)$ is oriented towards the convex side of the curve. A principal binormal is denoted $\mathbf{e_b} = \mathbf{e_t} \times \mathbf{e_n} = (0, 1, 0)$. The distributed gravitational body force $\mathbf{f}(s)$ has components $(0, 0, -\pi r^2(s) g\varrho_s)$.

We are looking for the reaction force vector $\mathbf{F}(s) = T(s)\mathbf{e_t} + V(s)\mathbf{e_n}$ and for the reaction bending moment vector $\mathbf{M}(s) = M(s)\mathbf{e_b}$, where $T(s)$ and $V(s)$ are the sizes of the tension and shear forces on the stem cross section and $M(s)$ is the size of the bending moment on the stem cross section. The point $P$ is fixed at a distance $s_0$ from the base of the stem. We have a truncated cone shaped segment $S$ of length $\Delta s$ that begins at the point $P$. The segment $S$ acts on the higher segment with the force $\mathbf{F}(s_0 + \Delta s)$. The lower segment acts on the segment $S$ with the force $\mathbf{F}(s)$. The weight of the segment is $\int_{s_0}^{s_0+\Delta s} \mathbf{f}(\sigma)\,\mathrm{d}\sigma$. All external forces have to be in static equilibrium:

$$-\mathbf{F}(s_0 + \Delta s) + \mathbf{F}(s_0) + \int_{s_0}^{s_0+\Delta s} \mathbf{f}(\sigma)\,\mathrm{d}\sigma = \mathbf{0}. \tag{1}$$

Similarly, the segment $S$ acts on the higher segment with the bending moment $\mathbf{M}(s_0 + \Delta s)$. The lower segment acts on the segment $S$ with the bending moment $\mathbf{M}(s_0)$. The force $-\mathbf{F}(s_0 + \Delta s)$ rotates the segment with the bending moment $[\Delta s.\mathbf{e_t}] \times [-\mathbf{F}(s_0 + \Delta s)]$. All bending moments have to be in static equilibrium:

$$-\mathbf{M}(s_0 + \Delta s) + \mathbf{M}(s_0) - [\Delta s.\mathbf{e_t}] \times \mathbf{F}(s_0 + \Delta s) = \mathbf{0}. \tag{2}$$

We divide the equations (1) and (2) by $\Delta s$ and take the limit of $\Delta s$ to 0. We obtain two equations

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}s} = \mathbf{f} \qquad \text{and} \qquad \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}s} + \mathbf{e_t} \times \mathbf{F} = \mathbf{0}. \tag{3}$$

From the first equation (3) we can readily construct a formula for the vector $\mathbf{F}$ that satisfies the condition $\mathbf{F}(L) = (-D, 0, W_c)$:

$$\mathbf{F}(s) = \left(-D, 0, \left(1 - \frac{s}{L}\right) K(s) W_s + W_c\right), \quad \text{where} \quad K(s) = \frac{r^2(s) + r(s)a + a^2}{A^2 + Aa + a^2}.$$

For the forces $T(s)$ and $V(s)$ we have

$$\begin{aligned} T(s) &= W_s(1 - \frac{s}{L})K(s)\cos(\Theta(s)) + W_c\cos(\Theta(s)) - D\sin(\Theta(s)), \\ V(s) &= -W_s(1 - \frac{s}{L})K(s)\sin(\Theta(s)) - W_c\sin(\Theta(s)) - D\cos(\Theta(s)). \end{aligned}$$

We can simplify the second equation (3) to

$$M'\mathbf{e_b} = -\mathbf{e_t} \times (T(s)\mathbf{e_t} + V(s)\mathbf{e_n}) = -V(s)\mathbf{e_b}.$$

If we use the Bernoulli-Euler approximation of the bending moment $M(s) = EI(s)\Theta'(s)$ we obtain a non-linear differential equation

$$EI(s)\Theta'' + EI'(s)\Theta' = K(s)W_s\left(1 - \frac{s}{L}\right)\sin\Theta + W_c\sin\Theta + D\cos\Theta. \tag{4}$$

We suppose that the pine is vertical and well clamped with the root system. We can add two conditions $\Theta(0) = \Theta'(0) = 0$ at the base of the stem.

Now we derive the equations for the horizontal and vertical displacements $x(s)$ and $z(s)$ in the direction of the axes $x$ and $z$. For the segment $S$ (mentioned above) we have

$$\begin{aligned} x(s_0 + \Delta s) - x(s_0) &= \Delta s \cdot \sin(\Theta(s_0)), \\ z(s_0 + \Delta s) - z(s_0) &= \Delta s \cdot \cos(\Theta(s_0)). \end{aligned}$$

We divide both the equations by $\Delta s$ and take the limit of $\Delta s$ to 0:

$$x' = \sin\Theta \qquad \text{and} \qquad z' = \cos\Theta. \tag{5}$$

We add the conditions $x(0) = z(0) = 0$ at the base of the stem.

The formulation of the bending problem is now complete. Our aim is to find the angle $\Theta$ from the equation (4) which satisfies the zero clamp condition at the base of the stem and then find the displacements $x$ and $z$ from the equations (5) which satisfy the zero clamp conditions.

Since the right hand sides of equations (4) and (5) are Lipschitzian in $\Theta$ and $x$ or $z$, a solution of this problem exists and is unique by a theorem for existence and uniqueness of solution for ordinary differential equations.

## 4. Solution

Because we are not able to express the solution of (4) and (5) explicitly we use numerical calculation. Our aim is to find a discrete solution $\Theta_D$ of equation (4) in the set of points $\{s_0, \dots, s_N\}$ using the Runge-Kutta method. We smooth the function $\Theta_D$ with a spline $\Theta_{31}$. Finally we integrate equations (5) from 0 to $s$ and compute these integrals numerically. Instead of the function $\Theta$ we use the approximate spline $\Theta_{31}$.

First of all we list some properties of the functions $\Theta$, $x$ and $z$, which we will want to keep during computation. We can think of the angle $\Theta$ in the range $\langle 0, \frac{\pi}{2} \rangle$ and then for all $s \in \langle 0, L \rangle$

(i) the function $\Theta$ is convex, nondecreasing and nonnegative
(ii) the functions $x$ and $z$ are nondecreasing and nonnegative
(iii) the function $x$ is convex
(iv) the function $z$ is concave.

We select an arbitrary equidistant discretization $D\{0, L\}$ of the interval $\langle 0, L \rangle$. We denote the nodes $0 = s_0, \dots, s_N = L$. For solving equation (4) we use the Runge-Kutta method. As output points we choose $s_i$, $i = 0, \dots, N$. We denote this solution by $\Theta_D$.

For interpolation we use the cubic spline $\Theta_{31}$. We have many possibilities how to choose two boundary conditions. We can fix the first and second differentials at $s = 0$ or we can fix the first differential at $s = 0$ and $s = L$. The third possibility, which we are going to use, is to fix the second differential at $s = 0$ and $s = L$. We obtain these two values from equation (4), where for $s$ we substitute 0 or $L$, for $\Theta$ we substitute $\Theta_D(0)$ or $\Theta_D(L)$ and for $\Theta'$ we substitute the values obtained from the Runge-Kutta calculation at the points 0 or $L$.

We integrate equation (5) from 0 to $s$ and using zero boundary conditions we obtain the equations in the integral form

$$x(s) = \int_0^s \sin(\Theta(\sigma))\,\mathrm{d}\sigma \qquad \text{and} \qquad z(s) = \int_0^s \cos(\Theta(\sigma))\,\mathrm{d}\sigma.$$

Instead of the unknown function $\Theta$ we use the spline $\Theta_{31}$ there. Although the function $\Theta_{31}$ is partly polynomial of the third degree, we are not able to find the primitive function. We evaluate integrals numerically. We define discrete functions $x_D$ and $z_D$ at the nodes of discretization $D$ by

$$\begin{aligned} x_D(0) &= 0 \qquad & x_D(s_{i+1}) &= x_D(s_i) + K(s_i, s_{i+1}, \sin\Theta_{31}(s)) \\ z_D(0) &= 0 \qquad & z_D(s_{i+1}) &= z_D(s_i) + K(s_i, s_{i+1}, \cos\Theta_{31}(s)), \end{aligned}$$

where $K(a, b, f(x))$ is the quadrature formula for the integral $\int_a^b f(x)\,\mathrm{d}x$.

| Parameter | Notation | Value |
|---|---|---|
| Pine height | $L$ | $20\,\mathrm{m}$ |
| Stem radius at base | $A$ | $0.13\,\mathrm{m}$ |
| Stem radius at canopy | $a$ | $0.06\,\mathrm{m}$ |
| Stem density | $\varrho_s$ | $760\,\mathrm{kg{\cdot}m^{-3}}$ |
| Young's modulus | $E$ | $10^{10}\,\mathrm{N{\cdot}m^2}$ |
| Canopy radius | $a_c$ | $1.5\,\mathrm{m}$ |
| Canopy density | $\varrho_s$ | $10\,\mathrm{kg{\cdot}m^{-3}}$ |
| Drag coefficient | $C_D$ | $1.0$ |
| Wind speed | $U$ | $20\,\mathrm{ms^{-1}}$ |

Table 1. Physical parameters of mature pine.

For computing $\Theta_D$ we have used the program RKF45, which uses the $4^{\text{th}}$ and $5^{\text{th}}$ order Runge-Kutta method of Fehlberg type with automatic step size selection for solving ordinary differential equations. The step size is acceptable if the relative local discretization error is less than the given value RELERR. We have selected the most precise possibility (in the given arithmetic) RELLER$=10^{-6}$. The number of nodes $N$ in the discretization $D$ has been the pine height (given in meters) plus one. For the computation we have used the physical parameters of a mature pine from Table 1.

For the function $\Theta_{31}$ we have required that it is twice differentiable and possesses the properties of the function $\Theta$ mentioned above. The cubic spline $\Theta_{31}$ has satisfied these requirements throughout all our computation.

For functions $x_D$ and $z_D$ we have required that they possess the properties of functions $x$ and $z$ mentioned above in the following sense:

$$x_D(s_i) \geqslant x_D(s_{i-1}), \qquad z_D(s_i) \geqslant z_D(s_{i-1}), \qquad i = 1 \dots N$$

(the functions $x$ and $z$ are nondecreasing) and

$$\left.\begin{aligned} x_D(s_{i+1}) + x_D(s_{i-1}) \geqslant 2x_D(s_i) \\ z_D(s_{i+1}) + z_D(s_{i-1}) \leqslant 2z_D(s_i) \end{aligned}\right\} i = 1 \dots N-1$$

(the function $x$ is convex, the function $z$ is concave). For all computations these conditions are satisfy if we use for the quadrature $K$ the first Newton-Cortes formula

$$K(a,b,f(x)) = h\left(\sum_{i=1}^{M-1} f(a+ih) + \frac{f(a)+f(b)}{2}\right), \qquad \text{where} \qquad h = \frac{b-a}{M}.$$

If the function $f$ is monotonous and positive then the error of quadrature is less than $|f(b)-f(a)|h/2$. We have used this estimate and we have chosen the step size in the quadrature $K$ for computing $x_D$ and $z_D$ on the interval $\langle s_i, s_{i+1}\rangle$ to be

$$h \leqslant \frac{2\cdot 10^{-3}}{N[\sin\Theta_{31}(s_{i+1}) - \sin\Theta_{31}(s_i)]}$$

or

$$h \leqslant \frac{2\cdot 10^{-3}}{N[\cos\Theta_{31}(s_i) - \cos\Theta_{31}(s_{i+1})]}.$$

The maximum error (in computation) in displacement is less than 1 mm.

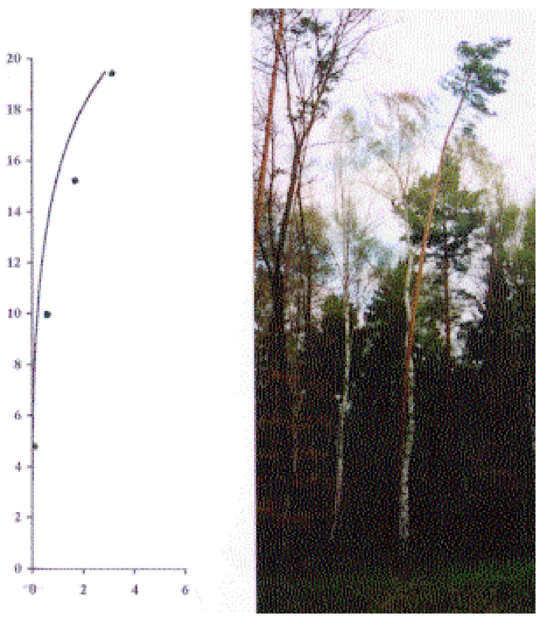


Figure 2. Computation and photo.
Discrete points in the chart are taken from the photo.

We have chosen a pine in a pine wood to verify the "quality" of the mathematical model and computation. We have compared a photo of the curved stem of the pine with the computed results (see Fig. 2). The photo was taken when a 15 m·s$^{-1}$ wind was blowing with $25\,\mathrm{m{\cdot}s^{-1}}$ wind gusts. The photo was taken during the biggest gusts, when $20\,\mathrm{m{\cdot}s^{-1}}$ wind blew at the height of canopy.

## 5. Pine break

We assume that the pine is so well clamped in the ground that it breaks rather than fall. For our model we say that the pine falls if at an arbitrary point the tension is greater than the bounds of stability. For wood we can suppose that the bound of elasticity is close to the bound of stability and use the Hooke law for all tensions.

First we are looking for the point at the cross section where the tension is maximal. For this it is necessary to study more deeply the Bernoulli-Euler approximation of the bending moment. The only non-zero member of the tension tensor is $\sigma_z$ and for it we have the expression

$$\sigma_z = \frac{T(s)}{\pi r^2(s)} - E\Theta'(s)x.$$

The first member in the expression of the tension $\sigma_z$ comes from the tension forces $T(s)$. The second member comes from the shear forces $V(s)$. In our case the principal reasons of bending the pine are the shear forces $V(s)$ and the first member in the expression of $\sigma_z$ is much smaller than the second, so we can neglect it. Now we can readily see that the biggest tension at the cross section is at the point $x = -r(s)$ (the biggest pull) or at the point $x = r(s)$ (the biggest push).

Now we are looking for $s$ such that the expression $E\Theta'(s)r(s)$ is maximal. We show that the differential of the function $E\Theta'(s)r(s)$ is positive so the extreme is for $s = L$. The function $\Theta'(s)$ is nonnegative. The function $r'(s)$ is negative if $A \neq a$. We can estimate

$$\left(E\Theta'(s)r(s)\right)' = E\Theta''(s)r(s) + E\Theta'(s)r'(s) \geqslant E\Theta''(s)r(s) + 4E\Theta'(s)r'(s).$$

Using $I = \frac{1}{4}\pi r^4(s)$ we can write

$$EI\Theta'' + EI'\Theta' = \frac{1}{4}\pi r^3\left(Er\Theta'' + 4Er'\Theta'\right).$$

From the last two equations and from equation (4) we derive

$$\left(E\Theta'(s)r(s)\right)' \geqslant \frac{4}{\pi r^3(s)}\left[K(s)\left(1-\frac{s}{L}\right)W_s\sin\Theta + W_c\sin\Theta + D\cos\Theta\right] > 0$$

for $\Theta \in \langle 0, \frac{\pi}{2}\rangle$.

The above account implies that the biggest tension is at the canopy, so the first point at which the pine would be broken is at the top. But there it can't break because it is the top. It sounds strange, because we know that pines have got broken in the woods. In the subsequent text we give some reasons leading to pines breaking, which our model has not described.

One of the points where the stability is fairly lower is the cross section where a knot is. By [3] a knot decreases the stability by more than 30 %. Usually in one cross section there is more than one knot so the bending stability is fairly lower. Our model has not described this properties because we have supposed that the material is isotropic.

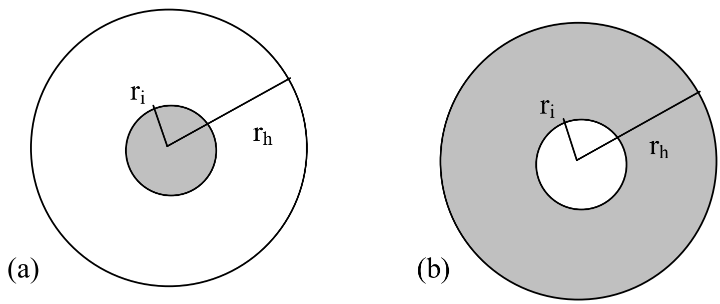


Figure 3. Destroyed pine cross sections.

Another reason leading to pine breaking is roting in the centre of the stem. We are going to show that if 70 % of the cross section is destroyed by roting the bending stability is less than 49 %. Quantities belonging to a healthy pine are marked with the index $h$ while those belonging to an ill one with the index $i$.

If $k \cdot 100\,\%$ of the cross section is rot (see Fig. 3a) then the radius of rot part of the cross section is $r_i^2(s) = kr_h^2(s)$ and for the bending moment we have

$$\frac{M_h(s) - M_i(s)}{M_h(s)} = \frac{EI_h(s)\Theta'(s) - EI_i(s)\Theta'(s)}{EI_h(s)\Theta'(s)} = \frac{I_h(s) - I_i(s)}{I_h(s)} =$$

$$= \frac{\frac{1}{4}\pi r_h^4(s) - \frac{1}{4}\pi(r_h^4(s) - r_i^4(s))}{\frac{1}{4}\pi r_h^4(s)} = k^2.$$

So the bending stability is lower by $100 \cdot k^2\,\%$ (see Fig. 4).

The third reason leading to pine breaking is the destruction of the pine by deer. We are going to show that if deer eats 30 % of a cross section then the bending stability is 51 % lower. The indexes mark the same as in the section above. If deer eat $k \cdot 100\,\%$ of a cross section (see Fig. 3b) then $r_i^2(s) = (1-k)r_h^2(s)$ and for the bending moments we have

$$\frac{M_h(s) - M_i(s)}{M_h(s)} = \frac{I_h(s) - I_i(s)}{I_h(s)} = \frac{\frac{1}{4}\pi r_h^4(s) - \frac{1}{4}\pi r_i^4(s)}{\frac{1}{4}\pi r_h^4(s)} = k(2-k).$$

So the bending stability is lower by $100 \cdot k(2-k)\,\%$ (see Fig. 4).

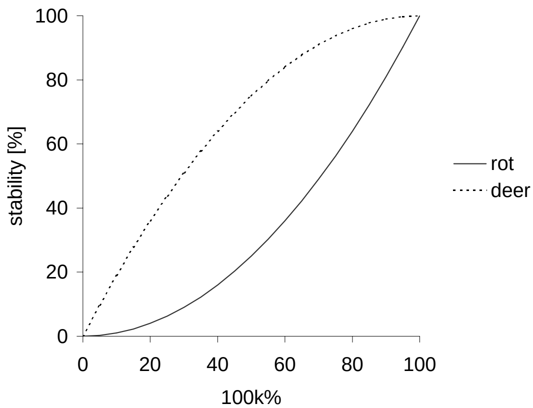


Figure 4. Bending stability of an ill pine.

At the end of this section we summarize all information. A mature wind-top-loaded pine could be broken only if the stem is mechanically damaged or if there are many knots in one cross section. This is the answer why only some pines in the wood are broken. If we could find the critical wind speed which would cause pine breaking then pine woods could not exist, because after a storm with wind speed higher than the critical one all pines would be broken.

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